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ON DIOPHANTINE APPROXIMATION BY ALGEBRAIC NUMBERS OF A GIVEN NUMBER FIELD : A NEW GENERALIZATION OF DIRICHLET APPROXIMATION THEOREM

by

Roland QUÊME

Introduction

It is well known that for all $\alpha \in \mathbb{R}$, $\alpha \notin \mathbb{Q}$ there are infinitely many p/q, $|p|$, $q \in \mathbb{N}$ such that $|\alpha - p/q| < 1/q^2$ (Dirichlet theorem), and that for any real algebraic number $\alpha \notin \mathbb{Q}$ and for any $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exist only finitely many p/q, $|p|$, $q \in \mathbb{N}$ such that $|\alpha - p/q| < 1/q^{2+\varepsilon}$ (Roth theorem).

Let K be a number field of degree n, signature (r, s) and absolute value of discriminant D.
Let B be the Minkowski constant of K ($B = (4/\pi)^s \, . \, (n!/n^n) \, . \, \sqrt{D}$).
Let $\sigma : K \to \mathbb{R}^r \times \mathbb{C}^s$ be the embedding defined by :

$$\sigma(\rho) = (\sigma_1(\rho), \dots, \sigma_r(\rho), \sigma_{r+1}(\rho), \dots, \sigma_{r+s}(\rho))$$

where, as usually, $K = \sigma_1(K)$.

For $x, y \in \mathbb{R}^r \times \mathbb{C}^s$ we note $x = (x_j, \ j = 1, \dots, r+s)$. Then we note $x + y = (x_j + y_j, \ j = 1, \dots, r+s)$ and $x.y = (x_j.y_j, \ j = 1, \dots, r+s)$.
We define, for $x \in \mathbb{R}^r \times \mathbb{C}^s$, the distance function and the norm function :

$$d(x) = |x_1| + \dots + |x_r| + 2|x_{r+1}| + \dots + 2|x_{r+s}|,$$
$$N(x) = |x_1| \cdots |x_r| . |x_{r+1}|^2 \cdots |x_{r+s}|^2 \ .$$

Let A be the ring of integers of K.

Then we obtain the diophantine approximation theorems :

(i) For $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$, there exist infinitely many $\beta = p/q$, $p, q \in A$ such that $0 < d(\alpha\sigma(q) - \sigma(p)) < n^2 \,.\, B^{2/n}/d(\sigma(q))$, with arbitrary large distance $d(\sigma(q))$.

(ii) For $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$, $\alpha_j \notin \sigma_j(K)$, $j = 1, 2, \ldots, r+s$, there exist infinitely many $\beta = p/q$, $p, q \in A$ such that $0 < N(\alpha - \sigma(p/q)) < (B/N_{K/\mathbb{Q}}(q))^2$.

We first summarize the state of the art with three types of generalizations found in the quoted literature for diophantine approximation by numbers of a given number field K. Let K be a number field of degree n, signature (r, s). For $\beta \in K$, let $P(\beta)$ be the field polynomial of β,

$$P(\beta) = (x - \sigma_1(\beta)) \cdots$$
$$(x - \sigma_r(\beta))(x - \sigma_{r+1}(\beta))\overline{(x - \sigma_{r+1}(\beta))} \cdots (x - \sigma_{r+s}(\beta))\overline{(x - \sigma_{r+s}(\beta))} \ .$$

Let $C \in \mathbb{N}$ such that $P_1(\beta) = CP(\beta) = b_n\beta^n + \cdots + b_1\beta + b_0$ is a polynomial with integer coprime coefficients b_i, $i = 0, 1, \ldots, n$. Then we define the height of $\beta \in K$ by $H_K(\beta) = \sup_{i=0,\ldots,n} |b_i|$.

The first generalization of Dirichlet theorem found in bibliography is :

Assume that $r > 0$ and choose a real embedding $\sigma_1 : K \to \mathbb{R}$. For every $\alpha \in \mathbb{R} - \sigma_1(K)$, then there exist infinitely many $\beta \in K$ such that $|\alpha - \sigma_1(\beta)| < C_1(K) \max(1, \alpha^2)/H_K(\beta)^2$ where $C_1(K)$ is a constant depending only on K (see SCHMIDT [8] p.253).

