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On the Representation of Large Integers by Integral Ternary Positive Definite Quadratic Forms

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A few years after the famous work of C. L. Siegel [17] on representation of integers by a genus of quadratic forms had appeared, Yu. V. Linnik [10] initiated a study of representation of integers by an individual ternary quadratic form. Due to the efforts of many authors (cf., for instance, [1], [3], [9], [11], [12], [15], [19], and references therein), we may now claim a success. Let $f(x) = \frac{1}{2} \sum_{1 \le i,j \le 3} a_{ij}x_ix_j$ be a positive definite quadratic form with integral rational coefficients, so that $a_{ij} = a_{ji}$, $a_{ij} \in \mathbb{Z}$, $2 \mid a_{ii}$ for $1 \le i,j \le 3$, and let $r_f(n) = \operatorname{card}\{u \mid u \in \mathbb{Z}^3, f(u) = n\}$ be the representation number of n by f; let $D = \det(a_{ij})$.

THEOREM 1. *Suppose that $n \in \mathbb{Z}$, $n \ge 1$ and $\gcd(n, 2D) = 1$. Then $r_f(n) = r(n, \operatorname{gen} f) + O(n^{\frac{1}{2}-\gamma})$ for $\gamma < \frac{1}{28}$, where $r(n, \operatorname{gen} f)$ denotes the number of representations of n by the genus of f averaged in accordance with Siegel's prescription* [17]. *Moreover, if n is primitively represented by f over the ring of p-adic integers for each rational prime p then $r(n, \operatorname{gen} f) \gg_{f,\varepsilon} n^{\frac{1}{2}-\varepsilon}$ for $\varepsilon > 0$.*

Proof. Let N be a positive integer such that $2D \mid N$ and $8 \mid N$, and let $\varphi \in S_0(\frac{3}{2}, N, \chi)$ with $\chi(d) = \left(\frac{2D}{d}\right)$, suppose furthermore that $\varphi \in \mathcal{U}^{\perp}$, in the notation of [15]. Thus φ is a 'good' cusp-form of weight $\frac{3}{2}$ (and character χ) that does not come from a θ-series. Therefore an argument due to H. Iwaniec [9] and W. Duke [3], supplemented by the considerations going back to G. Shimura [16] and B. A. Cipra [2], leads to an estimate for the

Fourier coefficients of φ (cf. also [7]), and on writing $\varphi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ we obtain: $a(n) \ll_{\varphi,\gamma} n^{\frac{1}{2}-\gamma}$ as soon as $(n, 2D) = 1$ and $\gamma < \frac{1}{28}$. By [15, Korollar 3], it follows then that $r_f(n) = r(n, \operatorname{spn} f) + O(n^{\frac{1}{2}-\gamma})$ for $(n, 2D) = 1$ and $\gamma < \frac{1}{28}$, where $r(n, \operatorname{spn} f)$ denotes the representation number of n averaged over the spinor genus containing f (cf. [15]). On the other hand, by [15, Korollar 2], if $(n, 2D) = 1$ then $r(n, \operatorname{spn} f) = r(n, \operatorname{gen} f)$. Finally, the estimate $r(n, \operatorname{gen} f) \gg n^{\frac{1}{2}-\varepsilon}$ for $\varepsilon > 0$ is a consequence of Siegel's work [17, 18] (cf. also [14, Satz (3.1)]), as soon as n is primitively representable by f over the p-adic integers. This completes the proof.

REMARK 1. The condition $(n, 2D) = 1$ in Theorem 1 has been used in the proof twice, to ensure the estimate $a(n) \ll n^{\frac{1}{2}-\gamma}$ and to deduce the identity $r(n, \operatorname{spn} f) = r(n, \operatorname{gen} f)$. The former use of this condition is due to the fact that $\varphi \in S(\frac{3}{2}, N, \chi)$ with $\chi = \left(\frac{2D}{d}\right)$ (see [13] for the details). It is an interesting question to what extent one can weaken the condition $(n, 2D) = 1$ in Theorem 1. The work of R. Schulze-Pillot [15] (cf. also [19] and references therein) is pertinent to this question.

THEOREM 2. *Let q be a rational prime congruent to* 5 *modulo* 8 *and let $f(x) = x_1^2 + x_2^2 + q^3 x_3^2$. Then $r_f(n) \gg_{q,\varepsilon} n^{\frac{1}{2}-\varepsilon}$ for $\varepsilon > 0$ and $n \equiv 7 \pmod 8$.*

Proof. Let $n = q^\ell n_1$, $q \mid n_1$ and suppose that $n \equiv 7 \pmod 8$. Consider the quadratic form $g(x) = x_1^2 + x_2^2 + q^m x_3^2$, where $m = 3 - \ell$ when $\ell \leq 3$ and $m = 0$ when $\ell \geq 3$; let $n_2 = nq^{m-3}$. Since $n_2 \equiv 3 \pmod 8$ if $\ell \geq 3$ and $n_2 \not\equiv 0 \pmod q$ when $\ell < 3$, it follows from Theorem 1 that $r_g(n_2) \gg n_2^{\frac{1}{2}-\varepsilon}$ for $\varepsilon > 0$. On writing $x_1^2 + x_2^2 = q^{3-m}(n_2 - q^m y_3^2)$ one notes that to each solution of the equations: $n_2 = g(y)$ with $y \in \mathbb{Z}^3$, $q^{3-m} = z_1^2 + z_2^2$ with $z_1 \in \mathbb{Z}$, $z_2 \in \mathbb{Z}$ there corresponds a unique solution of the equation $n = f(x)$ with $x \in \mathbb{Z}^3$. Since $q \equiv 1 \pmod 4$ it follows, in particular, that $r_f(n) \gg n^{\frac{1}{2}-\varepsilon}$ for $\varepsilon > 0$. This completes the proof.

