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MAPS BETWEEN p–COMPLETIONS OF THE CLARK–EWING SPACES X(W,p,n)

by

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Abstract. Let Z_p denote the ring of p–adic integers. Let $W \subset GL(n,Z_p)$ be a finite group such that p does not divide the order of W. The group W acts on $K((Z_p)^n,2)$. Let $X(W,p,n)_p$ be the p–completion of the space $K((Z_p)^n,2) \underset{W}{\times} EW$. We classified homotopy classes of maps between spaces $X(W,p,n)_p$.

0. INTRODUCTION

Let Z_p denote the ring of p–adic integers. Let Y_p denote the p–completion of a space Y.

Let T be a torus and let $W \subset GL(\pi_1(T)\otimes Z_p)$ be a fintie group. The group W acts on the space $(BT)_p$. Let

$$X(W,p,T) := ((BT)_p \times_W EW)_p$$

where EW is a contractible space equipped with a free action of W.

The aim of this paper is to apply the program from [1] to study maps between spaces X(W,p,T). The starting point was an attempt to generalize one result of Hubbuck (see [8] Theorem 1.1.). The plan of work will follow closely that of [3] and [13].

Example. *Let* G *be a connected, compact Lie group,* T *its maximal torus and* W *its Weyl group. If* p *does not divide the order of* W *then* $(BG)_p \approx (BT \times_W EW)_p$.

This example suggests the following defintion.

Definition. *Let us set* $X = X(W,p,T)$. *We shall call* T *a maximal torus of* X *and* W *a Weyl group of* X.

The projection $(BT)_p \times EW \longrightarrow (BT)_p \times_W EW$ *induces a map* $i : BT \longrightarrow X$. *We shall call* $i : BT \longrightarrow X$ *a structure map of* X.

We point out that in [5] A. Clark and J. Ewing studied cohomology algebras of spaces $(BT)_p \times_W EW$. We warn the reader that our notation is different from the notation used in [5]. The space $X(W,p,T)$ is the p–completion of the Clark–Ewing space $X(W,p,\text{rank } T)$.

Through the whole paper we shall assume that p is an odd prime. We need this assumption to show Proposition 1.1. It is clear that this assumption is not essential, however we were not able to overcome technical difficulties for $p = 2$.

Now we shall state our main results.

Let us set $X = X(W,p,T)$ and $X' = X(W',p,T')$.

THEOREM 1. Assume that p *does not divide the orders of* W *and* W'. *Then for any map* $f : X \longrightarrow X'$ *there is a map* $\Upsilon : (BT)_p \longrightarrow (BT')_p$ *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] & X' \\ (BT)_p \arrow[u, "i"] \arrow[r, "\Upsilon"] & (BT')_p \arrow[u, "i'"'] \end{tikzcd}$$

commutes up to homotopy. Moreover we have:

a) *if* $\Upsilon' : (BT)_p \longrightarrow (BT')_p$ *is such that* $f \circ i$ *is homotopic to* $i' \circ \Upsilon'$ *then there is* $w \in W'$ such that $w \circ \Upsilon'$ *is homotopic to* Υ,

b) *for any* $w \in W$ *there is* $w' \in W'$ *such that* $\Upsilon \circ w$ *is homotopic to* $w' \circ \Upsilon$.

The group W acts on $\pi_1(T) \otimes Z_p$, hence W acts on $\pi_1(T) \otimes R$ for any Z_p–module R.

DEFINITION 1. Let R *be a* Z_p*–algebra. We say that a homomorphism of* R*–modules*

$$\varphi : \pi_1(T) \otimes R \longrightarrow \pi_1(T') \otimes R$$

is admissible if for any $w \in W$ *there is* $w' \in W'$ *such that* $\varphi \circ w = w' \circ \varphi$.
We say that two admissible maps φ *and* ψ *from* $\pi_1(T) \otimes R$ *to* $\pi_1(T') \otimes R$ *are equivalent if there is* $w \in W'$ *such that* $w \circ \varphi = \psi$.

