Astérisque

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Astérisque, tome 191 (1990), p. 15-27

<http://www.numdam.org/item?id=AST_1990__191__15_0>

ON THE SPACE OF MAPS BETWEEN R-LOCAL CW COMPLEXES

by

D.J. Anick[1] and E. Dror Farjoun

1. Summary of Results and Notations

The papers [A1,A2] introduced and studied a differential graded Lie algebra (dgL) associated as a model to certain spaces. Building on that work, we construct in this note a simplicial skeleton for the space of pointed maps between two R-local simply-connected CW complexes ($R \subset \mathbb{Q}$). The construction entails two steps. First is the construction, in the category of dgL's, of a cosimplicial resolution and an associated "function complex" valid in a range of dimensions; and second is the connection with the topological mapping space via the above-mentioned models.

1.1. A function complex for dgL's. Let $R = \mathbb{Z}[(p-1)!]^{-1} \subset \mathbb{Q}$ for a prime p, and let L, M be free r-reduced dgL's over R having all generators in dimensions below rp ($r \geq 1$). We will construct a simplicial set, to be denoted hom(L,M), which serves in a range of dimensions as a function complex in the sense of Dwyer and Kan [DK]. Our construction is explicit, in terms of generators and differentials; it is something which could be implemented on a computer. When L and M arise as models for finite spaces X and Y, this means that a simplicial model for the pointed mapping space Y^X is computable in a range of dimensions.

1.2. The range of dimensions. When X and Y are R-local r-connected CW complexes ($r \geq 1$), whose dimensions m_X and m_Y are bounded above by m and by rp respectively ($m < rp$), we may associate to them the dgL models L_X and L_Y. Then Y^X has the d-type of $\hom(L_X, L_Y)$, where

$$d = \min(rp - 1, r + 2p - 3) - m .$$

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[1]Partially supported by a National Science Foundation grant.

Beyond dimension d, $\underline{\text{hom}}(L_X, L_Y)$ is still defined, but its connection with the geometry becomes much hazier.

1.3. Relation to tame homotopy. In view of [D] and [DK], one may associate to a pair of tame spaces (S,T) a function complex in the category of simplicial Lazard algebras. This function complex is homotopy equivalent (as a simplicial set) with the pointed mapping space T^S. When T is not tame, however, it is not obvious how one would obtain information about T^S through this technique. The desire to handle the non-tame case motivated this paper. Instead of requiring spaces to be tame, we require them to be R-local, and we restrict the dimensions where their cells may occur.

(The referee has proposed that Dwyer's functor may be able to be specialized suitably to the category of r-connected simplicial sets generated in dimension $\leq$ m. This specialization, call it S, might yield information about T^S when S belongs to CW_r^m. To accomplish this, one would attempt to use $\mathcal{S}$ in largely the same way that we have used $\mathcal{L}$ in this paper.)

1.4. Notations. We work over a fixed subring R of the rationals, and we denote by p the least non-inverted prime, i.e., $p = \inf\{n \in \mathbb{Z}_+ \mid n^{-1} \notin R\}$. In general, then, $\mathbb{Z}[(p-1)!]^{-1} \subseteq R \subseteq \mathbb{Q}$. As in tame homotopy, the relevant dimension ranges vary with a connectivity parameter r, where $r \geq 1$. Following [A1,A2] we introduce several categories.

- SS denotes the category of simplicial sets.
- TOP is the category of pointed topological spaces and pointed continuous maps.
- $CW_r^n(R)$ denotes the full subcategory of TOP, consisting of r-connected R-local CW complexes of dimension $\leq$ n. "Dimension" means as an R-local cell complex, e.g., the local n-sphere belongs to $\mathcal{O}b CW_r^n(R)$ even though it has topological dimension n + 1.
- $\mathcal{H}oCW_r^n(R)$ is the category obtained from $CW_r^n(R)$ by collapsing (pointed) homotopy classes of maps.
- DGL(R) is the category of connected dgL's over R. A dgL is <u>free</u> if it is free as a Lie algebra (ignoring the differential); in this case we write it as $(\mathbb{L}(V), \delta)$, where the <u>R-module of</u>

generators $V = \oplus_{i=1}^{\infty} V_i$ is free and positively graded, and the differential δ has degree -1.

