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Note on a conjecture of Szpiro

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1. Elliptic Curves. L. Szpiro has put forward the

Conjecture. For each $\epsilon > 0$ there is a constant $C(\epsilon)$ with the following property. Let E be any elliptic curve defined over the rationals with minimal discriminant D and conductor N . Then $|D| \leq C(\epsilon)N^{6+\epsilon}$.

This has a number of remarkable consequences (see for example [V] and [HS]), and so a proof would be of considerable interest. Perhaps also a disproof would have some significance. In the present note we show at least that the inequality of the conjecture cannot be much improved; in particular, it would be false in the form $|D| \leq CN^6(\log N)^k$ for any absolute constants C and k . This research was supported in part by the National Science Foundation.

Theorem. For any $\delta > 0$ and N_0 there is an elliptic curve E defined over the rationals whose minimal discriminant D and conductor $N \geq N_0$ satisfy

$$|D| \geq N^6 \exp\{(24-\delta)(\log N)^{1/2}, (\log \log N)^{-1}\}.$$

The proof of this result will be reduced to number theory using the following observation. First for a non-zero rational integer n we write $S(n)$ for the square-free kernel of n ; that is, the product of all distinct positive primes dividing n .

Lemma 1. Suppose a, b, c are coprime rational integers with

$$a+b+c=0, \quad a \equiv 1 \pmod{4}, \quad c \equiv 0 \pmod{32}.$$

Then the equation

$$y^2 = x(x-a)(x+b) \tag{1}$$

defines an elliptic curve E whose minimal discriminant D and conductor N satisfy

$$|D| = 2^{-8}(abc)^2, \quad N = S(abc).$$

Proof. In the standard notation ([S] p. 46) the equation (1) gives

$$c_4 = 16(a^2 + ab + b^2), \quad \Delta = 16(abc)^2.$$

Let p be an odd prime. It is easy to verify that if p divides Δ then p cannot divide c_4 . It follows (see [S] p. 172) that the equation (1) is minimal for all $p \neq 2$.

This is not so for $p = 2$. Indeed, the change of variables

$$x = 4x' + a, \quad y = 8y' + 4x'$$

leads to the equation

$$y'^2 + x'y' = x'^3 + (\alpha + 8\beta)x'^2 + 2a\beta x', \quad (2)$$

where the integers α and β are defined by

$$a = 4\alpha + 1, \quad c = -32\beta.$$

For this new equation we have

$$c'_4 = a^2 + ab + b^2, \quad \Delta' = 2^{-8}(abc)^2;$$

and since c'_4 is odd, we see now that (2) is minimal for $p = 2$.

The formula for D follows at once. The formula for N follows from the definition ([S] p. 361). For if p does not divide abc (in particular $p \neq 2$) then E has good reduction at p . If p divides abc and $p \neq 2$ then (1) is minimal and p does not divide c_4 , so E has multiplicative reduction ([S] p. 180). Finally if $p = 2$ then (2) is minimal, c'_4 is odd, and again E has multiplicative reduction. This completes the proof of Lemma 1.

It is clear that our Theorem is a consequence of Lemma 1 together with the following

Proposition. For any $\delta > 0$ and S_0 there are coprime rational integers a, b, c with

$$a + b + c = 0, \quad a \equiv 1 \pmod{4}, \quad c \equiv 0 \pmod{32}$$

and $S = S(abc) \geq S_0$ satisfying

$$|abc| \geq S^3 \exp\{(12-\delta)(\log S)^{1/2}(\log \log S)^{-1}\}. \quad (3)$$

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A similar result with the weaker inequality

$$\max(|a|, |b|, |c|) \geq S \exp\{(4-\delta)(\log S)^{1/2}(\log \log S)^{-1}\}$$

was established recently by C. Stewart and R. Tijdeman [ST]. In the next section we shall prove our Proposition by means of a small modification in their proof.

2. Number Theory. We require a preliminary lemma. For $y \geq 0$ write

$\theta(y) = \sum_{p \leq y} \log p$ as usual, and for $x \geq 0$ let $\Psi_0(x, y)$ be the number of positive odd integers not exceeding x that are divisible only by primes not exceeding y .