It is clear that the relation defined above is an equivalence relation on the set of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$. We shall denote by $\mathrm{Ahom}_R(T,T')$ the set of equivalence classes of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

Let us notice that the map $\pi_1(\Upsilon)$ induced by Υ from Theorem 1 on fundamental groups is admissible for $R = Z_p$. This map is unique up to the action of W', so any map $f : X \longrightarrow X'$ determines uniquely an equivalence class of $\pi_1(\Upsilon)$ in $\mathrm{Ahom}_{Z_p}(T,T')$ which we shall denote by $\chi(f)$.

THEOREM 2. Let us assume that p *does not divide the orders of* W *and* W'*. Then the natural map*

$$\chi : [X,X'] \longrightarrow \mathrm{Ahom}_{Z_p}(T,T')$$

is bijective.

For any space Y we set

$$H^*(Y,\mathbb{Q}_p) := H^*(Y,Z_p) \otimes \mathbb{Q} ,$$

where $\mathbb{Q}_p$ is a field of p–adic numbers.

THEOREM 3. Let us assume that p *does not divide the orders of* W *and* W'*. Then the natural map*

$$\phi : [X,X'] \longrightarrow \mathrm{Hom}(H^*(X',\mathbb{Q}_p), H^*(X,\mathbb{Q}_p))$$

is injective.

We denote by $K^0(\ ,R)$ the 0^{th}–term of complex K–theory with R–coefficients. Let $\mathcal{O}_R$ be the set of operations in $K^0(\ ,R)$. The functor $K^0(\ ,R)$ is equipped with the natural augmentation $K^0(\ ,R) \longrightarrow R$. Let $\mathrm{Hom}_{\mathcal{O}_R}(K^0(X',R),K^0(X,R))$ be the set of R–algebra homomorphisms which commute with the action of $\mathcal{O}_R$ and augmentations.

THEOREM 4. If p *does not divides the order of* W *and* W′*, then the natural map*

$$\psi : [X,X'] \longrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_p}}(K^0(X',Z_p),K^0(X,Z_p))$$

is bijective.

We can formulate our results in a nice categorical way.

We shall define a category $Z_p - \mathrm{Rep}$ in the following way. Objects of the category Z_p–Rep are representations $\rho : W \longrightarrow GL(M)$ where M is a free, finitely generated Z_p–module, W is a finite group and p does not divide the order of W. It remains to define morphisms in this category. If $\theta : W \longrightarrow GL(M)$ and $\theta' : W' \longrightarrow GL(M')$ are two objects of $Z_p - \mathrm{Rep}$, we say that a homomorphism of Z_p–modules $f : M \longrightarrow M'$ is admissible if for each $w \in W$ there is $w' \in W'$ such that $f \circ w = w' \circ f$. We say that two admissible homomorphisms f and g from M to M′ are equivalent if there is $w \in W'$ such that $f = w' \circ g$. We shall denote by $\mathrm{Ahom}(\theta,\theta')$ the set of equivalence classes of admissible homomorphisms from M to M′. The set $\mathrm{Ahom}(\theta,\theta')$ is the set of morphisms from θ to θ' in the category Z_p– Rep. The category $Z_p - \mathrm{Rep}$ is equipped with the product defined in the following way:

$$(\theta{:}W \longrightarrow GL(M)) \oplus (\theta'{:}W' \longrightarrow GL(M')) = \theta\oplus\theta' : W \times W' \longrightarrow GL(M \oplus M').$$

The product of morphisms is defined in the obvious way.

We denote by Ht(p) the category whose objects are spaces X(W,p,T) such that p does not divide the order of W. Morphisms in Ht(p) are homotopy classes of maps. The category Ht(p) has products defined in the obvious way.

THEOREM 5. *There is an equivalence of categories*

$$R : Z_p - \mathrm{Rep} \longrightarrow \mathrm{Ht}(p)$$

with products.