- $DGL_r^m(R)$ denotes the full subcategory of DGL(R) whose objects have the form $(\mathbb{L}(V),\delta)$ where $V = \oplus_{i=r}^{m} V_i$, i.e., they are free with all generators occurring in dimensions r through m, inclusive.
- L denotes the model, introduced in [A1], which carries $CW_r^{m+1}(R)$ to $DGL_r^m(R)$ when $m < rp$.

1.5. Distinguished morphisms in $DGL_r^m(R)$. The category $DGL_r^m(R)$ cannot be made into a closed model category, but we will find it convenient to distinguish three classes of morphisms anyway. Call $f \in Mor\,DGL_r^m(R)$ a weak equivalence if it induces an isomorphism on homology of universal enveloping algebras. It is a cofibration if it splits as an inclusion of free Lie algebras (ignoring the differential), and it is a fibration if it is surjective in dimensions above r. Trivial fibrations are simultaneously fibrations and weak equivalences.

2. Function Complexes in $DGL_r^m(R)$

We will now investigate the possibility of doing homotopy theory in $DGL_r^m(R)$. The dimension limitation, viz., the "m" in $DGL_r^m(R)$, spoils our hope of doing so in the sense of Quillen [Q] or even Baues [B]. We cannot dispense entirely with the bound m, because dgL's exhibit a variety of undesirable behaviors when generator dimensions are permitted to exceed rp. On the other hand, the canonical constructions of turning a map into a fibration or cofibration tend to increase the dimensions of generators, and thus they eventually bump us out of any fixed $DGL_r^m(R)$.

An alternate approach is suggested in [T] and [A1]. We may define for $m < rp$ a homotopy relation on morphisms by utilizing a certain cylinder construction, which raises by one the maximum generator dimension. The gap between m and rp then offers us a "breathing space" in which we can perform the standard constructions approximately $(rp - m)$ times, and thus higher homotopy information is obtainable up to dimension (approximately) $rp - m$. This cylinder construction, known as the Tanré cylinder, is recalled next.

2.1. The Tanré cylinder. This is developed in [T] and [A1] so we provide here only a brief overview. Given a dgL $L = (\mathbb{L}(V),\delta)$ in $DGL_r^m(R)$, where $m < rp$, Tanré associates to it another dgL in $DGL_r^{m+1}(R)$, denoted $IL = (I\mathbb{L}(V), I\delta)$. Taking the set of weak equivalences to be as in 1.5, the dgL IL is a valid cylinder object on L in the sense of [Q] or [B]. In particular, I comes with natural weak equivalences j_0, j_1: id $\to I$, and if $L \overset{f}{\underset{g}{\rightrightarrows}} M$ are two morphisms in $DGL_r^m(R)$, then f and g are homotopic if and only if $f \amalg g$ factors through IL. Collapsing homotopy classes gives us a category which we denote by $\mathit{Ho}DGL_r^m(R)$.

We remark that I is not a functor, although $If\colon IL \to IM$ exists non-canonically for each $f\colon L \to M$ in $\mathit{Mor}DGL_r^m(R)$. However, I does satisfy the weak naturality condition $If \circ j_0(L) = j_0(M) \circ f$, $If \circ j_1(L) = j_1(M) \circ f$.

