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Some Developments and Applications of the Abstract Mourre Theory

Anne Boutet de Monvel-Berthier and Vladimir Georgescu [1]

1. Introduction

In 1979 Eric Mourre introduced the concept of locally conjugate operator and invented a very efficient method of proving the limiting absorption principle (L.A.P.). His ideas opened the way to a complete solution of the N-body problem: detailed spectral properties have been obtained by Perry, Sigal and Simon and asymptotic completeness has been proved by Sigal and Soffer. The abstract side of Mourre theory has been further developped by Perry, Sigal and Simon [PSS] (they eliminated an assumption on the first commutator which was annoying in applications) and by Mourre [M] and Jensen and Perry [JP] (the L.A.P was established in better spaces).

In [ABG] efforts were made in order to avoid the use of the second commutator of the hamiltonian with the conjugate operator. Optimal, in some sense, results in this direction were obtained in [BGM2] and [BG1]. In [BGM2] the space $\mathscr{G}$ which appears below is the domain of the hamiltonian and the main theorem is easy to apply in the N-body case with short-range and long-range interactions of a very general nature. In [BG1,2] the space $\mathscr{G}$ is the form-domain of the hamiltonian (the domain is not assumed invariant under the group generated by the conjugate operator, this being compensated by a stronger condition on the first commutator) and the theory is applied to pseudo-differential operators. In both cases, the L.A.P. is established in "optimal" (in some sense) spaces, which allows one to get without any further effort very good criteria for the existence and completeness of relative, local wave operators.

The main part of this article is devoted to an exposition of several applications of a version of the locally conjugate operator method which we developed in [BG1,2]. In fact, theorems 3.1 and 3.2 below are the main results got in [BG1] and in sections 4 and 5 we show their force and also fineness. In the preliminary section 2 we introduce and discuss the most important notion we have isolated, that of operator of class $\mathscr{C}^1$ with respect to a unitary group. This is a quite general property and in section 5 we show in some simple cases that it is almost impossible to be replaced by a weaker one without loosing the strong form of the L.A.P. given

[1] Lecture delivered by A. Boutet de Monvel-Berthier

in theorem 3.1. Moreover, in section 5 we show how to deal with hamiltonians with very singular interactions (this part will be treated more thoroughly in a later publication). But section 4 contains the most important results. Although their formulation is abstract, it is trivial to apply them to many-body hamiltonians. After the Nantes conference, as A. Soffer raised the problem of the spectral analysis of hard-core N-body hamiltonians, we decided to formulate, in this paper, several consequences of theorem 3.1 such as to cover non-densely defined hamiltonians (in fact we use pseudo-resolvents in place of resolvents). The particular case of hard-core N-body hamiltonians is the subject of a in-preparation-joint-paper with A. Soffer. Finally, an appendix contains a technical estimate related to Littlewood-Paley theory which seemes to us quite powerful in various situations.

2. Unitary Groups in Friedrichs Couples

In our approach, the natural framework for the "locally conjugate operator method" is a triplet $(\mathscr{G},\mathscr{H};W)$ consisting of two Hilbert spaces $\mathscr{G},\mathscr{H}$ such that $\mathscr{G}\subset\mathscr{H}$ continuously and densely, and a strongly continuous unitary one-parameter group $W=\{W_\alpha\}_{\alpha\in\mathbb{R}}$ in $\mathscr{H}$ which leaves $\mathscr{G}$ invariant: $W_\alpha\mathscr{G}\subset\mathscr{G}$ for all $\alpha\in\mathbb{R}$ .The Hilbert spaces are always complex but not necessarily separable. In our applications, $\mathscr{G}$ will be either the domain of the hamiltonian, or its form domain, or it will be just $\mathscr{H}$ (although, in this last case, the hamiltonian could be unbounded and even non-densely defined).

A triplet $(\mathscr{G},\mathscr{H};W)$ with the preceding properties will be called a *unitary group in a Friedrichs couple* , the pair of spaces $(\mathscr{G},\mathscr{H})$ being called a *Friedrichs couple.* In this section we shall fix such a system $(\mathscr{G},\mathscr{H};W)$ and we shall study some notions related to it.

