Astérisque

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Astérisque, tome 209 (1992), p. 195-201
<http://www.numdam.org/item?id=AST_1992__209__195_0>

ON THE STRUCTURE AND THE NUMBER OF SUM-FREE SETS

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1. Introduction

A finite set A of positive integers is called *sum-free*, if $A \cap (A+A) = \emptyset$. In this note we study the structure of sum-free sets. For n odd, $\{1, 3, 5, \ldots, n\}$ and $\{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n\}$ are important examples of such sets.

For any non-empty finite set $K \subset \mathbb{Z}$, we denote by $\ell(K)$ and $m(K)$, respectively, the largest and smallest element of K, by $d(K)$ the greatest common divisor of the elements of K, and by $|K|$ the cardinality of K. For the sets A considered below, we set $m := m(A)$, $\ell := \ell(A)$, $a := |A|$, $2A := A+A$ and $A - m := \{x - m \mid x \in A\}$, $\ell - A := \{\ell - x \mid x \in A\}$. Denote $[m, n] = \{x \in \mathbb{Z} \mid m \leq x \leq n\}$. There is a general property of sum-free sets (from [CE], page 63) which we will use later: If B is a sum-free subset of $\{1, \ldots, n\}$ then B contains at most one of i and $\ell(B) - i$, for each positive integer $i < \ell(B)$; and if $\ell(B)$ is even, then $\frac{1}{2}\ell(B) \notin B$. Hence

$$|B| \leq \lceil \frac{1}{2}\ell(B) \rceil \leq \lceil \frac{1}{2}n \rceil \ . \tag{1}$$

We will show that if the cardinality of a sum-free set A does not differ much from $\frac{1}{2}\ell(A)$, then A does not differ much from one of the two examples mentioned above. More precisely, we will prove

Theorem 1. *Let A be a sum-free set of positive integers for which $a \geq \frac{5}{12}\ell+2$. Then either*

1) *All elements of A are odd, or*
2) *A contains both odd and even integers, $m \geq a$, and for $A_1 := A \cap [1, \frac{1}{2}\ell]$ we have*
$$|A_1| \leq \frac{\ell - 2a + 3}{4} .$$

Let $f(n)$ denote the number of sum-free subsets of $\{1, \ldots, n\}$.

P.J. Cameron and P. Erdös in their talk at the First Conference of the Canadian Number Theory Association [CE, page 64] conjectured that

$$f(n) = O(2^{\frac{n}{2}}) .$$

P. Erdös and A. Granville, and independently N. Calkin as well as N. Alon [Al] showed that
$$f(n) = 2^{\left(\frac{1}{2} + o(1)\right)n} .$$
The proof in [Al] is more general and in particular applies to any group.

As a simple corollary of Theorem 1 we will prove that the number of sum-free sets $A \subset [1, n]$ for which $a \geq \frac{5}{12}\ell + 2$ has the bound $O\left(2^{\frac{n}{2}}\right)$.

2. The Structure of Sum-Free Sets of Large Cardinality

As a main tool in the proof of Theorem 1 we will use the following two theorems from [F1].

Let M and N be finite sets of non-negative integers such that $m(M) = m(N) = 0$.

Theorem 2. *If $\ell(M) = \max(\ell(M),\ \ell(N))$ and $\ell(M) \leq |M| + |N| - 3$, then $|M + N| \geq \ell(M) + |N|$.*

Theorem 3. *If $\max(\ell(M),\ \ell(N)) \geq |M| + |N| - 2$ and $d(M \cup N) = 1$, then*

$$|M + N| \geq |M| + |N| - 3 + \min(|M|, |N|) .$$

We shall also use the following result from [F2]:

Lemma. *If $A \subset \mathbb{Z}$ is finite, then*

$$|2A| \geq 2|A| - 1 . \qquad (2)$$

Proof of Theorem 1. Let us call a set A difference-free if $A \cap (A - A) = \emptyset$. Note first that the notions of sum-free set and of difference-free set coincide. For if $x, y, z \in A$, then $x = y + z \iff y = x - z$. Thus if A is not sum-free then A is not difference-free and conversely.