2.2. Constructing the cosimplicial resolution. We construct next an initial segment of a cosimplicial resolution for objects in $DGL_r^m(R)$. We shall use it to define a function complex between two such dgL's. We follow as closely as possible the standard procedure, due to Dwyer and Kan [DK], for constructing cosimplicial resolutions in any closed model category. By a <u>cosimplicial resolution</u> for an object A we mean a (not necessarily functorial) diagram

$$A \rightleftarrows \Delta^1 A \;\; \Delta^2 A \; \ldots \; \Delta^n A \; \ldots \tag{1}$$

satisfying the usual cosimplicial identities. In (1), each arrow is a weak equivalence; the coface maps are cofibrations, while the codegeneracies are fibrations. (See [DK, Section 4.3] for a precise definition.)

Let us review the Dwyer-Kan construction for a closed model category $\mathcal{C}$. Given an object A, a <u>cylinder</u> on A is an object IA which provides the first stage of a cosimplicial resolution for A. That is, IA fits into a diagram

$$A \overset{i_0}{\underset{i_1}{\rightrightarrows}} A \amalg A \xrightarrow{c} IA \xrightarrow{q} A \tag{2}$$

such that c is a cofibration, q is a trivial fibration, and both composites are the identity on A. This I( ) need not be a functor, but we do assume the compatibility of $j_0 = ci_0$ and $j_1 = ci_1$ with any If. Typically I arises by factoring the folding morphism $A \amalg A \xrightarrow{\nabla} A$ into a cofibration followed by a trivial fibration.

Assuming one has such an I, let $\underset{\sim}{\Delta}^0$ be the identity functor and let $\dot{\underset{\sim}{\Delta}}^1$ be the functor $\dot{\underset{\sim}{\Delta}}^1 A = A \amalg A$. Then let $\underset{\sim}{\Delta}^1$ be the push-out of $A \xleftarrow{\nabla} \dot{\underset{\sim}{\Delta}}^1 A \xrightarrow{j_0} I\dot{\underset{\sim}{\Delta}}^1 A$. It is obvious how $\underset{\sim}{\Delta}^1 A$ serves as the first stage in the cosimplicial resolution (1).

Inductively, suppose the first $(n - 1)$ stages of (1) have been constructed. Let F_A be the functor from the category of faces of the simplicial complex $\dot{\Delta}^n$ and inclusions among them (see 3.2) to C, which takes a k-simplex to $\underset{\sim}{\Delta}^k A$, and an inclusion to the appropriate arrow of (1). Let $\dot{\underset{\sim}{\Delta}}^n A$ be $\mathrm{colim}(F_A)$ and let $\underset{\sim}{\Delta}^n A$ be the push-out of

$$A \longleftarrow \dot{\underset{\sim}{\Delta}}^n A \longrightarrow I\dot{\underset{\sim}{\Delta}}^n A . \qquad (3)$$

We wish to perform the Dwyer-Kan construction in the category $DGL^m_r(R)$, which is not a closed model category. Let us check precisely which axioms are used. Assuming the existence of I, we need: closure under finite colimits for diagrams of cofibrations; that the push-out of a (resp. trivial) cofibration exists and is a (resp. trivial) cofibration; that two out of three of f and g and gf being weak equivalences makes the third a weak equivalence; and the left lifting property for cofibrations with respect to trivial fibrations. When we take I to be I, the category $DGL^m_r(R)$ satisfies these four axioms, for $m \leq rp$.

However, as we have noted, the Tanré cylinder construction I applied to a dgL L having some m-dimensional generators will have some (m+1)-dimensional generators. Inductively, $\underset{\sim}{\Delta}^n L$ lies in $DGL^{m+n}_r(R)$. This dimension shift, along with the constraint $m + n \leq rp$, is what confines us to an initial segment of a cosimplicial resolution (1).

We have actually verified

LEMMA 2.3. When $m + n \leq rp$, there are constructions

$$\underset{\sim}{\Delta}^n,\ \dot{\underset{\sim}{\Delta}}^{n+1}: DGL^m_r(R) \to DGL^{m+n}_r(R).$$

Applied to a dgL $L \in Ob DGL^m_r(R)$, they come with homomorphisms that provide the first $rp-m$ stages of a cosimplicial resolution (1) for L.