Let $\mathscr{G}^*$ be the adjoint (or antidual) space of $\mathscr{G}$; identify $\mathscr{H}^*=\mathscr{H}$ by using Riesz lemma and embed as usual $\mathscr{G}\subset\mathscr{H}\subset\mathscr{G}^*$. Then define $\mathscr{G}^s=[\mathscr{G},\mathscr{G}^*]_{(1-s)/2}$ by complex interpolation for $-1\leq s\leq 1$, so that $\mathscr{G}^1=\mathscr{G}$, $\mathscr{G}^0=\mathscr{H}$ and $\mathscr{G}^{-1}=\mathscr{G}^*$. Observe that we have canonical identifications $(\mathscr{G}^s)^*=\mathscr{G}^{-s}$. We shall denote $\mathscr{X}=B(\mathscr{G},\mathscr{G}^*)$ the Banach space of continuous linear operators from $\mathscr{G}$ to $\mathscr{G}^*$ and $\|\cdot\|_{\mathscr{X}}$ its norm; observe that $\mathscr{X}$ is equipped with an isometric involution $T\mapsto T^*$. For each $s,t\in[-1,+1]$ we have canonical embeddings $B(\mathscr{G}^s,\mathscr{G}^t)\subset\mathscr{X}$. Then the norm in $\mathscr{G}^s$, resp. in $B(\mathscr{G}^s,\mathscr{G}^t)$, will be denoted $\|\cdot\|_s$, resp. $\|\cdot\|_{s,t}$, and we abbreviate $\|\cdot\|_0=\|\cdot\|$, $\|\cdot\|_{0,0}=\|\cdot\|$.

The following fact will be often used below:

LEMMA 2.1: *Let* E,F *be Hilbert spaces such that* $E\subset F$ *continuously and let* $W_\alpha(\alpha)=e^{iA\alpha}$, $\alpha\in\mathbb{R}$, *be a* C_0*-group in* F *which leaves* E *invariant:* $W_\alpha E\subset E$

$(\forall \alpha \in \mathbb{R})$. *Denote* $W^{\mathbf{E}}_{\alpha} = W_{\alpha}|_{\mathbf{E}}$ *considered as operator in* $\mathbf{E}$. *Then* $\{W^{\mathbf{E}}_{\alpha}\}_{\alpha \in \mathbb{R}}$ *is a* C_0-*group in* $\mathbf{E}$ *and its infinitesimal generator is the closed, densely defined operator* $A^{\mathbf{E}}$ *in* $\mathbf{E}$ *defined as the restriction of* A *to* $D(A^{\mathbf{E}}) = \{u \in D(A) \cap \mathbf{E} \mid Au \in \mathbf{E}\}$.

Proof: The lemma has been proved in [ABG] under the assumption that $\mathbf{E},\mathbf{F}$ are separable. We shall reduce ourselves to this case. The only problem is to prove the continuity of $\alpha \mapsto W_{\alpha}u \in \mathbf{E}$ when $u \in \mathbf{E}$. Let $\mathbf{E}_0$ (resp. $\mathbf{F}_0$) be the closed subspace of $\mathbf{E}$ (resp.$\mathbf{F}$) generated by $\{W_{\alpha}u \mid \alpha \in \mathbb{R}\}$. Then $\mathbf{E}_0 \subset \mathbf{F}_0$ continuously and densely, W leaves $\mathbf{E}_0$ and $\mathbf{F}_0$ invariant and it is strongly continuous in $\mathbf{F}_0$. Moreover, $\mathbf{F}_0$ is separable because $\alpha \mapsto W_{\alpha}u \in \mathbf{F}_0$ is continuous and its image is a total subset of $\mathbf{F}_0$. Since $\mathbf{F}_0^* \subset \mathbf{E}_0^*$ continuously and densely, we see that $\mathbf{E}_0^*$ is separable, hence $\mathbf{E}_0$ is separable too. Now we may apply lemmas 1.1.3 and 1.1.4 from [ABG1] to $(\mathbf{E}_0,\mathbf{F}_0;W|_{\mathbf{F}_0})$. ■