THEOREM 6. *In Theorems* 1,2,3 *and* 4 *we can drop the assumption* "p *does not divide the order of* W' " *if* $X' = (BG)_p$, *where* G *is a connected, compact Lie group.*

COROLLARY 7. *Let* $X = X(W,p,T)$ *and let* p *be a prime not dividing the order of* W. *Let us assume that the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$ *is irreducible. Then there is a finite number of self-maps* $I_1,\ldots,I_n$ *of* X *such that for any* $f : X \longrightarrow X$ *there is* k *for which* $f \circ I_k$ *is an Adams* ψ^α*—map i.e. the map induced by* $f \circ I_k$ *on* $H^{2i}(X,\mathbb{Q}_p)$ *is a multiplication by* α^i. *The number* n *is smaller or equal to the number of elements of* Aut(W)/Inn(W) *which preserve the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$.

Example. (see also [3])
Let $X = BSU(n)_p$. *The Weyl group of* SU(n) *is* Σ_n. *If* $n \neq 6$ *then* $\mathrm{Aut}\,\Sigma_n = \mathrm{Inn}\,\Sigma_n$ *and for* $n = 6$ *the outer automorphism does not preserve the natural representation of* Σ_6 *on* $\pi_1(T) \otimes \mathbb{Q}_p$. *This implies that the self-maps of* $BSU(n)_p$ *are Adams* ψ^α*—maps.*

We point out that Corollary 7 can be view as a generalization of a result of Hubbuck (see [8] Theorem 1.1.) The example is a special case of the result of Hubbuck. However, it concerns maps between p—completed spaces $BSU(n)_p$ while Hubbuck is dealing with classical spaces BG.

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cript.

1. THE LANNES T FUNCTOR FOR SPACES X(W,p,T)

Let $X = X(W,p,T)$. Let us assume that p does not divide the order of W. In this section we shall compute the cohomology of the mapping space map(BV,X) and its connected component $\mathrm{map}_f(BV,X)$ where V is an elementary abelian p–group and $f : BV \longrightarrow X$ is a map

It follows from [5] (see Proposition on p. 425) that

$$H^*(X,F_p) = H^*(BT,F_p)^W .$$

The map $f : BV \longrightarrow X$ induces a map $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$. Let us notice that $\mathrm{Im}\, f^*$ is contained in the kernel of the Bockstein homomorphism. Hence it suffices to look at the polynomial part of $H^*(BV,F_p)$ when extending f^* to $H^*(BT,F_p)$. It follows from [2] Proposition 1.10 that there is $g^* : H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ such that $f^* = g^* \circ i^*$ where $i^* : H^*(X,F_p) \longrightarrow H^*(BT,F_p)$ is the inclusion induced by a structure map $i : BT \longrightarrow X$.

For a torus T, the solutions in T of $t^p = 1$ make up a subgroup T(1). The map g^* is induced by a homomorphism $\varphi : V \longrightarrow T(1)$. This follows from [9] Theorem 0.4. Let $\Lambda_f : V \otimes T(1)^* \longrightarrow F_p$ be an adjoint map of φ. The group W acts on $\mathrm{Hom}(V \otimes T(1)^*,F_p)$ through its action on $T(1)^*$. Let W_f be the isotropy subgroup of Λ_f.

***PROPOSITION* 1.1.** *Let* $X = X(W,p,T)$. *Let us assume that* p *does not divide the order of* W. *Let* V *be an elementary abelian* p*–group and let* $f : BV \longrightarrow X$ *be any map. Then we have an isomorphism*

$$H^*(\mathrm{map}_f(BV,X);F_p) = H^*(BT,F_p)^{W_f} .$$

PROOF: For a vector space U over F_p let us denote by P(U) the polynomial algebra on U, by $\Lambda(U)$ the exterior algebra on U and by $A(U)$ the symmetric algebra on U divided by the ideal generated by all polynomials $x^p - x$ for $x \in U$. The polynomial $x^p - x$ splits completely over F_p. Hence we have an isomorphism of F_p–algebras $A(U) = \bigoplus_{a \in U^*} F_p$. We point out that $A(U)$ is concentrated in degree zero.