Definition 2.4. For $L \in Ob DGL^m_r(R)$, $M \in Ob DGL(R)$, let $\underset{\sim}{\Delta}^n$ be as in Lemma 2.3 for $n \leq rp - m$. Define the function complex between L and M, denoted hom(L,M), to be the simplicial set consisting of $Hom_{DGL(R)}(\underset{\sim}{\Delta}^n L, M)$ in dimension n when $n \leq rp - m$, and consisting of degeneracies only, above dimension $rp - m$.

Remark 2.5. Definition 2.4 may depend upon choices made during the construction of $\underset{\sim}{\Delta}^n L$. The results that we are interested in will hold regardless of which choices were made. More importantly, the definition depends upon m and r, in the sense that the relevant dimension range will vary according to which $DGL^m_r(R)$ we view a given L as lying in. In practice, of course, we will want to use the largest possible r and the smallest possible m. In this paper, the intended r and m will always be apparent from the context.

3. Constructing the Simplicial Map

Having constructed hom(L,M) for dgL's, we turn our attention to its connection with the pointed mapping space Y^X. We have mentioned the dgL model L for pointed R-local CW complexes. We will define a simplicial map $\hat{L}$ from a skeleton of Y^X to hom($L(X)$,$L(Y)$).

3.1. The model L. In [A1] the first author showed that for any $X \in Ob CW^{m+1}_r(R)$ with $m < rp$ there exists $L \in Ob DGL^m_r(R)$ such that UL is an Adams-Hilton model for X. We write $L(X)$ for this L. One has a similar assertion and notation for maps. The passage from X to L is not functorial, since X does not canonically determine

L; nor does a map $f\colon X \to Y$ uniquely determine $L(f)$, even after $L(X)$ and $L(Y)$ have been fixed. However, $L(f)$ is determined up to homotopy, and hence $L(X)$ is determined up to homotopy type. In spite of this indeterminacy, the function complex between such models always does the right thing up to a certain dimension.

The main advantage of L as a model for X is that it is built directly from a cellular decomposition of X, so it is fairly small and accessible to computations.

3.2. Review of Y^X. The pointed mapping space Y^X may be viewed as the simplicial set

$$Y^X = \{\mathrm{Hom}_{TOP}(|\Delta^n| \ltimes X, Y)\}_{n\geq 0} \ . \tag{4}$$

Here Δ^n is the standard simplicial complex whose geometric realization is the standard n-simplex, and $\ltimes$ denotes the half-smash. The subcomplex of Δ^n obtained by removing the n-simplex is denoted, as usual, by $\dot{\Delta}^n$.

Denote by $sd(\Delta^n)$ (resp. $sd(\dot{\Delta}^n)$) the first barycentric subdivision of Δ^n (resp. $\dot{\Delta}^n$). Whenever $X \in ObCW_r^m(R)$, then an easy Kunneth formula argument shows that $|sd(\Delta^n)|\ltimes X$ and $|sd(\dot{\Delta}^n)|\ltimes X$ belong to $ObCW_r^{m+n}(R)$ (cf. 4.4 for a discussion of CW structures). As long as $m + n \leq rp$, a model $L(|sd(\Delta^n)|\ltimes X)$ exists for $|\Delta^n|\ltimes X$.

LEMMA 3.3. For $X \in ObCW_r^m(R)$, $m + n \leq rp$, one can choose models such that there are isomorphisms

$$L(|sd(\Delta^n)|\ltimes X) \approx \underset{\sim}{\Delta}^n L(X) \ , \quad \text{and} \tag{5}$$
$$L(|sd(\dot{\Delta}^{n+1})|\ltimes X) \approx \underset{\sim}{\dot{\Delta}}^{n+1} L(X) \ .$$

Furthermore, the model L applied to the coface and codegeneracy maps

$$|sd(\Delta^n)|\ltimes X \leftrightarrows |sd(\Delta^{n+1})|\ltimes X$$

may be taken to be the coface and codegeneracy homomorphisms mentioned in Lemma 2.3, for $L = L(X)$.