The second generalization of Dirichlet theorem is :

Assume that $s > 0$ and choose a complex embedding $\sigma_2 : K \to \mathbb{C}$. For every $\alpha \in \mathbb{C} - \sigma_2(K)$, then there exist infinitely many $\beta \in K$ such that $|\alpha - \sigma_2(\beta)| < C_2(K)/H_K(\beta)$ where $C_2(K)$ is a constant depending only on K (see SCHMIDT [6] p.206).

The third generalization is :

Let $\beta_1, \ldots, \beta_\ell \in K$; let $\mathfrak{b}$ be the fractional ideal of K generated by $(1, \beta_1, \ldots, \beta_\ell)$.

We define the generalized height of the ℓ-tuple $(\beta_1,\dots,\beta_\ell)$ by :

$$\mathfrak{h}_K(\beta_1,\dots,\beta_\ell) = N_{K/\mathbb{Q}}(\mathfrak{b}) \prod_{j=1}^{r} \max(1,|\sigma_j(\beta_1)|,\dots,|\sigma_j(\beta_\ell)|) \prod_{j=r+1}^{r+s} \max(1,|\sigma_j(\beta_1)|,\dots,|\sigma_j(\beta_\ell)|)^2.$$

(i) if $r > 0$, let $\sigma_3 : K \to \mathbb{R}$ be a real embedding and $\alpha_1,\dots,\alpha_\ell \in \mathbb{R}$, not all in $\sigma_3(K)$; put in that case $\nu = 1$;

(ii) if $s > 0$, let $\sigma_3 : K \to \mathbb{C}$ be a complex embedding and $\alpha_1,\dots,\alpha_\ell \in \mathbb{C}$, not all in $\sigma_3(K)$; put in that case $\nu = 2$;

then there is a constant $C_3(K,\alpha_1,\dots,\alpha_\ell)$ depending only on K, $\alpha_1,\dots,\alpha_\ell$ such that there exist infinitely many $\beta = (\beta_1,\dots,\beta_\ell)$, $\beta_i \in K$, with

$$|\alpha_i - \sigma_3(\beta_i)|^\nu < C_3(K,\alpha_1,\dots,\alpha_\ell).\mathfrak{h}_K(\beta_1,\dots,\beta_\ell)^{-1-1/\ell},\ i = 1,2,\dots,\ell \qquad (1)$$

(see SCHMIDT [7] p.2).

The main difference between the quoted formulation and our theorem are summarized in the four next points :

1) In classical approximations above, $|\alpha-\beta|$ is obtained for *one* of the conjugates $\beta = \sigma_1(\beta)$. On the other hand, our estimate involves simultaneously *all* the conjugates of the same $\beta \in K$,

   for the distance function,

$$d(\alpha\sigma(q)-\sigma(p)) = |\alpha_1\sigma_1(q)-\sigma_1(p)| + \cdots + |\alpha_r\sigma_r(q)-\sigma_r(p)| + 2|\alpha_{r+1}\sigma_{r+1}(q)-\sigma_{r+1}(p)| + \cdots + 2|\alpha_{r+s}\sigma_{r+s}(q)-\sigma_{r+s}(p)|$$

   for the norm function,

$$N(\alpha-\sigma(p/q)) = |\alpha_1-\sigma_1(p/q))| \cdots |\alpha_r-\sigma_r(p/q)|.|\alpha_{r+1}-\sigma_{r+1}(p/q)|^2 \cdots |\alpha_{r+s}-\sigma_{r+s}(p/q)|^2.$$

2) Our approximation theorem cannot be immediately connected to usual simultaneous approximation theorems, because in simultaneous approximation $|f(\alpha_1-\beta_1)|,\dots,|f(\alpha_\ell-\beta_\ell)|$ the simultaneous approximations $\beta_1,\dots,\beta_\ell$ are not conjugate of the same $\beta \in K$ (see for instance (1)).

3) Our result contains not only effective but *explicit* constants with *simple* relationship to the structure of the number fields (the Minkowski constant for instance, with the distance function choosen).

4) Our proof is the exact generalization of the approximation by $\mathbb{Q}$ to approximation by a given number field K, using geometry of numbers properties of number fields embedding in $\mathbb{R}^n$.

