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DIVISOR FUNCTIONS OF INTEGER MATRICES:

EVALUATIONS, AVERAGE ORDERS AND APPLICATIONS

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Divisor functions have been extensively studied in classical number theory. We examine the corresponding functions for matrices over the ring of integers. These functions not only are a natural generalisation of historically important concepts, but their study also yields applications and some new interpretations.

We deal only with $r \times r$ nonsingular matrices with entries from the ring of integers. Since there are an infinite number of unimodular matrices, it is necessary to identify *canonical factorisations* of matrices. In his study of arithmetic of matrices, Nanda [8] defines them as follows:

DEFINITION 1. The decomposition of the matrix A as

$$A = A_1 A_2 \dots A_k, \tag{1}$$

where $A_k, A_{k-1}, \dots, A_2$ are matrices in nonsingular Hermite Normal Form (HNF), is said to be a k-order (canonical) factorisation of A.

Since an HNF matrix uniquely represents a class under one-sided equivalence, two factorisations defined as in (1) are *inequivalent*.

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DEFINITION 2. The function $\tau_k^{(r)}$ counts the number of k-order factorisations of an $r \times r$ matrix A, i.e.,

$$\tau_k^{(r)}(A) = \sum_{A=A_1A_2\ldots A_k} 1. \tag{2}$$

When we restrict ourselves to the case $k = 2$, A_2 above is called a *divisor* of A. As in classical theory, we define the functions $\sigma_a^{(r)}$.

DEFINITION 3. For a non-negative integer a, let $\sigma_a^{(r)}(A)$ be the sum of the a^{th} powers of the determinants of divisors of a matrix A, i.e.,

$$\sigma_a^{(r)}(A) = \sum_{A=A_1A_2} (\det A_2)^a. \tag{3}$$

We notice that $\tau_2^{(r)} = \sigma_0^{(r)}$.

Evaluations

For evaluating divisor functions, it is enough to consider A to be in non-singular Smith Normal Form (SNF) with prime-power determinant. We use the notation F_r for the matrix

$$\left(\text{diag}\left[p^{f_1}, p^{f_1+f_2}, \ldots, p^{f_1+f_2+\ldots+f_r}\right]\right)$$

and may also denote it by $\langle f_1, f_2, \ldots, f_r\rangle$, where $f_1 \geq 1,\ f_i \geq 0$ for $i > 1$, and p is a prime number. We write G_r for the matrix $\langle f_1, f_2, \ldots, f_{r-1}, f_r - 1\rangle$.

It is clear that the evaluation of $\sigma_a^{(r)}$ involves the solving of a system of diophantine equations with rather stringent side conditions and is therefore not easy.

In [4] we proved the result:

$$\tau_2(F_r) - \tau_2(G_r) = \sigma_1(F_{r-1}), \tag{4}$$

which helped us evaluate $\tau_2^{(2)}$ and $\tau_2^{(3)}$. Here we generalise (4) and prove a recurrence between $\sigma_a^{(r)}$ and $\sigma_{a+1}^{(r-1)}$. Thus it becomes possible to reduce the problem to evaluations on F_1 or on *prime matrices* $P_r = \langle 1, 0, \ldots, 0\rangle$. We prove:

THEOREM 1. $\sigma_a(F_r) - p^a\sigma_a(G_r) = \sigma_{a+1}(F_{r-1})$.

Proof. Let $\nu = f_1 + f_2 + \ldots + f_r$. Now

$$F_r = \begin{pmatrix} A_{r-1} & 0 \\ Y & p^t \end{pmatrix} \begin{pmatrix} B_{r-1} & 0 \\ X & p^{\nu-t} \end{pmatrix}, \qquad 0 \le t \le \nu \tag{5}$$

$$G_r = \begin{pmatrix} A_{r-1} & 0 \\ Y & p^t \end{pmatrix} \begin{pmatrix} B_{r-1} & 0 \\ X & p^{\nu-t-1} \end{pmatrix}, \qquad 0 \le t < \nu \tag{6}$$