In the set $A - A$, to each positive difference $x - y$ there corresponds the negative difference $y - x$. Denote by $(A - A)_+$ and $(A - A)_-$, respectively, the set of positive and negative differences.

Since $A - A = (A - A)_+ \cup (A - A)_- \cup \{0\}$ and $|(A - A)_+| = |(A - A)_-|$, we have

$$|A - A| = 2|(A - A)_+| + 1 \; . \tag{3}$$

The sets A and $(A - A)_+$ are both contained in the interval $[1, \ell]$. Since A is difference-free, it follows that

$$|A| + |(A - A)_+| \leq \ell \; . \tag{4}$$

This inequality is very restrictive for large $a = |A|$, and we will use it in conjunction with a lower bound for $|(A - A)_+|$ to be obtained from Theorems 2 and 3, to prove Theorem 1.

Let us study various cases according to the value of $d(A - m)$.

We first observe that $d(A - m) \leq 2$, for if $d(A - m) \geq 3$ then $a \leq \frac{\ell}{3} + 1$ which contradicts the condition $a \geq \frac{5}{12}\ell + 2$.

In case $d(A - m) = 2$ first consider the subcase when m is odd. Then all the numbers of A are odd and we have Case 1 of Theorem 1.

If $d(A - m) = 2$, then m cannot be even, under the hypothesis of Theorem 1. Indeed, if m is even and $d(A - m) = 2$ then all the integers in A are even and the set $\frac{A}{2} := \{x \mid x = \frac{a}{2},\ a \in A\}$ is sum-free, with largest element $\ell_1 = \frac{\ell}{2}$. Also if $a \geq \frac{5}{12}\ell + 2$ then (1), applied to $B = \frac{A}{2}$, would yield $\frac{5}{12}\ell + 2 \leq a = |A| = |\frac{A}{2}| = |B| \leq \frac{\ell_1 + 1}{2} = \frac{\ell + 2}{4}$, which is absurd.

The only case left is that in which $d(A - m) = 1$. Clearly the elements of A cannot then all be of the same parity. We define sets M and N by $M := A - m$ and $N := \ell - A$. Then $m(M) = m(N) = 0$, $\ell(M) = \ell(N) = \ell - m$, $|M| = |N| = a$, $|M + N| = |A - A|$; and $d(M \cup N) = 1$ since $d(M) = 1$. If we had

$$\ell - m \geq 2a - 2 \; , \tag{5}$$

Theorem 3 would apply, giving $|A - A| = |M + N| \geq 3a - 3$, whence $|(A - A)_+| \geq \frac{3a}{2} - 2$ by (3). Using this in (4) together with $a \geq \frac{5}{12}\ell + 2$ would yield the absurd

$$\ell \geq |(A - A)_+| + a \geq \frac{5a}{2} - 2 > \frac{25}{24}\ell \; .$$

Hence (5) is impossible: $\ell - m < 2a - 2$ if $d(A - m) = 1$ and $a \geq \frac{5}{12}\ell + 2$.

Theorem 2 applies, and gives $|A - A| \geq \ell - m + a$, whence $|(A - A)_+| \geq \frac{1}{2}(\ell - m + a - 1)$ by (3).

Using this inequality, (4) and $a \geq \frac{5}{12}\ell + 2$, we get

$$m > \frac{\ell}{4} \ . \tag{6}$$

Having obtained this lower bound for m, we can strengthen it as follows.