Let us apply this lemma in the case of the unitary group W in the Friedrichs couple $(\mathcal{G},\mathcal{H})$. Denote A the self-adjoint operator in $\mathcal{H}$ such that $W_{\alpha} = e^{iA\alpha}$. The notations $W^{\mathcal{G}}_{\alpha}$, $A^{\mathcal{G}}$ have the same signification as in the preceding lemma. Now let $W^{\mathcal{G}^*}_{\alpha} = (W^{\mathcal{G}}_{-\alpha})^* \in B(\mathcal{G}^*)$. Since for a group weak and strong continuity are equivalent, $\{W^{\mathcal{G}^*}_{\alpha}\}_{\alpha \in \mathbb{R}}$ will be a C_0-group in $\mathcal{G}^*$; we denote $A^{\mathcal{G}^*}$ its generator (closed, densely defined operator in $\mathcal{G}^*$ such that $W^{\mathcal{G}^*}_{\alpha} = \exp(i\alpha A^{\mathcal{G}^*})$).

It is easily shown that $W^{\mathcal{G}^*}_{\alpha}|_{\mathcal{H}} = W_{\alpha}$ and an application of lemma 2.1 shows that A is just the restriction of $A^{\mathcal{G}^*}$ to $\{u \in D(A^{\mathcal{G}^*}) \cap \mathcal{H} \mid A^{\mathcal{G}^*}u \in \mathcal{H}\}$. Interpolating between $\mathcal{G}$ and $\mathcal{G}^*$, we see that $W^{\mathcal{G}^*}$ induces a C_0-group $W^{\mathcal{G}^s}$ in each $\mathcal{G}^s$, the infinitesimal generators of these groups being the natural restrictions of $A^{\mathcal{G}^*}$. It will be obvious in later arguments that no confusion arises if we drop the index which indicates the space in which the operators are considered. We summarize these facts in:

PROPOSITION 2.2: *Let* $(\mathcal{G},\mathcal{H};W)$ *be a unitary group in a Friedrichs couple. Then , for each* $\alpha \in \mathbb{R}$, *the operator* W_{α} *in* $\mathcal{H}$ *is continuous when* $\mathcal{H}$ *is equipped with the topology induced by* $\mathcal{G}^*$ *and, if we denote again by* W_{α} *its unique extension to a continuous operator on* $\mathcal{G}^*$, *the application* $\alpha \mapsto W_{\alpha} \in B(\mathcal{G}^*)$ *is a* C_0-*group in* $\mathcal{G}^*$ *which leaves invariant and induces a* C_0-*group in each space* $\mathcal{G}^s$. *Let* A *be the infinitesimal generator of the group* W *in* $\mathcal{G}^*$, i.e. A *is the unique closed, densely defined operator in* $\mathcal{G}^*$ *such that* $W_{\alpha} = e^{iA\alpha}$; *denote* $D(A;\mathcal{G}^*)$ *its domain. Then for each* $s \in [-1,+1]$, *the restriction of* A *to*

(2.1) $\quad D(A;\mathcal{G}^s) = \{u\in\mathcal{G}^s \mid u\in D(A;\mathcal{G}^*)$ *and* $Au\in\mathcal{G}^s\}$

is a closed, densely defined operator in $\mathcal{G}^s$ *which is just the infinitesimal generator of the* C_0*-group* $W_\alpha|_{\mathcal{G}^s}$.

We shall always consider $D(A;\mathcal{G}^s)$ as a Hilbert space, the norm being the graph norm associated to A in $\mathcal{G}^s$: $\|u\|_s^A = [\ \|u\|_s^2 + \|Au\|_s^2\,]^{1/2}$. It follows from a well-known lemma of Nelson (see theorem 1.9 in [D]) that $D(A;\mathcal{G})\subset D(A;\mathcal{G}^s)\subset\mathcal{G}^s$ continuously and *densely* for all $s\in[-1,+1]$. Moreover, the operator A with domain $D(A;\mathcal{H})$ is self-adjoint in $\mathcal{H}$.