are factorisations of F_r and G_r whenever B_{r-1} is a divisor of F_{r-1} and $YB_{r-1} + Xp^t = 0$. Then

$$\sigma_a(F_r) = \sum_{\substack{YB_{r-1}+Xp^t=0 \\ 0\le t<\nu}} (\det B_{r-1})^a\, p^{(\nu-t)a} + \sum_{YB_{r-1}+Xp^\nu=0} (\det B_{r-1})^a \tag{7}$$

and

$$\sigma_a(G_r) = \sum_{\substack{YB_{r-1}+Xp^t=0 \\ 0\le t<\nu}} (\det B_{r-1})^a\, p^{(\nu-t-1)a} \tag{8}$$

Thus

$$\sigma_a(F_r) = p^a\sigma_a(G_r) + \sum_{YB_{r-1}+Xp^\nu=0} (\det B_{r-1})^a. \tag{9}$$

But, as in the proof of (4), we can show that the equation $YB_{r-1} + Xp^\nu = 0$ has solutions for all possible choices of X by considering the cases where the matrix

$$\begin{pmatrix} B_{r-1}^{-1} F_{r-1} & 0 \\ -p^\nu X B_{r-1}^{-1} & p^\nu \end{pmatrix}$$

is integral. Since X can be chosen in $\det(B_{r-1})$ ways, we have:

$$\sigma_a(F_r) = p^a\sigma_a(G_r) + \sum_{B_{r-1}\,|\,F_{r-1}} (\det B_{r-1})^{a+1}.$$

□

It is easy to evaluate $\sigma_a(P_r)$ by generalising Nanda's result [7] for $a = 0$:

THEOREM 2. $\sigma_a(P_r) = \sum_{j=0}^{r} p^{aj} \begin{bmatrix} r \\ j \end{bmatrix}$, *where* $\begin{bmatrix} r \\ j \end{bmatrix}$ *are the Gaussian polynomials in* p.

Proof. Nanda [7] gives a combinatorial argument to show that the number of divisors of P_r with p occurring exactly j times on the diagonal is given by $\begin{bmatrix} r \\ j \end{bmatrix}$. The factor p^{aj} is obviously the a^{th} power of the determinant of the divisor. □

The above theorems give us a method for evaluation, but the calculations involved are cumbersome, as the following example shows.

EXAMPLE.

$$\begin{aligned}\sigma_1(F_3) = {} & (p^{f_3} + p^{f_3-1} + \ldots + 1)\sigma_2(F_2) \\ & + p^{f_3+1}(p+1)\big(\sigma_2\langle f_1, f_2 - 1\rangle + p^2\sigma_2\langle f_1, f_2 - 2\rangle + \ldots + p^{2f_2-2}\sigma_2\langle f_1, 0\rangle\big) \\ & + p^{f_3+2f_2+1}\sigma_2\langle f_1 - 1, 1\rangle + p^{f_3+2f_2+2}(p+1)\sigma_2\langle f_1 - 1, 0\rangle \\ & + p^{f_3+2f_2+4}\sigma_2\langle f_1 - 2, 1\rangle + p^{f_3+2f_2+5}(p+1)\sigma_2\langle f_1 - 2, 0\rangle + \ldots \\ & + p^{f_3+2f_2+3f_1+2}(p+1)\sigma_2\langle 1, 0\rangle + p^{f_3+2f_2+3f_1+4}\sigma_1\langle 1, 0\rangle,\end{aligned}$$

where

$$\begin{aligned}\sigma_2(F_2) = {} & (p^{2f_2} + p^{2f_2-2} + \ldots + 1)\sigma_3\langle f_1\rangle + p^{2f_2+2}(p^2+1)\sigma_3\langle f_1 - 1\rangle \\ & + p^{2f_2+6}(p^2+1)\sigma_3\langle f_1 - 2\rangle + \ldots + p^{2f_2+4f_1-6}(p^2+1)\sigma_3\langle 1\rangle \\ & + p^{2f_2+4f_1-2}\sigma_2\langle 1\rangle.\end{aligned}$$

Average orders

In [3] we evaluated $T(x) = \sum_{\det A \le x} \tau_2^{(2)}(A)$, where A is in SNF. The absence of a zeta function does not allow satisfactory extension of this result.