Proof. This is easily deduced by induction on n. At each stage, L can be chosen to commute with colimits of inclusions of CW complexes [A1, Theorem 8.5i], with cylinders [A2, Lemma 5], and with push-outs in which one map is CW and the other is an inclusion into a cylinder [A2, Lemma 6].

PROPOSITION 3.4. Let $X \in ObCW_r^m(R)$ where $m \leq rp$, and let $Y \in ObCW_r^{rp}(R)$. There is a homomorphism of simplicial sets

$$(6) \qquad \hat{L}: (Y^X)^{rp-m} \to \underline{hom}(L(X), L(Y)).$$

The source of (6) is the (rp-m)-skeleton of the simplicial set (4). For each $f \in (Y^X)^{rp-m}$, $\hat{L}(f)$ may be interpreted as a valid L-model for f.

Proof. We build $\hat{L}$ dimension by dimension. Assume we have the simplicial map

$$\hat{L}^{n-1}: (Y^X)^{n-1} \to \underline{hom}(L(X), L(Y)).$$

For each element $f: |\Delta^n| \ltimes X \to Y$, view f as a map from the CW complex $|sd(\Delta^n)| \ltimes X$ to Y. Consider

$$(7) \qquad \underset{\sim}{\Delta}^n L(X) \xrightarrow[\text{by (5)}]{\approx} L(|sd(\Delta^n)| \ltimes X) \xrightarrow[L(f)]{} L(Y).$$

This composite belongs to the dimension n part of $\underline{hom}(L(X), L(Y))$ if $n \leq rp - m$. Thus we may extend $\hat{L}^{n-1}$ to $\hat{L}^n: (Y^X)^n \to \underline{hom}(L(X), L(Y))$ by defining $\hat{L}^n(f)$ to be the composite (7). The only subtlety is the requirement that $\hat{L}^n$ is to be a simplicial map, i.e., compatible with faces and degeneracies. This in turn requires that we utilize the flexibility inherent in our choices for $L(f)$.

We are supposing that $\hat{L}^{n-1}$ is simplicial, i.e., these choices have been made compatibly below dimension n. Given $f: |\Delta^n| \ltimes X \to Y$, let $\dot{f}$ denote the restriction $\dot{f}: |sd(\dot{\Delta}^n)| \ltimes X \to Y$, and for $0 \leq i \leq n$ let $f_i: |sd(\Delta^{n-1})| \ltimes X \to Y$ denote the further restriction to the i^{th} face of $|\Delta^n|$ half-smashed with X. By our inductive assumption, the $L(f_i)$ are compatible with faces; by [A1, theorem 8.5j] their colimit serves as a valid choice for $L(\dot{f})$. Lastly, use [A1, theorem 8.5h] to extend this choice for $L(\dot{f})$ to some valid model $L(f)$. By Lemma 3.3, the resulting choice for $\hat{L}(f)$ remains compatible with faces and degeneracies.

PROPOSITION 3.5. Let $X \in Ob\,CW_r^t(R)$, $Y \in Ob\,CW_r^{rp}(R)$, where $t = \min(rp - 1, r + 2p - 3)$. Then $\hat{L}$ induces a bijection

$$\pi_0(\hat{L}): \pi_0(Y^X) \to \pi_0(\underline{\hom}(L(X), L(Y))).$$

If instead $X \in Ob\,CW_r^{t+1}(R)$, then $\pi_0(\hat{L})$ is a surjection .