Finally, let us remark that the equality $W_\alpha^* = W_{-\alpha}$ has to be interpreted in the following sense: if $-1\le s\le 1$, then the adjoint of the operator $W_\alpha|_{\mathcal{G}^s}\in B(\mathcal{G}^s)$ is equal to $W_{-\alpha}|_{\mathcal{G}^{-s}}\in B(\mathcal{G}^{-s})$, the identification $(\mathcal{G}^s)^* = \mathcal{G}^{-s}$ being assumed.

Let us consider now the group of automorphisms of the Banach space $\mathcal{X}=B(\mathcal{G},\mathcal{G}^*)$ induced by W, namely $\mathcal{W}_\alpha(T)=W_\alpha T W_\alpha^*$ for $T\in\mathcal{X}$. Observe that $\alpha\mapsto\mathcal{W}_\alpha(T)\in\mathcal{X}$ is continuous only when $\mathcal{X}$ is equipped with the strong operator topology, hence $\{\mathcal{W}_\alpha\}_{\alpha\in\mathbb{R}}$ is not a C_0-group on $\mathcal{X}$. However, one has $\mathcal{W}_\alpha=e^{i\mathcal{A}\alpha}$, with $\mathcal{A}(T)=[A,T]$, in a sense which we shall explain below.

DEFINITION 2.3: *Let* $0<\theta\le 1$. *We shall say that an operator* $T\in B(\mathcal{G},\mathcal{G}^*)$ *is of class* $C^\theta(A;\mathcal{G},\mathcal{G}^*)$, *and we shall write* $T\in C^\theta(A;\mathcal{G},\mathcal{G}^*)$, *if the function* $\alpha\mapsto\mathcal{W}_\alpha(T)\in\mathcal{X}$ *is Hölder continuous of order* θ, *i.e. there is* $c<\infty$ *such that* $\|W_\varepsilon T W_\varepsilon^* - T\|_{\mathcal{X}}\le c|\varepsilon|^\theta$ *for* $|\varepsilon|\le 1$. *For* $\theta=+0$ *we replace Hölder-continuity by Dini-continuity, more precisely we write* $T\in C^{+0}(A;\mathcal{G},\mathcal{G}^*)$ *if* $\int_0^1 \|W_\varepsilon T W_\varepsilon^* - T\|_{\mathcal{X}}\,\varepsilon^{-1}d\varepsilon<\infty$.

Remark that we could replace here $W_\varepsilon T W_\varepsilon^* - T$ by the commutator $[T,W_\varepsilon]=TW_\varepsilon - W_\varepsilon T=(W_\varepsilon T W_\varepsilon^* - T)W_\varepsilon$. One can refine the notion and define $T\in C^\theta(A;\mathcal{G}^s,\mathcal{G}^t)$ for some $-1\le s,t\le 1$ by replacing the norm $\|\cdot\|_{\mathcal{X}}$ with the norm $\|\cdot\|_{s,t}$.

If $T:\mathcal{G}\to\mathcal{G}^*$ is a linear continuous operator, we shall denote $[A,T]\equiv-[T,A]$ the continuous sesquilinear form on $D(A;\mathcal{G})$ defined by the formula $\langle u|[A,T]v\rangle=\langle Au|Tv\rangle-\langle u|TAv\rangle$. Taking into account that W is a C_0-group in $\mathcal{G}$

and that $\alpha \mapsto W_\alpha u \in \mathcal{G}$ is strongly differentiable for each $u \in D(A;\mathcal{G})$ it is trivial to see that

$$(2.2) \qquad W_\alpha T W_\alpha^* - T = i \int_0^\alpha W_\tau [A,T] W_\tau^* \, d\tau$$

as sesquilinear forms on $D(A;\mathcal{G})$. In particular, denoting $A_\alpha = (i\alpha)^{-1}(W_\alpha - 1)$ for $\alpha \neq 0$, we get

$$(2.3) \qquad [A_\alpha, T] = \alpha^{-1} \int_0^\alpha W_\tau [A,T] W_{\alpha-\tau} \, d\tau.$$

as forms on $D(A;\mathcal{G})$. In the next lemma we shall summarize some easy consequences of these formulas.