Here we instead allow A to be in HNF and consider the following *zeta function* [5]:

$$Z_r(s) = \sum_{A \text{ in HNF},\ r\times r} (\det A)^{-s} = \sum \alpha_r(n)\, n^{-s}, \qquad s = \sigma + it, \quad \sigma > 1, \tag{10}$$

where $\alpha_r(n)$ is the HNF class-number, i.e., the number of $r \times r$ HNF matrices with determinant n.

Thus

$$Z_r(s) = \zeta(s)\zeta(s-1)\ldots\zeta(s-r+1), \tag{11}$$

where $\zeta(s)$ is the Riemann zeta-function. $Z_r(s)$ is a holomorphic function with simple poles at $s = r, r-1, r-2$. It is, in fact, a special case of the Koecher zeta-function [10].

Authors (e.g. [9]) have evaluated $\alpha_r(n)$, and in [3] we obtained an average order for $\alpha_r(n)$ by using elementary methods.

Here we give a few asymptotic results for divisor functions obtained by using $Z_r(s)$ for

$$A_r(x) = \sum_{n \le x} \alpha_r(n), \quad T_k(x) = \sum_{A \text{ in HNF, } \det A \le x} \tau_k(A)$$

and $S_a^{(r)}(x) = \sum_{A \text{ in HNF, } \det A \le x} \sigma_a^{(r)}(A)$. These are only first results. We should be able to prove sharper results in [5].

THEOREM 3.

$$A_r(x) = \frac{1}{r} L_r x^r - \frac{1}{2(r-1)} L_{r-1} x^{r-1} + o(x^{r-1}),$$

where $L_k = \prod_{j=2}^{k} \zeta(j)$.

Proof. From Perron's formula we have:

$$A_r(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{Z_r(s)\, x^s}{s} ds.$$

The main terms can be obtained as residues at $s = r$, $r-1$. □

THEOREM 4. $T_k(x) \sim P_r(x)$, *where $P_r(x)$ is a polynomial in x of degree r.*

Proof. Obvious. □

THEOREM 5.

$$S_a^{(r)}(x) = \frac{1}{a+r} L_r M_{a,0} x^{a+r} - \frac{1}{2(a+r-1)} L_{r-1} M_{a,1} x^{a+r-1} + o(x^{a+r-1}),$$

where

$$M_{b,t} = \prod_{j=1}^{r} \zeta(b+j-t).$$

Proof. Similar to that of Theorem 3. □

Restricted divisor functions

Here we consider factorisations on the subsets
(a) $\mathcal{S}_r$, of all $r \times r$ matrices in SNF, and
(b) $\mathcal{D}_r$ of all $r \times r$ diagonal matrices,
and show their connection with average orders of functions of integers.

DEFINITION 4. The decompositions

$$\begin{aligned} S &= S_1 S_2 \dots S_k, \qquad && S, S_i \in \mathcal{S}_r \\ \text{and} \quad D &= D_1 D_2 \dots D_k, \qquad && D, D_i \in \mathcal{D}_r \end{aligned}$$

are said to be k-order SNF (resp., *diagonal*) *factorisations* of S (resp., D).

In [3] we have defined the SNF *zeta function*

$$\widehat{Z}_r(s) = \sum \beta_r(n)/n^s = \zeta(s)\zeta(2s)\dots\zeta(rs) \tag{12}$$

and proved:

THEOREM 6.

$$\sum_{\substack{S \in \mathcal{S}_r \\ \det S \leq x}} \widehat{\sigma}_a(r)(S) = \frac{1}{a+1} L_r^{(1)} K_1 x^{a+1} + \frac{1}{a+\frac{1}{2}} L_r^{(1/2)} K_2 x^{a+\frac{1}{2}} + \frac{1}{a+\frac{1}{3}} L_r^{(1/3)} K_3 x^{a+\frac{1}{3}} + O(x^{a+\varepsilon}).$$

where ε is the best known exponent in the error of the three-dimensional divisor problem, and the constants are defined as

$$L_r^{(i)} = \prod_{\substack{j=1 \\ ji \neq 1}}^{r} \zeta(ji), \qquad K_i = \prod_{j=1}^{r} \zeta(j(a+i)).$$