Let us notice that we have the following natural identifications

$$H^*(BT,F_p) = P(T(1)^*)$$

and

$$H^*(BV,F_p) = P(V^*) \otimes \Lambda(\beta^{-1}V^*).$$

To simplify the notation let us set $A := A(V \otimes T(1)^*)$ and $H := H^*(BT,F_p) = P(T(1)^*)$. It follows from Corollary 2 in [4] that for any unstable A_p–algebra M and any A_p–algebra homomorphism $h : P((Z/p)^*) \longrightarrow M \otimes H^*(BZ/p,F_p)$ we have

$$h(t^*) = m_{t*} \otimes 1 + m_{v*} \otimes v^*.$$

This implies that we have a natural isomorphism

$$\Phi_M : \mathrm{Hom}(H;M \otimes H^*(BV)) \approx \mathrm{Hom}\,(A \otimes H;M).$$

where $\mathrm{Hom}(\ ;\ )$ is in the category of unstable A_p–algebras. If $h(t^*) =$

$m_{t*} \otimes 1 + \sum_{v* \in V*} m_{v*} \otimes v^*$ then $\Phi_M(h)([v \otimes t^*] \otimes 1) = \sum_{v* \in V*} m_{v*} \cdot v^*(v)$

and $\Phi_M(h)(1 \otimes t^*) = m_{t*}$.

Hence it follows that

$$(*) \qquad T_V(H) = A \otimes H.$$

If $M = F_p$ then we have an isomorphism

$\Phi_{F_p} : \mathrm{Hom}(H;H^*(BV)) \approx \mathrm{Hom}(A \otimes H;F_p)$. The group W acts on H and A through its action on $T(1)^*$. The isomorphism (*) and the fact that the functor $T_V(-)$ is exact implies that

$$(**) \qquad T_V(H^W) = (A \otimes H)^W$$

(see [4] Proposition 3).

Let $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$ be the map induced by f on cohomology. Let $\lambda : T_V(H^*(X,F_p)) \longrightarrow F_p$ be the adjoint map of f^* and let $\overline{\lambda} : T_V(H) \longrightarrow F_p$ be the adjoint map of g^* We recall that $g^* : H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ is such that $f^* = g^* \circ i^*$. The restriction of $\overline{\lambda}$ to $V \otimes T(1)^*$ is equal to Λ_f, where
$\Lambda_f : V \otimes T(1)^* \longrightarrow F_p$ is an adjoint map of $\varphi : V \longrightarrow T(1)$.

It follows from [6] 2.3 Theorem and the equality (**) that

$$H^*(\mathrm{map}_f(BV,X),F_p) \approx T_V(H^*(X,F_p)) \underset{T_V^0(H^*(X,F_p))}{\otimes} F_p \approx (A \otimes H)^W \underset{A^W}{\otimes} F_p .$$

If $V^* \otimes T(1) = \coprod_{W'} W/W'$, as a W–set then $A \approx \bigoplus_{W'} F_p[W/W']$ as a W–module. This follows from the isomorphism $A(U) = \bigoplus_{a \in U^*} F_p$ mentioned at the beginning of the proof. For any $W' \subset W$, $F_p[W/W']^W \approx F_p$. The maps $\overline{\lambda}$ and λ induce $\tilde{\lambda} : A \longrightarrow F_p$ and $\tilde{\tilde{\lambda}} : A^W = \oplus F_p \longrightarrow F_p$. The algebra homomorphism $\tilde{\tilde{\lambda}}$ is the identity on one's of F_p's and it is zero on all others. We recall that the isotropy subgroup of Λ_f is W_f. The fact that $\tilde{\lambda}$ restricts to Λ_f on $V \otimes T(1)^*$ implies that $\tilde{\tilde{\lambda}}$ is the identity on $F_p[W/W_f]^W$. Hence we have the following isomorphisms

$$(A \otimes H)^W \underset{A^W}{\otimes} F_p \approx (F_p[W/W_f] \otimes H)^W \underset{F_p}{\otimes} F_p \approx H^{W_f} . \qquad \square$$