LEMMA 2.4: *An operator* $T \in B(\mathcal{G}, \mathcal{G}^*)$ *is of class* $C^1(A;\mathcal{G},\mathcal{G}^*)$ *if and only if one of the following equivalent properties is fulfilled*:

(a) $\liminf_{\varepsilon \to +0} \|[A_\varepsilon, T]\|_{\mathcal{X}} < \infty$;

(b) *the function* $\alpha \mapsto W_\alpha T W_\alpha^* \in B(\mathcal{G}, \mathcal{G}^*)$ *is weakly derivable at* $\alpha = 0$;

(c) *the preceding function is strongly continuously derivable* ;

(d) *the sesquilinear form* $[A,T]$ *is continuous for the topology induced by* $\mathcal{G}$ *on* $D(A;\mathcal{G})$;

(e) $\lim_{\varepsilon \to 0} [A_\varepsilon, T]$ *exists weakly in* $B(\mathcal{G}, \mathcal{G}^*)$;

(f) $\lim_{\mu \to +0} \int_\mu^1 (W_{2\varepsilon} T W_{2\varepsilon}^* - 2 W_\varepsilon T W_\varepsilon^* + T) \, \varepsilon^{-2} d\varepsilon$ *exists weakly* (*hence also strongly*) *in* $B(\mathcal{G}, \mathcal{G}^*)$.

Under these conditions, if we denote by the same symbol $[A,T]$ *the continuous sesquilinear form on* $\mathcal{G}$ *which extends the form* $[A,T]$ *given on* $D(A;\mathcal{G})$ *and the continuous operator* $\mathcal{G} \to \mathcal{G}^*$ *associated to it, then:*

$$(2.4) \qquad [A,T] = -i \frac{d}{d\alpha} W_\alpha T W_\alpha^* \big|_{\alpha=0} = \lim_{\varepsilon \to 0} [A_\varepsilon, T] \, ,$$

the derivative and the limit being taken in the strong operator topology of $B(\mathcal{G}, \mathcal{G}^*)$. *Moreover, we shall have* $[A,T] \in B(\mathcal{G}^s, \mathcal{G}^t)$ *for some* $-1 \le s,t \le 1$, *if and only if* $T \in C^1(A;\mathcal{G}^s,\mathcal{G}^t)$ *and in this case* (2.2) *will hold strongly in* $B(\mathcal{G}^s, \mathcal{G}^t)$.

Proof: (2.2) and (2.3) show that $\varepsilon^{-1}(W_\varepsilon T W_\varepsilon^* - T) \to [iA,T]$ and $[A_\varepsilon, T] \to [A,T]$ weakly as forms on $D(A;\mathcal{G})$ (W is strongly continuous on $D(A;\mathcal{G})$ also). So (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (d) $\Leftrightarrow$ (c) (use (2.2) again). From (a) and the compacity of closed balls of $\mathcal{X}$ in the weak operator topology, we see that $\varepsilon_j^{-1}(W_{\varepsilon_j} T W_{\varepsilon_j}^* - T)$ is weakly convergent in $B(\mathcal{G}, \mathcal{G}^*)$ for some sequence $\varepsilon_j \to 0$, so we get (d) again. It remains to show that (f) is equivalent with the other assertions (see [BB] for the technique which we shall use). Let $J_\tau:\mathcal{X}\to\mathcal{X}$ be defined by $J_\tau=\tau^{-1}\int_0^\tau \mathcal{W}_\alpha d\alpha$ and let $S_\mu=(\ell n2)^{-1}\int_\mu^{2\mu}(\mathcal{W}_\varepsilon(T)-T)\varepsilon^{-2}d\varepsilon$. A simple calculation gives:

$$(2.5)\qquad J_\tau(S_\mu)=(\ell n2)^{-1}\int_\mu^{2\mu} J_\alpha[\tau^{-1}(\mathcal{W}_\tau(T)-T)]\,\alpha^{-1}d\alpha.$$