REMARK 2. Theorem 2 confirms a conjecture of D. R. Heath-Brown [8, p. 137–138], that every large integer congruent to 7 modulo 8 is represented by the form $x_1^2 + x_2^2 + q^3 x_3^2$ when $q \equiv 5 \pmod 8$ and q is a rational prime.

DEFINITION. Let $n \in \mathbb{Z}$. We say that n is square-full if $n > 0$ and $p \mid n \Rightarrow p^2 \mid n$ for each rational prime p.

COROLLARY. *Every sufficiently large positive integer is a sum of at most three square-full numbers.*

Proof. By a classical theorem of Gauß, each positive integer n is either a sum of three squares or it is of the shape $n = 4^{\ell}(8k+7)$ with $\ell \in \mathbb{Z}$, $k \in \mathbb{Z}$. In the latter case, however, Theorem 2 shows that the integer n is represented, for instance, by the form $x_1^2 + x_2^2 + 125x_3^2$ if k is sufficiently large. Other possibilities are also easily eliminated since the form $x^2 + y^2 + 2z^2$ is easily seen to represent n as soon as $n \equiv 4 \pmod 8$, cf. [8, p. 137]. This completes the proof.

REMARK 3. This corollary has been first proved by D. R. Heath-Brown [8], by a different method; according to [8, p. 137], it answers a question posed by P. Erdős and A. Ivić.

REMARK 4. This note contains the text of my lecture at the 16$^{\text{th}}$ Journées Arithmétiques (Marseilles, July 1989). Since then a new important paper by W. Duke and R. Schulze-Pillot [5] has appeared, which allows, in particular, to weaken the condition $(n, 2D) = 1$ in Theorem 1 of this note (cf. also Remark 1). Unfortunately, the authors suppress the details of the proof of their crucial Lemma 2 [5, p. 50–51]; following [7], where incidentally the proof of the corresponding assertion is also omitted, we are content with a weaker statement [13, p. 17–19] that leads to the results described above. Finally, we cite here two articles [4], [6], throwing further light on our subject.

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References

[1] Cassels, J.W.S.: Rationale quadratische Formen. Jahresbericht der Deutschen Mathematiker-Vereinigung, 82 (1980), 81–93.

[2] Cipra, B.A.: On the Niwa-Shintani theta-kernel lifting of modular forms. Nagoya Mathematical Journal, 91 (1983), 49–117.

[3] Duke, W.: Hyperbolic distribution problems and half-integral weight Maass forms. Inventiones Mathematicae, 92 (1988), 73–90.

[4] Duke, W.: Lattice points on ellipsoids. Séminaire de Théorie des Nombres de Bordeaux, le 20 mai 1988, Année 1987–88, Exposé no. 37.

[5] Duke, W. & Schulze-Pillot, R.: Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids. Inventiones Mathematicae, 99 (1990), 49–57.

[6] Fomenko, O.M.: Estimates of the norms of cusp-forms and arithmetic applications. Zapiski LOMI, 168 (1988), 158–179.

[7] Fomenko, O.M. & Golubeva, E.P.: Asymptotic distribution of integral points on a two-dimensional sphere. Zapiski LOMI, 160 (1987), 54–71.
[8] Heath-Brown, D.R.: Ternary quadratic forms and sums of three square-full numbers; in: Séminaire de Théorie des Nombres, Paris 1986/87 (edited by C. Goldstein), Birkhäuser, 1988; pp. 137–163.
[9] Iwaniec, H.: Fourier coefficients of modular forms of half-integral weight. Inventiones Mathematicae, 87 (1987), 385–401.
[10] Linnik, Yu. V.: On the representation of large integers by positive definite quadratic forms. Izvestiya Akademii Nauk SSSR (ser. mat.), 4 (1940), 363–402.
[11] Linnik, Yu. V.: Ergodic properties of algebraic fields. Springer-Verlag, 1968.
[12] Malyshev, A.V.: Yu. V. Linnik's ergodic method in number theory. Acta Arithmetica, 27 (1975), 555–598.
[13] Moroz, B. Z.: Recent progress in analytic arithmetic of positive definite quadratic forms. Max-Planck-Institut für Mathematik, preprint, 1989.
[14] Peters, M.: Darstellungen durch definite ternäre quadratische Formen. Acta Arithmetica, 34 (1977), 57–80.
[15] Schulze-Pillot, R.: Thetareihen positiv definiter quadratischer Formen. Inventiones Mathematicae, 75 (1984), 283–299.
[16] Shimura, G.: On modular forms of half-integral weight. Annals of Mathematics, 97 (1973), 440–481.
[17] Siegel, C.L., Über die analytische Theorie der quadratischen Formen; in: Gesammelte Abhandlungen, Bd. I. Springer-Verlag, 1966; 326–405.
[18] Siegel, C.L., Über die Klassenzahl quadratischer Zahlkörper; in: Gesammelte Abhandlungen, Bd. I. Springer-Verlag, 1966; 406–409.
[19] Teterin, Yu. G.: Representations of integers by spinor genera of translated lattices. Zapiski LOMI, 151 (1986), 135–140.

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