2. MAPS FROM BP TO X

Let T be a torus. For a torus T the solutions in T of $t^{p^n} = 1$ make up a subgroup T(n); let $T(\infty) = \bigcup_n T(n)$. Let us set $M = \pi_1(T) \otimes Z_p$. Let $W \subset GL_{Z_p}(M)$ be a finite group. The action of W on M extends to the action of W on $M \otimes \mathbb{Q}$. The lattice M in $M \otimes \mathbb{Q}$ is invariant therefore W acts also on $M \otimes \mathbb{Q}/_M$. Observe that $M \otimes \mathbb{Q}/_M = T(\infty)$. From the action of W on $T(\infty)$ we can recover the original action of W on M if we take the induced action of W on $(H^2(BT(\infty);Z_p))^*$. Hence any finite subgroup of $GL_{Z_p}(M)$ can be realized as a subgroup of $\mathrm{Aut}(T(\infty))$.

PROPOSITION 2.1. *Let* W *be a finite subgroup of* $\mathrm{Aut}(T(\infty))$. *Let us assume that* p *does not divide the order of* W. *If* P *is a finite* p*-group then any map* $f: BP \longrightarrow (B(T(\infty)\tilde{\times}W))_p$ *is induced by a homomorphism* $\varphi: P \longrightarrow T(\infty)\tilde{\times}W$.

We were informed that a similar result was also known to W. Dwyer. This proposition is an analog of the theorem of Dwyer and Zabrodsky (see [7] 1.1. Theorem). The proof will follow closely the proof of the Dwyer and Zabrodsky theorem contained in [14], which depends very much on [10].

Let us set $G = T(\infty) \tilde{\times} W$.

LEMMA 2.2. *Let* $V = Z/p$, *let* $\varphi: V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\varphi$ *in* G *and let* $\varphi_0: V \longrightarrow G_0$ *be the map induced by* φ. *Then the map*

$$\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p) \longrightarrow \mathrm{map}_{B\varphi}(BV,(BG)_p)$$

is a homotopy equivalence.

PROOF: It follows from Proposition 1.1 that

$H^*(\mathrm{map}_{B\varphi}(BV,(BG)_p),F_p) \approx P^{W_0}$ where $P \approx H^*(BT,F_p)$ and $W_0 = G_0/T(\infty)$ is the isotropy subgroup of $\varphi: V \longrightarrow T(\infty)$. In the same way we get

$H^*(\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p),F_p) = P^{W_0}$. Hence the map considered by us is a homotopy equivalence. □

LEMMA 2.3. *Let* P *be a* p*–group, let* $Z/p = V$ *be a subgroup of the center of* P. *Let* $\varphi : V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\,\varphi$ *in* G *and let* $\varphi_0 : V \longrightarrow G_0$ *be the induced homomorphism. Let*

$$[BP,(BG)_p](B\varphi) = \{f \in [BP,(BG)_p] : f_{|BV} \sim B\varphi\}$$

and let $[BP,(BG_0)_p](B\varphi_0)$ *be defined in an analogous way. Then the inclusion map* $i : G_0 \longrightarrow G$ *induces a bijection*

$$(*) \qquad [BP,(BG_0)_p](B\varphi_0) \longrightarrow [BP,(BG)_p](B\varphi) .$$

PROOF: We have a fibration $BV \longrightarrow BP \longrightarrow B(P/V)$. Let $BV \longrightarrow EP/V \longrightarrow E(P/V)$ be the fibration induced by pulling back over $pr : E(P/V) \longrightarrow B(P/V)$. The group P/V acts on EP/V through maps homotopics to the identity and the space EP/V is a model for BV. It follows from Lemma 2.2 that the map

$$\mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi_0}(EP/V,(BG_0)_p)\to \mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi}(EP/V,(BG)_p))$$