If (b) is fulfilled, taking into account that $\lim_{\tau\to 0} J_\tau=1$ in the strong operator topology of $\mathcal{X}=B(\mathcal{G},\mathcal{G}^*)$, we get $S_\mu=(\ell n2)^{-1}\int_\mu^{2\mu} J_\alpha(i[A,T])\alpha^{-1}d\alpha$ which easily implies that $\lim_{\mu\to 0}S_\mu=[iA,T]$ strongly. Now observe that:

$$(2.6)\qquad 2\ell n2\ S_\mu=2\int_\mu^1(\mathcal{W}_\varepsilon(T)-T)\,\varepsilon^{-2}d\varepsilon-2\int_{2\mu}^1(\mathcal{W}_\varepsilon(T)-T)\,\varepsilon^{-2}d\varepsilon=$$
$$=2\int_{1/2}^1(\mathcal{W}_\varepsilon(T)-T)\varepsilon^{-2}d\varepsilon-\int_\mu^{1/2}(\mathcal{W}_{2\varepsilon}(T)-2\mathcal{W}_\varepsilon(T)+T)\,\varepsilon^{-2}d\varepsilon\ ,$$

hence the limit in (f) exists strongly. Reciprocally, assume (f). Then (2.6) shows that $\lim_{\mu\to 0} S_\mu\equiv S$ exists weakly. But (2.5) implies (with no assumption on T) that $\lim_{\mu\to 0} J_\tau(S_\mu)=\tau^{-1}(\mathcal{W}_\tau(T)-T)$ strongly. So we get $\tau^{-1}(\mathcal{W}_\tau(T)-T)=J_\tau(S)\to S$ strongly as $\tau\to 0$, in particular (b) is fulfilled. ■

COROLLARY 2.5 (Virial theorem): *If* $T:\mathcal{G}\to\mathcal{G}^*$ *is symmetric and of class* $C^1(A;\mathcal{G},\mathcal{G}^*)$, *and if* $u,v\in\mathcal{G}$ *are such that* $Tu=\lambda u$, $Tv=\lambda v$ *for some* $\lambda\in\mathbb{R}$, *then* $\langle u|[A,T]v\rangle=0$.

Proof: Using the second equality in (2.5) we have:

$$\langle u|[A,T]v\rangle=\lim_{\varepsilon\to 0}\langle u|[A_\varepsilon,T]v\rangle=\lim_{\varepsilon\to 0}(\langle u|A_\varepsilon Tv\rangle-\langle Tu|A_\varepsilon v\rangle)=0.\ ■$$

In order to arrive at deeper aspects of Mourre theory (namely a precise form of the limiting absorption principle) the C^1 regularity property is not enough. One can introduce a stronger notion, namely to ask that $\alpha\mapsto W_\alpha TW_\alpha^*\in B(\mathcal{G},\mathcal{G}^*)$ be *norm* derivable at $\alpha=0$; *we then say that* T *is of class* $C_u^1(A;\mathcal{G},\mathcal{G}^*)$ (i.e. it is of class C^1 in the uniform topology). This is equivalent with asking, besides $T\in C^1(A;\mathcal{G},\mathcal{G}^*)$, that $\alpha\mapsto W_\alpha[A,T]W_\alpha^*$ be norm-continuous. Unfortunately, even this assumption is not strong enough, as our example from section 5 shows. However, the sufficient assumption we have been able to isolate, is only slightly stronger than this one. In fact, the proof of lemma 2.4 shows that $T\in C_u^1(A;\mathcal{G},\mathcal{G}^*)$ *if and only if the limit in* (f) *exists in norm.* Our condition is the following:

DEFINITION 2.6: *An operator* $T\in B(\mathcal{G},\mathcal{G}^*)$ *is said to be of class* $\mathcal{C}^1(A;\mathcal{G},\mathcal{G}^*)$ *if:*

$$(2.7)\qquad \int_0^1\|W_{2\varepsilon}TW_{2\varepsilon}^*-2W_\varepsilon TW_\varepsilon^*+T\|_{\mathcal{X}}\ \varepsilon^{-2}d\varepsilon<\infty.$$