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Prerequisites-Notations

K : number field

n : degree of K

(r, s) : signature of K

x : $x \in \mathbb{R}^r \times \mathbb{C}^s$, $x = (x_j \mid j = 1, \ldots, r+s)$

$x + y$: $x + y = (x_j + y_j \mid j = 1, \ldots, r+s)$

$x.y$: $x.y = (x_j.y_j \mid j = 1, \ldots, r+s)$

$d(x)$: for $x \in \mathbb{R}^r \times \mathbb{C}^s$, the distance function is defined by :

$$d(x) = |x_1| + \cdots + |x_r| + 2|x_{r+1}| + \cdots + 2|x_{r+s}|$$

$N(x)$: for $x \in \mathbb{R}^r \times \mathbb{C}^s$, the norm form is defined by :

$$N(x) = |x_1| \cdots |x_r|.|x_{r+1}|^2 \cdots |x_{r+s}|^2$$

$U(o, \tau)$: for $\tau \in \mathbb{R}_+$, convex body of $\mathbb{R}^n$ defined by

$$U(o, \tau) = \{x \mid x \in \mathbb{R}^r \times \mathbb{C}^s, d(x) < n\tau\}$$

where $\mathbb{R}^r \times \mathbb{C}^s$ is isomorphically identified to $\mathbb{R}^n$ by

$$x_{r+i} = (R(x_{r+i}), I(x_{r+i})) \ , \ i = 1, \ldots, s \ ,$$

where R and I are the real and imaginary part.

The volume of $U(o,\tau)$ is $v(U(o,\tau)) = 2^r(\pi/2)^s n^n \tau^n/n!$

(see for instance SAMUEL [5] p.70).

A : ring of algebraic integers in K.

$\sigma(A)$: embedding of A in $\mathbb{R}^r \times \mathbb{C}^s$ defined, for $a \in A$, by

$$\sigma(a) = (\sigma_1(a), \dots, \sigma_r(a), \sigma_{r+1}(a), \dots, \sigma_{r+s}(a))$$

where $\mathbb{R}^r \times \mathbb{C}^s$ is isomorphically identified to $\mathbb{R}^n$ by

$$\sigma_{r+i}(a) = (R(\sigma_{r+i}(a)), I(\sigma_{r+i}(a))).$$

$\sigma(A)$ is a lattice.

D_0 : Let $w_1, \dots w_n \in A$ such that $\sigma(w_1), \dots, \sigma(w_n)$ is a basis of the lattice $\sigma(A)$.

we define classically the fundamental domain D_0 by :

$$D_0 = \{x \mid x \in \mathbb{R}^r \times \mathbb{C}^s \ , \ x = u_1\sigma(w_1) + \cdots + u_n\sigma(w_n) \ , \ 0 \le u_i < 1\}.$$

$D(\sigma(a))$: fundamental domain of $\sigma(A)$ deduced from the fundamental domain D_0 by the translation $0 \to \sigma(a)$:

$$D(\sigma(a)) = \{(y_j) \in \mathbb{R}^r \times \mathbb{C}^s \ |(y_j - \sigma_j(a))| \ j = 1, \dots, r+s) \in D_0\}.$$

Results

THEOREM 1. *Let K be a number field of degree n, signature (r,s), and absolute value of discriminant D. Let B be the Minkowski bound of K ($B = (4/\pi)^s.(n!/n^n).\sqrt{D}$). Let A be the ring of integers of K. Let $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$. Then, for any $m \in \mathbb{R}$, $m > 0$, there are infinitely many different $\beta = p/q$ where $p, q \in A$, such that $d(\sigma(q)) > m$ and*

$$0 < d(\alpha.\sigma(q) - \sigma(p)) < (n^2.B^{2/n})/d(\sigma(q)).$$

Proof :

1) Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$,

$$\lambda = (1+2\varepsilon)^{1/n}.B^{2/n}/2 = (1+2\varepsilon)^{1/n}.(n!/n^n)^{2/n}.(4/\pi)^{(2s)/n}.D^{1/n}/2.$$

Let $m \in \mathbb{R}_+$, arbitrary large and $\mu = \lambda m^{-1/n}$.
Consider the set $E = U(o, m^{1/n}) \cap \sigma(A)$ where U and σ have the meaning of notations paragraph. From $v(U(o, m^{1/n})) = 2^r(\pi/2)^s n^n m/n!$ and $v(D(o)) = 2^{-s}\sqrt{D}$, we deduce

$$t = \text{Card } (E) = (2^r(\pi/2)^s n^n m)/(n! 2^{-s}\sqrt{D}) + O(m^{1-1/n}).$$