is a homotopy equivalence. There is a bijective correspondence between P/V–maps $E(P/V) \longrightarrow \mathrm{map}_{B\varphi_0}(EP/V,(BG_0)_p)$ and maps $E(P/V) \underset{(P/V)}{\times} EP/V \longrightarrow (BG_0)_p$ which composed with $E(P/V) \times EP/V \longrightarrow E(P/V) \underset{(P/V)}{\times} EP/V$ are homotopic to $B\varphi_0$. The same bijection holds if we replace φ_0 by φ and G_0 by G. This implies that the induced map on π_0 is the map (*). This finishes the proof. □

LEMMA 2.4. (see [15] 1.5. Lemma) *Let* $\varphi : L \longrightarrow K$ *be a simplicial map. Let* $V_0^{\varphi}(L,X)$ *be the subspace of the space* map.(L,X) *of pointed maps from* L *to* X *consisting of maps* $f : L \longrightarrow X$ *such that*

$f_{|\varphi^{-1}(k)} \sim *$ *for every* $k \in K$. *Let* $\mathrm{map}_*(\varphi^{-1}(k),X)$ *be the path component of the constant map in the space of pointed maps* $\mathrm{map}.(\varphi^{-1}(k),X)$. *Let us assume that for every* $k \in K$, *the space* $\mathrm{map}_*(\varphi^{-1}(k),X)$ *is weakly homotopy equivalent to* $*$. *Then* φ *induces a weak homotopy equivalence*

$$\varphi^* : \mathrm{map}.(K,X) \xrightarrow{\approx} V_0^{\varphi}(L,X) .$$

PROOF OF PROPOSITION 2.1: Let us assume that $P = Z/p$. It follows from [2] Proposition 1.10 that $f^* : H^*(BG,F_p) \longrightarrow H^*(BP,F_p)$ factors through $H^*(BT(\infty),F_p)$. But any morphism $H^*(BT(\infty),F_p) \longrightarrow H^*(BP,F_p)$ is of the form $B\varphi$ (see [9] Theorem 0.4). Hence f is induced by a homomorphism.

Let us suppose that any map $f : BP \longrightarrow (BG)_p$ is induced by a homomorphism if the order of P is less or equal to p^{n-1}.

Let the order of P be equal to p^n and let $f : BP \longrightarrow (BG)_p$ be a map. Let $V = Z/p$ be contained in the center of P and let $i : V \longrightarrow P$ be the inclusion.

Assume that the composition

$$BV \xrightarrow{Bi} BP \xrightarrow{f} X$$

is null homotopic. We want to show that f is homotopic to $f_1 \circ Bpr$ where $pr : P \longrightarrow P/_V$ is the natural homomorphism and $f_1 : B(P/_V) \longrightarrow X$ is a map. First we show that the space of pointed maps homotopic to $*$ $\mathrm{map}_*(BV,X)$ is weakly contractible. This space is p–local because BV and X are p–local. Let $\mathrm{map}_{const}(BV,X)$ be the connected component containing a constant map of map (BV,X). It follows from Proposition 1.1 that

$$H^*(\mathrm{map}_{const}(BV,X),F_p) = H^*(BT(\infty),F_p)^W .$$

The last group is of course $H^*(X,F_p)$. Hence the evaluation map $\mathrm{map}_{const}(BV,X) \longrightarrow X$ is a weak homotopy equivalence and consequently the space $\mathrm{nap}_*(BV,X)$ is weakly contractible. Lemma 2.4 implies that f is homotopic to $f_1 \circ Bpr$. By the inductive assumption f_1 is induced by a homomorphism.