It is clear that the expression under the norm above may be replaced by the more symmetrical $W_\varepsilon TW_\varepsilon^*+W_{-\varepsilon}TW_{-\varepsilon}^*-2T$ or by $[W_\varepsilon,[W_\varepsilon,T]]$. In fact $W_{2\varepsilon}TW_{2\varepsilon}^*-2W_\varepsilon TW_\varepsilon^*+T=[W_\varepsilon,[W_\varepsilon,T]]W_{2\varepsilon}^*$. Using the notation $A_\varepsilon=(i\varepsilon)^{-1}(W_\varepsilon-1)$ introduced above, (2.7) can be expressed in the equivalent from

(2.8) $\int_0^1 \|[A_\varepsilon,[A_\varepsilon,T]]\|_{\mathscr{X}}\, d\varepsilon < \infty.$

The remark we made just before the definition implies $\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)\subset C_u^1(A;\mathscr{G},\mathscr{G}^*)$. In order to compare the assumption $T\in\mathscr{C}^1$ with other assumptions made in the development of Mourre theory, it is useful to introduce the classes $C^s(A;\mathscr{G},\mathscr{G}^*)$ for $1<s\leq 2$ or $s=1+0$.

DEFINITION 2.7: *Let* $s\in]1,2]$ *or* $s=1+0$; *denote* $\theta=s-1$ *in the first case and* $\theta=+0$ *in the second one. We shall say that* $T\in B(\mathscr{G},\mathscr{G}^*)$ *is of class* $C^s(A;\mathscr{G},\mathscr{G}^*)$ *if* $T\in C^1(A;\mathscr{G},\mathscr{G}^*)$ *and* $[A,T]\in C^\theta(A;\mathscr{G},\mathscr{G}^*)$.

So $T\in C^{1+0}(A;\mathscr{G},\mathscr{G}^*)$ means that $\alpha\mapsto W_\alpha TW_\alpha^*\in B(\mathscr{G},\mathscr{G}^*)$ is derivable and its derivative is a Dini-continuous function. We have for $0<\theta\leq 1$:

(2.9) $\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)\supset C^{1+0}(A;\mathscr{G},\mathscr{G}^*)\supset C^{1+\theta}(A;\mathscr{G},\mathscr{G}^*).$

Only the first inclusion is not completely trivial, but it follows easily from:

$$W_\varepsilon TW_\varepsilon^*+W_{-\varepsilon}TW_{-\varepsilon}^*-2T = i\int_0^\varepsilon \{W_\tau[A,T]W_\tau^*-W_{-\tau}[A,T]W_{-\tau}^*\}\, d\tau.$$

By lemma 2.4, $T\in C^2(A;\mathscr{G},\mathscr{G}^*)$ means that $[A,T]$ and $[A,[A,T]]$ belong to $B(\mathscr{G},\mathscr{G}^*)$; this is, essentially, the situation considered by Mourre and Perry, Sigal and Simon. The case $0<\theta<1$ was studied in [ABG] while the class $\mathscr{C}^1$ is implicit in the definition of "admissibility" given in section 4 of [BGM].

We shall not explain here how the assumption $T\in\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)$ is verified in applications. In fact this is quite easy if one uses the technique presented in [BG2] together with the estimate proved in the appendix at the end of this paper (see [BG2] for examples).

3. The Limiting Absorption Principle

In this section we shall summarize the results of our Note [BG1]. Let $(\mathscr{G},\mathscr{H};W)$ be a unitary group in a Friedrichs couple and H a self-adjoint operator in $\mathscr{H}$ with $\mathscr{G}$ as form-domain (i.e. $\mathscr{G}=D(|H|^{1/2})$ algebraically; by closed graph theorem the equality will hold on a topological level too). Then H extends to a continuous symmetric operator (denoted by the same symbol) $H:\mathcal{G}\to\mathcal{G}^*$ and, if E is the spectral measure of H, then $E(J)\in B(\mathcal{G})\cap B(\mathcal{G}^*)$ for any Borel set $J\subset\mathbb{R}$.