Therefore, for m sufficiently large, we have $t > \{2^r \pi^s n^n m/(n!\sqrt{D})\}.\{1-\varepsilon\}$. For any $a \in A$, for all $q_i \in A$ with $\sigma(q_i) \in E$, it is possible to define $p_i(a) \in A$ and $\rho_i(a) \in \mathbb{R}^r \times \mathbb{C}^s$, $i = 1, 2, \ldots, t$, such that $\rho_i(a) = \alpha\sigma(q_i) - \sigma(p_i(a))$, $i = 1, 2, \ldots, t$ and $\rho_i(a) \in D(\sigma(a))$. Notice that the approximation function $d(x)$ is meaningful because $d(\alpha\sigma(q) - \sigma(p)) = 0$ leads to $p = q = 0$: from the definition of $d(x)$, $d(\alpha\sigma(q) - \sigma(p)) = 0$ implies $\alpha_j\sigma_j(q) - \sigma_j(p) = 0$, $j = 1, \ldots, r+s$, and thus $\alpha_j = \sigma_j(p/q)$, $j = 1, \ldots, r+s$ and therefore $\alpha \in \sigma(K)$, which is in contradiction with hypothesis. Thus the $\rho_i(a)$, $i = 1, \ldots, t$, are different each others.

Consider the set $G = \{U(\rho_i(a), \mu/2) \mid i = 1, 2, \ldots, t \ , \ \forall a \in A\}$. G cannot be a packing of $\mathbb{R}^n$ (for packing definition, see for instance LEKKERKERKER [2] p.169) because

$$tv(U(o, \mu/2)) > \{(1-\varepsilon)2^r\pi^s n^n m/(n!\sqrt{D})\}.$$
$$\{2^r(\pi/2)^s n^n (1+2\varepsilon)(n!/n^n)^2(4/\pi)^{2s} D m^{-1}/(2^n 2^n n!)\}$$
$$tv(U(o, \mu/2)) > (1-\varepsilon)(1+2\varepsilon)2^{-s}\sqrt{D} > v(D(o)).$$

Therefore, for m sufficiently large, there exist $\rho_i(a)$ and $\rho_{i'}(b)$ with

$$\rho_i(a) = \alpha\sigma(q_i) - \sigma(p_i(a)) \qquad (1)$$
$$\rho_{i'}(b) = \alpha\sigma(q_{i'}) - \sigma(p_{i'}(b)) \qquad (2)$$

such that $U(\rho_i(a), \mu/2) \cap U(\rho_{i'}(b), \mu/2) \neq \emptyset$.

Then $d(\rho_i(a) - \rho_{i'}(b)) < n\mu$ from the definition of the convex set $U(\rho(a), \mu/2)$.
Let $p = p_i(a) - p_{i'}(b)$, $p \in A$ and $q = q_i - q_{i'}$, $q \in A$. Then, from the value of μ, we deduce

$$d(\alpha\sigma(q) - \sigma(p)) < n\mu = (n(1+2\varepsilon)^{1/n}.B^{2/n}/2)m^{-1/n}. \qquad (2')$$

Consider the sequence of values of ε defined by $\varepsilon_1 = 1$, $\varepsilon_2 = 1/2, \ldots, \varepsilon_k = 1/k, \ldots$ Therefore, for m given, for any ε_k there exist $p(\varepsilon_k)$, $q(\varepsilon_k) \in A$ such that

$$d(\alpha\sigma(q(\varepsilon_k)) - \sigma(p(\varepsilon_k))) < (nB^{2/n}/2).m^{-1/n}.(1+2\varepsilon_k)^{1/n}. \tag{2''}$$

From $\sigma(q(\varepsilon_k)) \in 2E$, we deduce that $d(\sigma(q(\varepsilon_k)))$ is bounded above independently of ε_k. From inequality (2''), we then deduce that $d(\sigma(p(\varepsilon_k))$ is also bounded above independently of ε_k. Like $\sigma(A)$ is a lattice, it is possible to take out an infinite subsequence $k_1, k_2, \ldots, k_j$ such that $p(\varepsilon_{k_1}) = p(\varepsilon_{k_2}) = \cdots = p(\varepsilon_{k_j}) = p$ and $q(\varepsilon_{k_1}) = q(\varepsilon_{k_2}) = \cdots = q(\varepsilon_{k_j}) = q$ and then

$$d(\alpha\sigma(q) - \sigma(p)) \le (nB^{2/n}/2)m^{-1/n}. \tag{3}$$

From $\sigma(q_i) \in E$ in (1), we have $d(\sigma(q_i)) < nm^{1/n}$,
From $\sigma(q_{i'}) \in E$ in (2), we have $d(\sigma(q_{i'})) < nm^{1/n}$,
and thus $d(\sigma(q)) < 2nm^{1/n}$ or $m^{-1/n} < 2n/d(\sigma(q))$.