Proof. For $L, M \in Ob\,DGL_r^{rp-1}(R)$, $f \amalg g: L \amalg L \to M$ extends over IL if and only if it extends over $\underset{\sim}{\Delta}^1 L$. Thus $\pi_0(\underline{\hom}(L,M))$ coincides with the (Tanré-induced) set of homotopy classes $[L;M]$. Also, this diagram commutes:

$$\begin{array}{ccccc} \pi_0(Y^X) & \xleftarrow{\approx} & \pi_0((Y^X)^1) & \xrightarrow{\pi_0(\hat{L})} & \pi_0(\underline{\hom}(L(X), L(Y))) \\ \downarrow \approx & & \downarrow & & \downarrow \approx \\ [X;Y] & \xleftarrow{=} & [X;Y] & \xrightarrow{(L)_\#} & [L(X); L(Y)] \end{array} \tag{8}$$

where we have put $m = rp - 1$. By [A2, Theorem 3] the arrow $(L)_\#$ of (8) is a bijection. When $\dim(X) = t + 1$, use $(Y^X)^0$ in place of $(Y^X)^1$ in (8); then the upper left arrow and $(L)_\#$ are surjections, hence so is $\pi_0(\hat{L})$.

4. The d-type of Y^X

We conclude by showing that the simplicial map $\hat{L}$ of (6) is a homotopy equivalence in a range of dimensions. We fix the notation

(9) $$t = \min(rp - 1, r + 2p - 3).$$

4.1. Simplicial d-type. Let A and B denote simplicial sets, and let $d \geq 0$. A d-equivalence is a simplicial map $g: A \to B$ such that, for every choice of base point $a_0 \in (A)_0$, g induces a bijection on π_n for $n < d$ and a surjection on π_d. We say that B and B' have the same (d-1)-type if and only if there is a simplicial set A which comes with d-equivalences $B \xleftarrow{g} A \xrightarrow{g'} B'$. "Same (d-1)-type" is an equivalence relation because, if $B'' \longleftarrow A \xrightarrow{g} B \xleftarrow{g'} A' \longrightarrow B'$ are d-equivalences, letting A" be the fiber-homotopy pull-back of g

and g' leads to d-equivalences $A \leftarrow A'' \rightarrow A'$. (For an alternate approach to (d-1)-type, see [B, p. 364].) Note that the skeleton inclusion $A^d \rightarrow A$ is always a d-equivalence. Lastly, the condition on π_0 amounts to the requirement that g induce a bijection on path-components (resp., a surjection, if $d = 0$).

Two spaces having the same d-type tells us that their homotopy groups $\pi_n(\ )$ are isomorphic for $n \le d$, but it tells us much more than this. For instance, the spaces S^2 and $CP^\infty \times S^3$ have isomorphic π_n for all n; they have the same 2-type $(S^2 \leftarrow S^2 \vee S^3 \rightarrow CP^\infty \times S^3)$ but not the same 3-type.

We assert (see 4.7) that Y^X and $\underline{\text{hom}}(L(X),L(Y))$ have the same d-type, for a certain d.

4.2. Relative homotopy in $DGL_r^m(R)$. We need the concept of a relative homotopy, for dgL's. First let us review the concept for spaces. Let W be a pointed space and let X be a subspace; we fix a pointed map $\phi: X \rightarrow Y$. Denote by $\text{Hom}_{TOP}(W,Y)_\phi$ the set of all extensions of ϕ over W. Two maps in $\text{Hom}_{TOP}(W,Y)_\phi$ are <u>homotopic rel X</u>, denoted $f \underset{\phi}{\simeq} g$, if and only if there is a homotopy $F: W\times[0,1] \rightarrow Y$ such that $F|_{W\times 0} = f$, $F|_{W\times 1} = g$, and $F(w,s) = \phi(w)$ for $w \in X$. Denote by $[W;Y]_\phi$ the set of $\underset{\phi}{\simeq}$-equivalence classes. We will be especially interested in the case where W is a CW complex and X is a subcomplex.

Let $L \rightarrow K$ be a cofibration in $DGL_r^m(R)$, $m < rp$; we identify L with a sub-dgL of K. Let $M \in \mathcal{O}b DGL(R)$, and fix a dgL homomorphism $\lambda: L \rightarrow M$. Denote by $\text{Hom}_{DGL(R)}(K,M)_\lambda$ the set of all extensions of λ over K.