We can similarly define the *diagonal zeta function* as

$$Z_{\mathcal{D}}(s) = \sum_{D \in \mathcal{D}_r} (\det D)^{-s} = \zeta^r(s) \tag{13}$$

and obtain the average order $B_a^{(r)}(x)$ for diagonal divisor functions. For example, when $a = r = 2$, we get

THEOREM 7.

$$\begin{aligned} B_2^{(2)} =& \tfrac{1}{6} x \log^3 x + (2\gamma - \tfrac{1}{2}) x \log^2 x + (6\gamma^2 - 4\gamma + 4\gamma_1 + 1) x \log x \\ &+ (-1 + 4(\gamma - \gamma_1 + \gamma_2) - 6\gamma^2 + 4\gamma^3 + 12\gamma\gamma_1) x + O(x^{1/2}) \end{aligned}$$

where the γ's are constants.

Proof. We use the classical estimate for $\zeta^4(s)$. □

Applications

We can interpret some of the above results in different contexts. We give a few illustrations:

a) We consider the set $\Gamma = \mathrm{GL}(r, \mathbb{Z})$ of all $r \times r$ integer unimodular matrices as the unimodular group of degree r over the ring of integers [6]. The divisor functions of matrices then appear in the formal power series over the *Hecke algebra*, which is a polynomial ring over $\mathbb{Z}$ in infinitely many independent indeterminates. Thus, formally,

$$\sum \Gamma A \Gamma (\det A)^{-s} \sum \Gamma A \Gamma (\det A)^{a-s} = \sum \Gamma A \Gamma \sigma_a(A) (\det A)^{-s}, \tag{14}$$

where $A \in \mathcal{S}_r$.

b) We consider Ramanujan's classical formula

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}$$

for the case $a = 1$, $b = 2$, to get

$$Z_4(s) = \zeta(2s-3)\sum_{n=1}^{\infty}\sigma_1(n)\sigma_2(n)/n^s. \tag{15}$$

c) We can evaluate $\sigma_a(P_r)$ in terms of two different *partition functions*. We can show that to get the number of divisors of P_r, we may look at the number of $\underline{0}$'s to the right of each $\underline{1}$ in every string of length r with entries from $\{0,1\}$. This can be expressed in terms of the function $\hat{q}(n,k,r)$, which counts the number of partitions of n into at most k parts (not necessarily distinct) each at most r, i.e.,

$$\sigma_a(P_r) = \sum\sum q(n-ka,k,r-k)p^n. \tag{16}$$

This, we note, agrees with Theorem 2 via Sylvester's generating function for $\hat{\sigma}$ (see, e.g. Andrews [1]). We can also evaluate $\sigma_a(P_r)$ using the function $q(n,k,r)$ which counts the number of partitions of n into exactly k distinct parts. In [2] we have shown that the number of divisors of P_r that are equivalent to P_k, $k \le r$, is given by

$$\pi_{k,r} = \sum_{n=\frac{k(k+1)}{2}}^{kr-\frac{k(k-1)}{2}} q(n,k,r)\,p^{kr-\frac{k(k-1)}{2}-n}. \tag{17}$$

Thus we get

$$\sigma_a(P_r) = \sum\sum q\left(kr - \tfrac{k(k-1)}{2} - n - ak, k, r\right)p^n. \tag{18}$$

We see that there is no contradiction between these two results, for we can independently show that

$$\hat{q}(n,k,r-k) = q\left(kr - \tfrac{k(k-1)}{2} - n, k, r\right) \tag{19}$$

by writing n as a sum of parts $n_1, n_2, \ldots, n_k$; $r-k \ge n_i \ge n_{i+j} \ge 0$ and replacing n_i by $m_i = r - n_i - i + 1$.

As in [2], it is possible to obtain *polynomial* and *partition identities* from these results.

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