We then have from (3)

$$d(\alpha\sigma(q) - \sigma(p)) < (nB^{2/n}/2)(2n/d(\sigma(q)))$$
$$d(\alpha\sigma(q) - \sigma(p)) < n^2B^{2/n}/d(\sigma(q))\ . \tag{3'}$$

2) We shall now prove that there are infinitely many different $\beta = p/q$ with

$$d(\alpha\sigma(q) - \sigma(p)) < n^2B^{2/n}/d(\sigma(q)). \tag{4}$$

Let m_1, $m_2 \in \mathbb{R}_+$, m_1 given, $m_1 < m_2$ with $m_2 \to +\infty$. We have $m_2 > m_1$ and $\mu_1 > \mu_2$ with the meaning of m and μ above.
From (3') inequality, we have

$$d(\alpha\sigma(q_1) - \sigma(p_1)) < n^2B^{2/n}m_1^{-1/n}, \tag{5}$$
$$d(\alpha\sigma(q_2) - \sigma(p_2)) < n^2B^{2/n}m_2^{-1/n}, \tag{6}$$

If $\beta_2 = \beta_1$ then $p_2/q_2 = p_1/q_1$ and $\sigma_j(p_2/q_2) = \sigma_j(p_1/q_1)$, $j = 1, \ldots, r+s$. $\alpha_j\sigma_j(q_2) - \sigma_j(p_2) = \sigma_j(q_2)(\alpha_j - \sigma_j(p_1/q_1))$ and thus $N(\alpha\sigma(q_2) - \sigma(p_2)) = N_{K/\mathbb{Q}}(q_2)N(\alpha - \sigma(p_1/q_1))$, $N(\alpha\sigma(q_2) - \sigma(p_2)) \ge N(\alpha - \sigma(p_1/q_1))$. From the geometric mean inequality, $d(\alpha\sigma(q_2) - \sigma(p_2)) \ge nN(\alpha - \sigma(p_1/q_1))^{1/n}$ and then $\mu_2 > N(\alpha - \sigma(p_1/q_1))^{1/n}$, which is possible only for m_2 bounded above. Then, for any $\beta_1 = p_1/q_1$ given which verify (5), there are finitely many couples (p_2, q_2) such that $\beta_2 = p_2/q_2 = p_1/q_1$ and such that (2) is verified. Relation (4) is

verified by infinitely many couples (p, q), because in (3) $d(\alpha\sigma(q) - \sigma(p))$ can be made arbitrary small for m sufficiently large. Therefore there are infinitely many different $\beta = p/q$ such that (4) is verified. 3) We shall prove that there are finitely many different $\beta = p/q$ for one value of q given :
Let $\beta_1 = p_1/q$ and $\beta_2 = p_2/q$. If $q_1 = q_2 = q$ then

$$d(\alpha\sigma(q) - \sigma(p_1)) < n^2 B^{2/n}/d(\sigma(q)) \text{ and } d(\alpha\sigma(q) - \sigma(p_2)) < n^2 B^{2/n}/d(\sigma(q)).$$

Then, we deduce $d(\sigma(p_1 - p_2)) < 2n^2 B^{2/n}/d(\sigma(q))$, which is possible only, for p_1 given, for a finite number of p_2.

4) From 2) and 3), there are infinitely many different q, thus with arbitrary large $d(\sigma(q))$ such that

$$d(\alpha\sigma(q) - \sigma(p)) < (n^2 B^{2/n})/d(\sigma(q)), \quad \text{Q.E.D.}$$

Remark : If α is such that $\alpha_1 = \alpha_2 = \cdots = \alpha_{r+s}$, then an immediate consequence of the Dirichlet approximation theorem is that there are infinitely many p/q, $p, q \in \mathbb{Z} \subset A$ such that $d(\alpha\sigma(q) - \sigma(p)) < n/q = n^2/d(\sigma(q)) < (n^2 B^{2/n})/d(\sigma(q))$: in that particular case, the theorem 1 is an immediate consequence of Dirichlet theorem.

