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1. Introduction: terminology and facts.– Let E be an elliptic curve defined over $\mathbb{Q}$ which is supposed to be modular, i.e. E is a Weil curve, and denote by $F(z) = \sum A_n e^{2\pi i n z}$ the weight 2 newform attached to E by the Eichler-Shimura congruences.

Fix a prime $p > 7$. We shall be interested in which cases E has additive reduction at p, excluding the Kodaira reduction types I_ν^* $(\nu \geq 0)$ which are related to the potentially semi-stable case. Thus p divides exactly twice the geometric conductor N_E of the elliptic curve E.

After [Ed 89] we say that E is p-vertical if E has bad but potentially good ordinary reduction at p, and that E is p-horizontal if E has bad but potentially good supersingular reduction at p. Recall that these conditions can be given in terms of the Hasse invariant (cf. [Hu 87], pag.248) of E and, moreover, one gets:

$$E \text{ is } p\text{-vertical} \iff p \equiv 1 \pmod{e},$$

where e is the least common multiple of the multiplicities of the irreducible components of the special fibre of the stable model; that is

$$e = \begin{cases} 6 & \text{if } p\text{-type}\,(E) = II, II^*\,; \\ 4 & \text{if } p\text{-type}\,(E) = III, III^*\,; \\ 3 & \text{if } p\text{-type}\,(E) = IV, IV^*\,. \end{cases}$$

The Galois module E_p of the p-torsion points of E gives rise to a continuous and odd representation

$$\rho :\ \mathrm{G}_{\mathbb{Q}} \to \mathrm{Aut}(E_p) \simeq \mathrm{GL}_2(\mathbb{F}_p)\,,$$

which is almost always absolutely irreducible (cf. [Ma 78]).

Serre's conjecture (3.2.4?) in [Se 87] predicts, in this case, the existence of a Hecke cusp form (mod p)

$$f(q) = \sum a_n q^n$$

of type $(N_\rho, k_\rho, \varepsilon_\rho)$ satisfying

$$a_n \equiv A_n \pmod{p}, \qquad \text{for all } n \text{ prime to } N_E.$$

The level N_ρ, the weight k_ρ and the character ε_ρ are given by a precise recipe in [Se 87] and, depending on the Néron model of E, they have been computed, for instance in [Ba-La 91].

In [Ba-La 91], we verify (3.2.4?) for the Galois representations defined by the p-torsion points of the p-vertical Weil curves. In this paper our purpose is to emphasize the difference between the p-vertical and the p-horizontal cases in order to check Serre's conjecture. Several numerical examples, collected by computer calculations, lead us to give a conjecture which implies (3.2.4?) for the horizontal case.

2. Lowering the level (ordinary case).– First, we shall consider a general situation. Let

$$F(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$$

be a newform of type (N, k, ε), defined over $\overline{\mathbb{Q}}$. If $\alpha : (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}^*$ is a Dirichlet character modulo M, then the twisted form

$$F \otimes \alpha\,(z) = \sum_{n=1}^{\infty} A_n\, \alpha(n) e^{2\pi i n z} \qquad (\alpha(n) = 0 \text{ if g.c.d.}\,(n, M) \neq 1)$$

is a Hecke cusp form of type $(N', k, \varepsilon\,\alpha^2)$, where

$$N' = \text{l.c.m.}\,(N, M \cdot \text{conductor}\,(\varepsilon), M^2).$$

If M is prime to the level N, then $F \otimes \alpha$ is a newform of type $(NM^2, k, \varepsilon\,\alpha^2)$; otherwise, the form $F \otimes \alpha$ can either be or not be a newform! The question is to decide when it is.

After Li's work [Li 75], one has a nice criterion to decide whether a Hecke cusp form is new or not. More precisely, consider the operators K and H_N defined by

$$G|H_N = G|\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \quad \text{and} \quad G|K(z) = \overline{G(-\overline{z})},$$

for each cusp form $G(z) = \sum_{n=1}^{\infty} B_n e^{2\pi i n z}$ of type (N, k, ε). We have

PROPOSITION. (cf. [Li 75]). *Let $G \in S_k(N, \varepsilon)$ be a Hecke cusp form. Then G is a newform of type (N, k, ε) if and only if the functional equation $G|K|H_N = \gamma G$ holds for a certain complex constant γ of absolute value 1.*

Let us go back to the case of elliptic curves. Let F be the newform attached to the Weil curve E as above and let $N_E = Np^2$ be the conductor of E. Choose an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and let $\psi : (\mathbb{Z}/p\mathbb{Z})^* \to \overline{\mathbb{Q}}$ be the Dirichlet character which satisfies

$$\psi(n)\, n \equiv 1 \pmod{\mathfrak{P}},$$

where $\mathfrak{P}$ is the place of $\overline{\mathbb{Q}}$ dividing p fixed by our embedding.