Let us suppose that $f \circ Bi$ is induced by a homomorphism $\varphi : V \longrightarrow G$ and $\varphi(V) \neq 0$. Let G_0 be the centralizer of $\varphi(V)$ in G. It follows from Lemma 2.3 that up to homotopy there is a unique map $f_0 : BP \longrightarrow (BG_0)_p$ such that $BP \xrightarrow{f_0} (BG_0)_p \longrightarrow (BG)_p$ is homotopic to f and f_0 restricted to BV is induced by φ. Let $\rho : G_0 \longrightarrow G_0/\varphi(V)$ be the natural projection. The composition

$$BV \longrightarrow BP \xrightarrow{f_0} (BG_0)_p \xrightarrow{(B\rho)_p} (BG_0/\varphi(V))_p$$

is null–homotopic hence $(B\rho)_p \circ f_0$ factors uniquely as

$$BP \xrightarrow{Bpr} B(P/V) \xrightarrow{f_1} B(G_0/\varphi(V))_p .$$

This follows from the previous discussion.

One has the homotopy pullback

$$\begin{tikzcd}
BP \arrow[r, "f_0"] \arrow[d, "Bpr"] & (B(G_0)_p \arrow[d, "(B\rho)_p"] \\
B(P/V) \arrow[r, "f_1"] & (B(G_0/\varphi(V)))_p
\end{tikzcd}$$

because $\varphi(V)$ is contained in the center of G_0. By the inductive assumption f_1 is induced by a homomorphism $\varphi_1 : P/V \longrightarrow G_0/\varphi(V)$. We have a pullback of groups

$$\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d, "pr"] & G_0 \arrow[d, "\rho"] \\
P/V \arrow[r, "\varphi_1"] & G_0/\varphi(V) .
\end{tikzcd}$$

After applying the functor $(B\ )_p$ we get a homotopy pullback

$$\begin{tikzcd}
BP \arrow[r, "(B\psi)_p"] \arrow[d, "Bpr"] & (BG_0)_p \arrow[d, "(B\rho)_p"] \\
B(P/V) \arrow[r, "(B\varphi_1)_p"] & B(G_0/\varphi(V))_p
\end{tikzcd}$$

The map f_0 is homotopic to $(B\psi)_p$ hence f is homotopic to $(B\rho)_p \circ (B\psi)_p$. □

COROLLARY 2.5. *Let* T' *by any torus. Then any map* $g : BT'(\infty) \longrightarrow (BG)_p$ *is induced by a homomorphism* $\alpha : T'(\infty) \longrightarrow T(\infty)$.

PROOF. It follows from Proposition 2.1 that for any n the restriction of g to $BT'(n)$, $g_n : BT'(n) \longrightarrow (BG)_p$ is induced by a homomorphism. Let $S_n = \{\beta : T'(n) \longrightarrow G \mid (B\beta)_p \sim g_n\}$. The restriction of $\beta : T'(n) \longrightarrow G$ to $T'(n-1)$ maps S_n into S_{n-1}. Each set S_n is non–empty and finite. This implies that $\varprojlim_n S_n$ is non–empty. Hence there is a homomorphism $\alpha : T'(\infty) \longrightarrow G$ such that α induces g and factorizes through $T(\infty)$. □

3. PROOFS.

We start with the following lemma.

Lemma 3.1 Let $X = X(W,p,T)$, let $i : BT(\infty) \longrightarrow X$ be a structure map of X and let $w : BT(\infty) \longrightarrow BT(\infty)$ be a map induced by $w \in W$. Then the maps i and $i \circ w$ are homotopic.

Proof. Let $\tilde{w} : BT(\infty) \times EW \longrightarrow BT(\infty) \times EW$ be w on $BT(\infty)$ and a translation by w^{-1} on EW. Observe that $\tilde{w}$ is a covering transformation of the projection $pr : BT(\infty) \times EW \longrightarrow BT(\infty) \times_W EW$. The composition

$BT(\infty) \times EW \xrightarrow{pr} BT(\infty) \times_W EW \longrightarrow (BT(\infty) \times_W EW)_p$ is homotopic to i. Hence i and $i \circ w$ are homotopic. □

PROOF OF THEOREM 1:

It follows from Corollary 2.5 that $f \circ i$ is induced by a homomorphism $\varphi : T(\infty) \longrightarrow T'(\infty)$. We set $\Upsilon = (B\varphi)_p$. □