COROLLARY 2 : *Let K be a number field of degree n, signature (r, s) and absolute value of discriminant D. Let B be the Minkowski bound of K ($B = (4/\pi)^s (n!/n^n)/\sqrt{D}$). Let $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$, $\alpha_j \notin \sigma_j(K)$, $j = 1, \ldots, r+s$. Then there are infinitely many $\beta = p/q$, $p, q \in A$ such that*

$$0 < N(\alpha - \sigma(p/q)) < (B/N_{K/\mathbb{Q}}(q))^2.$$

Proof : From geometric mean inequality, we deduce from the theorem 1

$$n^n N(\alpha\sigma(q) - \sigma(p)) < n^{2n} B^2/d(\sigma(q))^n.$$

From geometric mean inequality $n^n N(\sigma(q)) < d(\sigma(q))^n$, and then

$$N(\alpha - \sigma(p/q)) < (B/N(\sigma(q)))^2 = (B/N_{K/\mathbb{Q}}(q))^2.$$

From $\alpha_j \notin \sigma_j(K)$ we deduce $|\alpha_j\sigma_j(q) - \sigma_j(p)| > 0$, $j = 1, \ldots, r+s$, and then

$$N(\alpha - \sigma(p/q)) > 0, \quad \text{Q.E.D.}$$

COROLLARY 3 : *Let K be a number field of degree n, signature (r,s) and absolute value of discriminant D. Let A be the ring of integers of K. For $x \in \mathbb{R}^r \times \mathbb{C}^s$, let $d_2(x)$ be the distance function defined by*

$$d_2(x) = (|x_1|^2 + \cdots + |x_r|^2 + 2|x_{r+1}|^2 + \cdots + 2|x_{r+s}|^2)^{1/2}.$$

(i) *then, for every $m \in \mathbb{R}$, $m > 0$ and every $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$, there exist infinitely many different p/q with $p, q \in A$ such that*

$$0 < d_2(\alpha\sigma(q) - \sigma(p)) < n\{\Gamma(1+n/2)(4/(\pi n))^{n/2}\sqrt{D}\}^{2/n}/d_2(\sigma(q))$$

with $d_2(\sigma(q)) > m$.

(ii) *then, for $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$, $\alpha_j \notin \sigma_j(K)$, $j = 1, \ldots, r+s$, there exist infinitely many $\beta = p/q$ where $p, q \in A$ such that :*

$$0 < N(\alpha - \sigma(p/q)) < \{\Gamma(1+n/2)(4/(\pi n))^{n/2}\sqrt{D}/N_{K/\mathbb{Q}}(q)\}^2.$$

Proof : it is exactly of the same nature than the proofs of theorem 1 and corollary 2 with function $d_2(x)$ instead of function $d(x)$.

Some generalizations

It is possible to study some generalizations of preceding results : we mention some obtained generalizations or problems to solve.

1) In the corollaries 2 and 3, it would be possible to search for a proof that not only $d(\sigma(q))$, but also $N_{K/\mathbb{Q}}(q)$, can be choosen arbitrary large.

2) A "Roth type" theorem could have one of the formulations :

   (i) Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, for $\alpha \in \mathbb{R}^r \times \mathbb{C}^s - \sigma(K)$, α_j, $j = 1, \ldots, r+s$ algebraic, then there would be only finitely many $\beta = p/q$, $p, q \in A$ such that $d(\alpha\sigma(q) - \sigma(p)) < 1/d(\sigma(q))^{1+\varepsilon}$.

   (ii) if the assertion 1) is true (arbitrary large $N_{K/\mathbb{Q}}(q)$), then for $\varepsilon \in \mathbb{R}_+$, for $\alpha \in \mathbb{R}^r \times \mathbb{C}^s$, $\alpha_j \notin \sigma_j(K)$ $j = 1, \ldots, r+s$, α_j algebraic $j = 1, \ldots, r+s$, there would be only finitely many norms $N_{K/\mathbb{Q}}(q)$ such that

$$0 < N(\alpha - \sigma(p/q)) < 1/N_{K/\mathbb{Q}}(q)^{2+\varepsilon}.$$

Compare to MAHLER [4] result (appendix C) : let $\alpha \in \mathbf{C}^n$, let $\beta \in K$ and $H_K(\beta)$ the height of β as previously defined.