Although we have stressed that the Tanré cylinder I is not natural, there is a cofibration $IL \rightarrow IK$ which extends the given cofibration $L \amalg L \rightarrow K \amalg K$. Let $q_L: IL \rightarrow L$ denote the trivial fibration which extends the fold map $L \amalg L \xrightarrow{\nabla} L$. Two dgL homomorphisms in $\text{Hom}_{DGL(R)}(K,M)_\lambda$ are <u>homotopic rel L</u>, denoted $f \underset{\lambda}{\simeq} g$, if and only if there exists $F: IK \rightarrow M$ whose restriction to $K \amalg K$ is $f \amalg g$ and whose restriction to IL is λq_L. Denote the set of $\underset{\lambda}{\simeq}$-equivalence classes by $[K;M]_\lambda$.

PROPOSITION 4.3. Let $W \in Ob\mathrm{CW}_r^t(R)$, let X be a subcomplex, and let $Y \in Ob\mathrm{CW}_r^{rp}(R)$. Fix a map ϕ: $X \to Y$ and fix a model $\lambda = L(\phi)$: $L(X) \to L(Y)$. Then L induces a bijection

$$(10) \qquad Ho(L)\colon [W;Y]_\phi \to [L(W);L(Y)]_\lambda ,$$

in which a $\underset{\phi}{\simeq}$-class $[f]$ is sent to the $\underset{\lambda}{\simeq}$-class $[L(f)]$. If instead $W \in Ob\mathrm{CW}_r^{t+1}(R)$, then (10) is a surjection.

Proof. One may easily adapt the proof of [A2, Theorem 3] to cover this situation as well. One needs only to be careful always to choose $L(f)$ for f: $W \to Y$ so as to extend the model λ for $f|_X$.

4.4. Homomorphisms induced by $\hat{L}$. We intend to study the homomorphisms induced by the $\hat{L}$ of (6) on homotopy groups. Let $X \in Ob\mathrm{CW}_r^m(R)$, $m \leq rp$, and $Y \in Ob\mathrm{CW}_r^{rp}(R)$. Fix a map ϕ: $X \to Y$ and view Y^X as the simplicial set (4); thus $\phi \in (Y^X)_0$. Fix $n \geq 0$ and take as base point the 0^{th} vertex $v_0 \in |sd(\dot{\Delta}^{n+1})|$. Henceforth, when we write S^n, we will intend S^n to be viewed as the CW realization $|sd(\dot{\Delta}^{n+1})|$ with base point v_0 (i.e., as a CW complex, S^n has one cell for each non-degenerate simplex of $sd(\dot{\Delta}^{n+1})$). Let $W = S^n \ltimes X$. The CW structures on S^n and on X give us a CW structure on W; note that $W \in Ob\mathrm{CW}_r^{m+n}(R)$. We identify X with the subcomplex $v_0 \times X$ of W. Clearly, $[W;Y]_\phi$ makes sense.

We consider the same setup in $\mathrm{DGL}_r^{rp}(R)$. Let $L \in Ob\mathrm{DGL}_r^m(R)$, $m < rp$, $M \in Ob\mathrm{DGL}(R)$. When $m + n \leq rp$, $\underset{\sim}{\Delta}^n L$ is defined, and we may include L into $\underset{\sim}{\Delta}^n L$ "at the 0^{th} vertex" (see (1)). Thus L is viewed as a sub-dgL of $K = \underset{\sim}{\dot{\Delta}}^{n+1} L$, and $[K;M]_\lambda$ makes sense for any given λ: $L \to M$. When $L = L(X)$, we may by Lemma 3.3 identify K with $L(W)$. Then the inclusion of the sub-dgL L into K is a valid L-model for the subcomplex inclusion $X \to W$ described above.