The proof of point a) is the same as the proof of Theorem 1.7 in [1]. Point b) follows from a) and Lemma 3.1. □

PROOF OF THEOREM 3:

Let $f,g : X \longrightarrow X'$ be two maps such that $H^*(f,\mathbb{Q}_p) = H^*(g,\mathbb{Q}_p)$. Let $i : BT_p \longrightarrow X$ be the map induced by a structure map $i : BT \longrightarrow X$. Corollary 2.5 implies that $f \circ i$ and $g \circ i$ are induced by two homomorphisms $\varphi,\Psi : T(\infty) \longrightarrow T'(\infty) \tilde{\times} W'$. We must show that φ and Ψ are conjugate.

For a finite group π let $R(\pi)$ be its complex representation ring. Let

$$R(T(\infty)) := \varprojlim_n R(T(n)) \text{ and } R(T'(\infty) \tilde{\times} W) := \varprojlim_n R(T'(n) \tilde{\times} W').$$

The Chern character $ch : K^0(\ ;Z_p) \longrightarrow \prod_i H^{2i}(\ ;\mathbb{Q}_p)$ is injective for spaces $BT(\infty)$ and $B(T'(\infty) \tilde{\times} W') = BT'(\infty) \underset{W}{\times} EW$. The group $R(T(\infty))$ is mapped injectively into $K^0(BT(\infty);Z_p)$. Hence we have

$$R(\varphi) = R(\Psi) : R(T'(\infty) \tilde{\times} W') \longrightarrow R(T(\infty)).$$

For each subgroup $S = Z/p^n$ of $T(\infty)$ the restrictions of φ and to S are conjugate by an element of W' because S is cyclic. The fact that W' is finite implies that the restrictions of φ and to any subgroup Z/p^∞ of $T(\infty)$ are conjugate by some element of W'. Once more the fact that W' is finite and the set of subgroups of the form Z/p^∞ in $T(\infty)$ is uncountable if rank $T > 1$ implies that φ and are conjugate by an element of W'. Hence $f \circ i$ and $g \circ i$ are homotopic. It follows from [12] Theorem 1 that f and g are homotopic. □

PROOF OF THEOREM 2:

We set $\chi(f) = \pi_1(\Upsilon)$ where Υ is the map from Theorem 1. The injectivity of χ follows from Theorem 3. Next one observe that $K^0(X';Z_p) = K^0((BT')_p;Z_p)^W$.

Then the proof of surjectivity is the same as in Theorem 1.5 in [13]. It is a standard application of Theorem 1 from [12]. □

PROOF OF THEOREM 4:
The fact that ψ is injective follows from Theorem 3 and the injectivity of Chern character. The proof of surjectivity is the same as in Theorem 1.5 in [13]. □

PROOF OF THEOREM 5:
Theorem 5 is a direct consequence of Theorem 2. □

PROOF OF THEOREM 6:
Let G be a connected, compact Lie group. Observe that any map $BT(\infty) \longrightarrow (BG)_p$ is induced by a homomorphism $T(\infty) \longrightarrow G$ what is an immediate consequence of [7] 1.1. Theorem. This was the crucial point to prove Theorems 1,2,3 and 4 for $X' = X(W',p,T')$. The proofs of Theorems 1,2 and 3 for $X' = (BG)_p$ are the same. Observe that $K^0((BG)_p;Z_p) = K^0((BT)_p;Z_p)^W$. Hence the proof of Theorem 4 carry over to the case $X' = (BG)_p$. □

PROOF OF COROLLARY 7: If the natural representation of W on $\pi_1(T) \otimes \mathbb{Q}_p$ is irreducible then $\pi_1(\Upsilon) : \pi_2((BT)_p) \longrightarrow \pi_2((BT)_p)$ is an isomorphism or a trivial map. The correspondence $w \longrightarrow w'$ from Theorem 7 point b) is then an isomorphism. The rest is obvious. □

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