Let $f(\beta) = \prod_{j=1}^{n} \min(1, |\alpha_j - \sigma_j(\beta)|)$.

Let $\delta \in \mathbf{R}$, $\delta > 0$. There are only finitely many β in K with

$$f(\beta) < H_K(\beta)^{-2-\delta}.$$

3) Let $\alpha \in \mathbf{R}^r \times \mathbf{C}^s - \sigma(K)$. It is always possible to find $q_1 \in A$ such that

$$d(\alpha\sigma(q_1) - \sigma(p_1)) < n^2 B^{2/n} / d(\sigma(q_1))$$

and such that for all $q' \neq q_1$, $q' \in A$ with $d(\alpha\sigma(q') - \sigma(p')) < n^2 B^{2/n}/d(\sigma(q'))$ then $d(\sigma(q')) > d(\sigma(q_1))$: $\sigma(A)$ is a lattice, therefore

$$d(\sigma(q_1)) = \min\{d(\sigma(q)) \mid q \in A \ , \ \exists p \ , \ d(\alpha\sigma(q) - \sigma(p)) < n^2 B^{2/n}/d(\sigma(q))\}$$

exists. It is always possible to find in the same way $q_2 \in A$ such that

$$d(\alpha\sigma(q_2) - \sigma(p_2)) < d(\alpha\sigma(q_1) - \sigma(p_1)) \text{ with}$$
$$d(\sigma(q_2)) = \min\{d(\sigma(q')) \mid d(\alpha\sigma(q') - \sigma(p')) < d(\alpha\sigma(q_1) - \sigma(p_1))\}.$$

It is then possible to consider $(\sigma(p_1), \sigma(q_1)), \ldots, (\sigma(p_i), \sigma(q_i)), \ldots$ as a sequence of best approximations of $\alpha \in \mathbf{R}^r \times \mathbf{C}^s - \sigma(K)$ by elements of $\sigma(K)$, generalizing the concept of sequences of best approximations of elements $\alpha \in \mathbf{R} - \mathbf{Q}$ by elements of $\mathbf{Q}$. This concept is studied in [10].

4) It is possible to generalize theorem 1 and corollaries 2 and 3 to simultaneous approximation. For instance, let $(\alpha^1, \ldots, \alpha^\ell) \in (\mathbf{R}^r \times \mathbf{C}^s)^\ell - \sigma(K)^\ell$. Then, there exist infinitely many ℓ-tuples $(q_1, \ldots, q_\ell) \in A^\ell$ and $p \in A$ such that

$$0 < d(\alpha^1\sigma(q_1) + \cdots + \alpha^\ell\sigma(q_\ell) - \sigma(p)) < n^{\ell+1} B^{\ell+1} / d(\sigma(q_m))^\ell$$

where $d(\sigma(q_m)) = \max_{i=1,\ldots,\ell}(d(\sigma(q_i)))$.

REFERENCES

[1] A. Baker. *Transcendental number theory*, Cambridge Univ. Press, (1979).

[2] C.G. Lekkerkerker. *Geometry of Numbers*, North Holland, (1969).

[3] K.F. Roth. Rational approximation to rational numbers, *Mathematica* **2** (1955), corrigendum, Ibid (1968).

[4] K. Mahler. *Lectures on diophantine approximation*, Notre Dame Univ., (1961).

[5] P. Samuel. *Théorie Algébrique des Nombres*, Herman, (1971).

[6] W.M. Schmidt. Approximation to algebraic numbers, *Enseignement Math.* **17** (1971), 183-253.

[7] W.M. Schmidt. Simultaneous approximation to algebraic numbers by elements in a number field, *Monatsh. Math.* **79** (1975), 55-66.

[8] W.M. Schmidt. *Diophantine approximation*, Lecture notes in Math. 785, Springer Verlag, (1980).

[9] K.B. Stolarsky. *Algebraic numbers and diophantine approximation*, Marcel Dekker, New-York, (1974).

[10] R. Quême. *A generalization of best approximations method to algebraic number fields*, draft manuscript, 11/89.

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