We ask for which values of $j \in \{0, \dots, p-2\}$ the twisted form $F \otimes \psi^j$ fails to be new. Prof. D. B. Zagier suggested to us to apply the following test which is an immediate consequence of Li's result.

COROLLARY. *Keep the above notations. If there exist a complex number $z \in \mathbb{H}$ such that*

$$\left| \frac{\sum_{n=1}^{\infty} \overline{A}_n \overline{\psi}^j(n)\, e^{-2\pi i n/N_E z}}{N_E\, z^2 \sum_{n=1}^{\infty} A_n \psi^j(n)\, e^{2\pi i n z}} \right| - 1 \neq 0,$$

then $F \otimes \psi^j$ is not new.

On a VAX 8600 at the Facultat d'Informàtica de Barcelona we have obtained the following numerical examples, by taking $z = 2i/\sqrt{N_E} \in \mathbb{H}$ and a few number (around 500) of Fourier coefficients for $F \otimes \psi^j$.

For the elliptic curve 338 **A1** (cf. [Cre 91]),

$$E : y^2 + xy = x^3 - x^2 + x + 1$$

of conductor $N_E = 2 \cdot 13^2$, we get the following data:

j	TEST
0	0.0000000000
1	0.0000000000
2	2.6699959395
3	0.0000000000
4	0.0000000000
5	0.0000000000
6	0.0000000000
7	0.0000000000
8	0.0000000000
9	0.0000000000
10	2.6699959395
11	0.0000000000

Observe that the Hecke cusp forms $F \otimes \psi^2$ and $F \otimes \psi^{10}$ don't seem to be newforms. Indeed, as we shall see later, they are not newforms.

Another example is provided by the elliptic curve

$$E \,:\, y^2 + xy + y = x^3 - 39x - 27$$

of conductor $N_E = 43^2$ (cf. [Ed-Gr-To 90]). In this case we get:

j	TEST
14	5.5513673695
28	5.5513673695 ,

and zero for all the others values of j. Now, the Hecke cusp forms $F \otimes \psi^{14}$ and $F \otimes \psi^{28}$ are not newforms.

In the previous examples E is vertical at $p = 13$, 43, respectively; indeed, in both cases we have $p \equiv 1 \pmod 3$ and, following Tate's algorithm [Ta 75], we find that p-type (E) is equal to II and IV, respectively.

Actually, we are able to say what happens in the general p-vertical case. If ℓ denotes a prime which does not divide the conductor of E, then one can prove that the restriction to an inertia group I_p at p of the ℓ-adic representation ρ_ℓ attached to $F \otimes \psi^{\frac{p-1}{e}}$ is given by

$$\rho_\ell(I_p) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} .$$

Therefore, we obtain

PROPOSITION. (cf. [Ba-La 91]). *If E is a p-vertical Weil curve as above, there exists a newform*

$$G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p-1}{e}}),$$

having the same eigenvalue system as the twisted form $F \otimes \psi^{\frac{p-1}{e}}$.

Similar arguments do not work when E is p-horizontal. Consider the following example: the elliptic curve 605 **A1**, in [Cre 91],

$$E \,:\, y^2 + xy = x^3 - x^2 - 1414x - 44027$$

of conductor $N_E = 5 \cdot 11^2$ has 11-type IV^*. Since $11 \equiv 2 \pmod 3$, E is 11-horizontal. Running our program we get:

j	TEST
0	0.0000000000
1	0.0000000000
2	0.0000000000
3	0.0000000000
4	0.0000000000
5	0.0000000000
6	0.0000000000
7	0.0000000000
8	0.0000000000
9	0.0000000000

Actually, all the twisted forms $F \otimes \psi^j$ are newforms.

3. Lowering the level (supersingular case) .– Keep the notations as above and let $v^{ss} = \frac{p+1}{e}$. Since we are interested in Serre's conjecture (3.2.4?), we shall assume without loss of generality that p-type$(E) = II, III, IV$. Indeed, the exclusion of the cases with asterisk remain justified by considering the minimal or companion representations as in [La 91].