For any positive integer i, the integers i and $m + i$ cannot both belong to $A(m \in A$ and A is sum-free). Hence the union $[\ell - 2m + 1, \ell]$ of the intervals $I = [\ell - 2m + 1,\ \ell - m]$ and $I + m$ contains at most m elements of A. Recall that $A_1 = A \cap \left[1, \frac{\ell}{2}\right]$. Let $A_2 = A \setminus A_1 = A \cap \left[\frac{\ell+1}{2}, \ell\right]$. Then by (6), $A_2 \subset \left[\frac{\ell+1}{2}, \ell\right] \subset [\ell - 2m + 1, \ell]$, and therefore

$$|A_2| \leq m \ . \tag{7}$$

Now $2A_1 \cap A_2 = \emptyset(A_2 \subset A$, and $2A_1 \cap A = \emptyset$ since A is sum-free) and by (6), $2A_1 \subset \left[\frac{\ell+1}{2}, \ell\right]$. Hence

$$|2A_1| + |A_2| \leq \left| \left[\frac{\ell+1}{2}, \ell\right] \right| \leq \frac{\ell+1}{2} \ .$$

By adding this inequality to (7) and using (2) and $|A_1| + |A_2| = a$ we get $2a \leq \frac{1}{2}(\ell + 3) + m$. Hence with $a \geq \frac{5}{12}\ell + 2$ we get

$$m > \frac{\ell}{3} + 2 \ . \tag{8}$$

From (8) we have $A \subset [m, \ell] \subset [\ell - 2m + 1, \ell]$. We have seen that this last interval contains at most m integers from A; it follows that $m \geq a$, which proves the first inequality in Case 2 of Theorem 1.

To establish the second inequality of Case 2, we observe that $\ell - A_1$, $2A_1$, and A_2 are pairwise disjoint subsets of $\left[\frac{\ell+1}{2}, \ell\right]$. We have already verified this for $2A_1$ and A_2. Also, $(\ell - A_1) \cap A_2 = \emptyset$ since A is sum-free and $(\ell - A_1) \cap 2A_1 = \emptyset$ because $\ell - A_1 \subset [0, \ell - m]$, $2A_1 \subset [2m, \ell]$ and $\ell - m < 2m$ by (8). Finally, $\ell - A_1 \subset \left[\frac{\ell}{2},\ \ell - 1\right]$ since $A_1 \subset \left[1,\ \frac{\ell}{2}\right]$; and $\frac{\ell}{2} \notin A$ if ℓ is even, because A is sum-free.

It now follows that $|\ell - A_1| + |2A_1| + |A_2| \leq \frac{\ell+1}{2}$, whence by (2), $3|A_1| + |A_2| - 1 \leq \frac{\ell+1}{2}$, or $|A_1| \leq \frac{\ell}{4} - \frac{a}{2} + \frac{3}{4}$. This completes the proof of Theorem 1.

3. Maximal Sum-Free Sets

We will call a sum-free subset of $[1,n]$ *maximal* if it is of maximal cardinality.

We will now prove the following theorem, stated without proof in [CE] on page 63.

Theorem 2. *For $n \geq 24$, the only maximal sets are*
(i) the set C of all odd numbers in $[1,n]$;
(ii) the set D of all numbers in $[1,n]$ which are greater than $\frac{n}{2}$,
(iii) if n is even, the set $E = D - 1 = \left[\frac{n}{2},\ n-1\right]$.

Proof: Clearly, the sets C, D and E are sum-free of cardinality $\lceil\frac{n}{2}\rceil$. Hence, by (1), a sum-free set A is maximal if and only if $|A| = \lceil\frac{n}{2}\rceil$. Let A be any maximal set. Then $a = |A| = \lceil\frac{n}{2}\rceil \geq \frac{n}{2} \geq \frac{5}{12}n + 2$ (if $n \geq 24$) $\geq \frac{5}{12}\ell + 2$, so the condition of Theorem 1 is satisfied.

In Case 1, $A \subseteq C$, so $A = C$.