Lemma 2. For any $\delta > 0$ and all sufficiently large x we have

$$e^{-\theta(y)} \Psi_0(x, y) \geq \exp\{(4-\delta)(\log x)^{1/2}(\log \log x)^{-1}\},$$

where $y = (\log x)^{1/2}$.

Proof. Let $\Psi(x, y)$ denote the usual number of positive integers not exceeding x that are divisible only by primes not exceeding y . Good estimates when $y = (\log x)^{1/2}$ were obtained by V. Ennola [E]; we use the version

$$\Psi(x, y) = \exp\{\pi(y) \log \log x - y + O(y(\log y)^{-2})\}$$

given by K.K. Norton ([N] p. 25). Here

$$\pi(y) = y(\log y)^{-1} + y(\log y)^{-2} + O(y(\log y)^{-3})$$

is the usual prime counting function, and we deduce that

$$\Psi(x, y) = \exp\{y + 2y(\log y)^{-1} + O(y(\log y)^{-2})\}. \quad (4)$$

Clearly also

$$\Psi(x, y) = \sum_{h=0}^{\infty} \Psi_0(2^{-h}x, y) = \sum_{h=0}^H \Psi_0(2^{-h}x, y) \leq (H+1)\Psi_0(x, y) \quad (5)$$

for $H = [(\log x)/(\log 2)]$. Finally

$$\theta(y) = y + O(y(\log y)^{-2}), \quad (6)$$

and this together with (4) and (5) leads to the inequality of Lemma 2.

Proof of Proposition. Select x large, put $y = (\log x)^{1/2}$, and let p be the least prime greater than y . Write $T = \Psi_0(x, y)$ and define the positive integer t by

$$x \leq 2^t < 2x.$$

From Lemma 2 we see that $T/p^t \rightarrow \infty$ as $x \rightarrow \infty$. Define the positive integer n by

$$\frac{1}{2}T \leq 2^n p^t < T,$$

and assume x is so large that $n \geq 5$. Since $T > 2^n p^t$, a simple application of the Box Principle enables us to find $t+1$ odd integers x_0, \dots, x_t , divisible only by primes not exceeding y , satisfying

$$1 \leq x_0 < x_1 < \dots < x_t \leq x,$$

and in the same residue class modulo $2^n p$. Since $2^t \geq x$, we can find i with $1 \leq i \leq t$ and

$$x_i \leq 2x_{i-1}. \tag{7}$$

Let d be the highest common factor of x_i and x_{i-1} , and write

$$a = \pm x_i/d, \quad b = \mp x_{i-1}/d, \quad c = \mp(x_i - x_{i-1})/d,$$

where the sign is chosen such that $a \equiv 1 \pmod{4}$. Since d is odd and $n \geq 5$, we also have $c \equiv 0 \pmod{32}$; and clearly $a+b+c = 0$. Further $p > y$ and so p does not divide x_i ; thus p does not divide d . Because p divides $x_i - x_{i-1}$, it must divide c , so that

$$S = S(abc) \geq p.$$

Therefore by assuming x sufficiently large we may suppose $S \geq S_0$ as required.

It remains to check (3). Now clearly $S(ab) \leq \frac{1}{2}e^{\theta(y)}$, and since 2^n divides c we have $S(c) \leq 2^{-(n-1)}|c|$. Thus

$$S \leq S(ab)S(c) \leq 2^{-n}e^{\theta(y)}|c|. \tag{8}$$

Also $|a| \geq |c|$, and (7) gives $|b| \geq \frac{1}{2}|a| \geq \frac{1}{2}|c|$, so that

$$|abc| \geq \frac{1}{2}|c|^3 \geq \frac{1}{2}S^3(2^n e^{-\theta(y)})^3.$$

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Further $p \leq 2y$ and so

$$2^n \geq T/(2pt) \geq T/(4yt) \geq (1/8)T(\log x)^{-3/2} .$$

Therefore

$$|abc| \geq 2^{-10} S^3 (\log x)^{-9/2} (e^{-\theta(y)} T)^3 .$$

Hence by Lemma 2, if x is sufficiently large we have

$$|abc| \geq S^3 \exp\{(12-\delta)(\log x)^{1/2} (\log \log x)^{-1}\} .$$

The Proposition follows on noting from (6) and (8) that if x is sufficiently large then

$$s \leq e^{\theta(y)} |c| \leq e^{2y} x \leq x^{1+\delta} .$$

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