Now let $X \in Ob\mathrm{CW}_r^m(R)$, $m \leq rp$, $Y \in Ob\mathrm{CW}_r^{rp}(R)$, as above. Choose an $\hat{L}$ as in Proposition 3.4. Let $\lambda = \hat{L}(\phi)$, which is a valid L-model for ϕ. For $n \leq rp - m$, consider the diagram

$$
(11) \quad
\begin{array}{ccc}
\mathrm{Hom}_{SS}(\dot{\Delta}^{n+1}, v_0; (Y^X)^{rp-m}, \phi) & \xrightarrow{(\hat{L})_*} & \mathrm{Hom}_{SS}(\dot{\Delta}^{n+1}, v_0; \underline{\mathrm{hom}}(L(X), L(Y)), \lambda) \\
\downarrow \alpha & & \\
\mathrm{Hom}_{SS}(\dot{\Delta}^{n+1}, v_0; Y^X, \phi) & & \downarrow \approx \\
\downarrow \approx & & \\
\mathrm{Hom}_{TOP}(|\dot{\Delta}^{n+1}| \ltimes X, Y)_\phi & & \mathrm{Hom}_{DGL(R)}(\underset{\sim}{\dot{\Delta}}^{n+1} L(X), L(Y))_\lambda \\
\downarrow = & & \downarrow \approx \\
\mathrm{Hom}_{TOP}(W, Y)_\phi & \overset{L'}{- - - \rightarrow} & \mathrm{Hom}_{DGL(R)}(L(W), L(Y))_\lambda \, .
\end{array}
$$

Because all the vertical arrows in (11) are bijections, there is a unique L' which makes the diagram commute. The following lemma follows easily from the construction of $\hat{L}$.

LEMMA 4.5. For any choice of $\hat{L}$ as in Proposition 3.4, the function L' of (11) satisfies this: for any $f \in \mathrm{Hom}_{TOP}(W, Y)_\phi$, $L'(f)$ is a valid L-model for f.

The reader may now check that the equivalence relations that we have on the various sets in (11) are compatible with the arrows, and lead to the diagram

$$
(12) \quad
\begin{array}{ccc}
\pi_n((Y^X)^{rp-m}, \phi) & \xrightarrow{(\hat{L})_\#} & \pi_n(\underline{\mathrm{hom}}(L(X), L(Y)), \lambda) \\
\downarrow \alpha_\# & & \\
\pi_n(Y^X, \phi) & & \downarrow \approx \\
\downarrow \approx & & \\
\mathrm{Hom}_{TOP}(W, Y)_\phi / (\underset{\phi}{\simeq}) & & \mathrm{Hom}_{DGL(R)}(\underset{\sim}{\dot{\Delta}}^{n+1} L(X), L(Y))_\lambda / (\underset{\lambda}{\simeq}) \\
\downarrow = & & \downarrow \approx \\
[W; Y]_\phi & \xrightarrow{(L')_\#} & [L(W); L(Y)]_\lambda \, .
\end{array}
$$

The following two facts are also clear.

LEMMA 4.6. (a) In (12), $(L')_\#$ coincides with $Ho(L)$ of (10). (b) In (12), $\alpha_\#$ is bijective if $m + n < rp$ and surjective if $m + n = rp$.

THEOREM 4.7. Let $X \in Ob\mathrm{CW}_r^m(R)$, $m \leq t + 1$, $Y \in Ob\mathrm{CW}_r^{rp}(R)$. Put $d = t - m$ (cf. (9)). The simplicial map $\hat{L}$ of (6) is a (d+1)-equivalence. Consequently, the simplicial sets Y^X and $\underline{\mathrm{hom}}(L(X), L(Y))$ have the same d-type.

Proof. The condition on π_0 is actually given by Proposition 3.5. When $t - m \geq n > 0$, $(\hat{L})_{\#}$ of (12) is bijective, by 4.3 and 4.6. When $n = t - m + 1$, $(\hat{L})_{\#}$ of (12) is surjective, again by 4.3 and 4.6.

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