In Case 2, $m \geq a$, so $A \subseteq \left[\lceil\frac{n}{2}\rceil,\ n\right] = F$, say. If n is odd, then $F = D$, so that $A = D$. If n is even, then $F = \left\{\frac{n}{2},\ \frac{n}{2}+1, \ldots, n\right\}$ which is a set of cardinality $\lceil\frac{n}{2}\rceil + 1$, so precisely one of its elements does not belong to A. If $\frac{n}{2} \notin A$, then $A = D$, and if $\frac{n}{2} \in A$, then $n \notin A$, therefore, $A = E$.

4. Some Examples of Sum-Free Sets

We now give two examples to show that each of the inequalities in Theorem 1 is best possible.

Example 1. Let us consider positive integers m and n such that $n \geq 36$ and $5n + 24 \leq 12m < 6n$ ($n \geq 36$ ensures the existence of at least one such m). Then define the set $A = ([n-m+1, n] \cup \{m\}) \setminus \{2m\}$. Then one has

1) A is a sum-free set,
2) $|A| = m$ and $\ell(A) = n$ so that the condition $a \geq \frac{5\ell}{12} + 2$ is fulfilled,
3) A contains an even number (because we have $n > 24$, so that $m > 12$ and $[n-m+1, n]$ contains at least two even numbers),
4) $m = a$.

Example 2. Let us consider two positive integers m and n satisfying $11n + 18 \leq 24m \leq 12n - 12$. (such that n is odd and $n \geq 53$), and let us define

$$A = \left[m, \frac{n-1}{2}\right] \cup ([n-m+1, n] \setminus [2m, n-1]) \ .$$

Then one has

1) A is a sum-free set,
2) $|A| = \frac{4m-n+1}{2}$ and $\ell(A) = n$, so that condition $a \geq \frac{5\ell}{12} + 2$ is fulfilled,

3) A contains an even number (because $[n-m+1,\ 2m-1]$ contains at least one even number),
4) therefore, we have $A \cap [1,\ \frac{\ell}{2}] = [m,\ \frac{n-1}{2}]$ so that $|A \cap [1,\ \frac{\ell}{2}]\,| = \frac{\ell}{4} - \frac{a}{2} + \frac{3}{4}$.

It can be shown that when m is sufficiently large, both equalities $m = a$ and $|A_1| = \frac{\ell}{4} - \frac{a}{2} + \frac{3}{4}$ cannot hold at the same time; and indeed deeper results can be established correlating the lower bound of m and the upper bound of $|A_1|$.

The hypothesis $a \geq \frac{5}{12}\ell + 2$ in Theorem 1 cannot be replaced by $a \geq \frac{2}{5}\ell$, as is seen from the example (for $n \in \mathbb{N}$ divisible by 5) of the set $[\frac{n}{5}+1,\ \frac{2n}{5}] \cup [\frac{4n}{5}+1, n]$. Furthermore, this set is locally maximal in the sense of the following definition.

Definition. A set A in $[1,n]$ is *locally maximal* if A is sum-free, but if $A \subseteq A' \subseteq [1,n]$ and$A' \neq A$, then A' is not sum-free.

There naturally arises the problem of determining all locally maximal sets.

5. On the Number of Sum-Free Sets

Theorem 1 immediately gives an upper bound for the number of sum-free sets for which $a \geq \frac{5}{12}\ell + 2$.

In Case 1 the number of sum-free sets with $|A| = a$ is less than or equal to $\binom{[\frac{n+1}{2}]}{a}$.

In Case 2 the number of sum-free sets is less than or equal to $\binom{n-a+1}{a}$. These upper bounds confirm the conjecture of Cameron and Erdös for the number of sum-free sets for which $a \geq \frac{5}{12}\ell + 2$.

It may be conjectured that the number of sum-free sets in $[1,n]$ of cardinality a is $O\left(\binom{\frac{n}{2}}{a}\right)$.

In conclusion, I am pleased to express my gratitude to Professor John Steinig, the referee of this paper, who has spent much of his time clarifying and making more precise an exposition of this paper.

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