CONJECTURE (ss.?) . *If E is a p-horizontal Weil curve as above, there exists a newform*

$$G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p+1}{e}}),$$

having the same eigenvalue system (mod $\mathfrak{P}$) *as the twisted form* $F \otimes \psi^{\frac{p+1}{e}}$*; i.e., such that for all n prime to N_E we have*

$$B_n \equiv A_n \psi^{v^{ss}}(n) \pmod{\mathfrak{P}}.$$

Moreover, G is $\mathfrak{P}$-ordinary if and only if $\rho|_{D_p}$ is not irreducible. Here D_p denotes a decomposition group for p.

First of all, we are going to show a numerical example of the lowering of the level predicted by the conjecture.

Consider the elliptic curve E given by the Weierstraß model

$$y^2 + xy = x^3 + 3x + 1;$$

it is the curve 242 **A1** in [Cre 91] and has conductor $N_E = 2 \cdot 11^2$. The special fibre of the Néron model over the local ring $\mathbb{Z}_{11}$ has reduction type II; therefore, E is 11-horizontal. Moreover, E has no $\mathbb{Q}$-rational isogenies of degree 11, and then

$$\rho : \mathrm{G}_{\mathbb{Q}} \to \mathrm{Aut}\,(E_{11}) \simeq \mathrm{GL}_2(\mathbb{F}_{11})$$

is irreducible; since $v_{11}(c_4) = 1$, we find that $\rho|D_{11}$ is reducible.

Choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_{11}$ such that the character

$$\psi : (\mathbb{Z}/11\mathbb{Z})^* \to \overline{\mathbb{Q}}^*, \quad \psi(2) = e^{2\pi i/10},$$

satisfies

$$\psi(2) \equiv 2^{-1} \pmod{\mathfrak{P}_{11}},$$

where $\mathfrak{P}_{11}$ is the place of $\overline{\mathbb{Q}}$, over 11, determined by the embedding.

Our conjecture (ss.?) predicts the existence of a cusp form

$$G(q) = \sum B_n q^n \in S_2(22, \psi^4)$$

satisfying the congruences

$$B_n \equiv A_n \psi^2(n) \pmod{\mathfrak{P}_{11}}$$

for all odd integers n prime to 11.

In this case, we find

$$\dim S_2(22, \psi^4) = 1\,.$$

The Eichler-Selberg trace formula (cf. [Hij-Pi-She 90]) allows us to obtain the Fourier coefficients of the unique normalized newform of this type.

An efficient implementation of this formula is due to J. Quer; its program, written in UBASIC, find the first coefficients of the newform $G(q) = \sum_{n\geq 1} B_n q^n$ in $S_2(22, \psi^4)$:

$$\begin{aligned}
B_3 &= -\zeta^4 - \zeta^2 - 2\zeta - 2\\
B_5 &= -2\zeta^3 - 2\\
B_7 &= -2\zeta^4 - 2\zeta^3 - 4\zeta^2 - 2\zeta\\
B_{13} &= -2\zeta^2 - 2\zeta - 2\\
B_{17} &= -4\zeta^4 - 4\zeta^3 - 5\zeta^2 - 5\zeta - 4\\
B_{19} &= -2\zeta^4 - 6\zeta^3 - 2\zeta^2 - 5\zeta - 5\\
&\dots
\end{aligned}$$

where $\zeta = e^{2\pi i/5}$. The coefficients B_n can be rewritten taking into account that $\cos\pi/5$ is the unique positive (double) root of the polynomial $16X^5 - 20X^3 + 5X + 1$ and that

$$i \cos\frac{\pi}{10} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} - \frac{\sqrt{5}-1}{4}\,.$$

On the other hand, computing the first coefficients of the L-series of the elliptic curve E, we find the following:

ℓ	2	3	5	7	11	13	17	19	23	29	31	37	41	43
A_ℓ	–	−2	−3	−2	–	−5	−3	−2	6	3	2	−7	−3	−8

Finally, since $\sqrt{5} \equiv 4 \pmod{11}$ and

$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} = e^{2\pi i/10} \equiv 2^{-1} \equiv 6 \pmod{\mathfrak{P}_{11}},$$

we get

ℓ	$\psi^2(\ell)$	B_ℓ	$A_\ell\,\psi^2(\ell) - B_\ell \pmod{\mathfrak{P}_{11}}$
2	$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$	–	–
3	$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$	$-\frac{\sqrt{5}+1}{2}(\psi^4(3)+1)$	0
5	$\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$	$(\sqrt{5}-1)(\psi^8(5)+1)-2$	0
7	$\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$	$-(\sqrt{5}-1)\psi^6(7)+2$	0
11	–	–	–
13	$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$	$-(\sqrt{5}-1)\psi^2(13)$	0
17	$\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$	$-\frac{\sqrt{5}+1}{2}\psi^8(17)+1$	0
19	$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$	$(2\sqrt{5}-5)(\psi^4(19)+1)$	0
23	1	$-(\sqrt{5}+1)$	0
29	$\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$	$(\sqrt{5}-5)\psi^6(29)+2\sqrt{5}$	0
31	$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$	$2\psi^2(31)$	0
37	$\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$	$6\psi^6(37)-3(\sqrt{5}+1)$	0
41	$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$	$\frac{7-5\sqrt{5}}{2}(\psi^4(41)+1)$	0
43	1	$\frac{3}{2}(3\sqrt{5}+1)$	0
47	$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$	$(2\sqrt{5}-6)(\psi^4(47)+1)$	0

Since a modular form of given weight and level $\pmod{p}$ cannot start with a very high power of q (cf. [Gr 90], pag.499), note that in this case we have truly proved the congruences $B_n \equiv A_n\psi^2(n) \pmod{\mathfrak{P}_{11}}$ for all n.

4. Galois representations attached to Weil curves.–

We deduce Serre's conjecture (3.2.4?) for the remaining case of p-horizontal Weil curves from our conjecture (ss.?).

THEOREM. *Assume conjecture (ss.?) is true. Then Serre's conjecture (3.2.4?) is true for all irreducible representations*

$$\rho : \mathrm{G}_{\mathbb{Q}} \to \mathrm{Aut}\,(E_p)$$

provided that E is a p-horizontal Weil curve as above.

PROOF. Consider the twisted representation

$$\rho(\frac{p+1}{e}) : = \rho \otimes \chi^{-\frac{p+1}{e}},$$

where χ is the pth cyclotomic character. It is easy to see that if $\rho(\frac{p+1}{e})$ satisfies Serre's conjecture so does ρ.

The reason to consider $\rho(\frac{p+1}{e})$ instaed of ρ is that the invariant weight for $\rho(\frac{p+1}{e})$ is less than $p+1$. Namely (cf. [Ba-La 91]),

$$k_{\rho(\frac{p+1}{e})} = \begin{cases} \frac{p+1}{e}(e-2) & \text{if } \rho|_{D_p} \text{ reduces;} \\ p+1-2\frac{p+1}{e} & \text{if } \rho|_{D_p} \text{ is irreducible .} \end{cases}$$

Let $G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p+1}{e}})$ be the newform attached to E as in conjecture (ss.?). We have (cf. Lemma 4, [Ba-La 91]) that

$$\mathrm{Tr}\,(G\,E_{1,\psi}^{-2\frac{p+1}{e}+(p-1)}) \in S_{k_{\rho(\frac{p+1}{e})}}(N,1)$$

with $\mathrm{Tr}\,(G\,E_{1,\psi}^{-2\frac{p+1}{e}+(p-1)}) \equiv G \pmod{\mathfrak{P}}$ if $\rho|_{D_p}$ is reducible, and

$$\mathrm{Tr}\,(G\,E_{1,\psi}^{-2\frac{p+1}{e}+2(p-1)}) \in S_{k_{\rho(\frac{p+1}{e})}}(N,1)$$

with $\mathrm{Tr}\,(G\,E_{1,\psi}^{-2\frac{p+1}{e}+2(p-1)}) \equiv G \pmod{\mathfrak{P}}$ if $\rho|_{D_p}$ is irreducible, where Tr denotes the trace operator and $E_{1,\psi}$ denotes the Eisenstein series of weight one attached to ψ.

Now, we see that the twisted representation $\rho(\frac{p+1}{e})$ arises from a Hecke cusp form (mod p) of type $(N, k_{\rho(\frac{p+1}{e})}, 1)$. If necessary, since N is prime to p, the results in [Jo-Li 89] bring the level N to N_ρ.

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