DEFINITION 3.1 : *We shall say that* A *is conjugate to* H *on an open subset* $J\subset\mathbb{R}$ *(in form sense) if* $H\in C^1(A;\mathcal{G},\mathcal{G}^*)$ *and there is a strictly positive number* a *and a compact operator* $K:\mathcal{G}\to\mathcal{G}^*$ *such that* $E(J)[iH,A]E(J)\geq aE(J)+K$ *(as operators* $\mathcal{G}\to\mathcal{G}^*$*). If* $K=0$*, we say that* A *is strictly conjugate to* H *on* J*. If* $\lambda\in\mathbb{R}$ *and* A *is (strictly) conjugate to* H *on a neighbourhood of* λ*, we say that* A *is (strictly) conjugate to* H *at* λ*. If* A *is (strictly) conjugate to* H *at all points of an open set* J*, then we say that* A *is locally (strictly) conjugate to* H *on* J.

Using the virial theorem (corollary 2.5) it is a trivial matter to show that, under the conditions of the first part of the preceding definition, H *has in* J *a finite number of eigenvalues (counting multiplicities)*. We shall denote J_0 the set of $\lambda\in J$ such that λ is not an eigenvalue of H. Then we put $\mathbb{C}^{\pm}=\{z\in\mathbb{C}\mid \pm\mathrm{Im}\, z>0\}$. Clearly $\mathbb{C}^{\pm}\ni z\mapsto (z-H)^{-1}\in B(\mathcal{G}^*,\mathcal{G})$ as a holomorphic function. In order to control its boundary values on J_0, we shall need the following space:

$$(3.1)\qquad \mathcal{E} = (\mathcal{G}^*,D(A;\mathcal{G}^*))_{1/2,1}\ .$$

Here $(\cdot,\cdot)_{\theta,p}$ is the real interpolation functor which makes sense if $0<\theta<1$ and $1\leq p\leq\infty$. Hence $\mathcal{E}$ is a Banach space such that $D(A;\mathcal{G}^*)\subset\mathcal{E}\subset\mathcal{G}^*$ continuously and densely. Taking adjoints we get $\mathcal{G}\subset\mathcal{E}^*$ continuously but not densely in general, because $\mathcal{E}$ could be non-reflexive. We shall denote $\mathring{\mathcal{E}}^*$ the closure of $\mathcal{G}$ in $\mathcal{E}^*$; it is known that $(\mathring{\mathcal{E}}^*)^*=\mathcal{E}$. Observe that we have a natural continuous embedding $B(\mathcal{G}^*,\mathcal{G})\subset B(\mathcal{E},\mathcal{E}^*)$, in particular we may consider the holomorphic function $\mathbb{C}^{\pm}\ni z\mapsto (z-H)^{-1}\in B(\mathcal{E},\mathcal{E}^*)$.

THEOREM 3.1: *Assume that* $H\in\mathcal{C}^1(A;\mathcal{G},\mathcal{G}^*)$ *and that* A *is conjugate to* H *on the open subset* $J\subset\mathbb{R}$*. Then the function* $\mathbb{C}^{\pm}\ni z\mapsto (z-H)^{-1}\in B(\mathcal{E},\mathcal{E}^*)$ *extends as a weak*-continuous function on* $\mathbb{C}^{\pm}\cup J_0$*. In particular,* H *has no singularly continuous spectrum in* J *and the function* $J_0\ni\lambda\mapsto(\lambda\pm io-H)^{-1}\in B(\mathcal{E},\mathcal{E}^*)$ *is well defined and weak*-continuous.*

THEOREM 3.2: *Let* $(\mathcal{G}_j,\mathcal{H};W_j)$, $j=1,2$, *be two unitary groups in Friedrichs couples with the same Hilbert space* $\mathcal{H}$*. Let* H_j *be a self-adjoint operator in* $\mathcal{H}$ *with* $\mathcal{G}_j$ *as form-domain and such that* $H_j\in\mathcal{C}^1(A_j;\mathcal{G}_j,\mathcal{G}_j^*)$*. Assume that* A_j *is conjugate to* H_j *on an open subset* $J\subset\mathbb{R}$ *(independent of* j*). Let* $\mathcal{E}_j=(\mathcal{G}_j^*,D(A_j;\mathcal{G}_j^*))_{1/2,1}$ *and assume that* *there is a continuous operator* $V:\overset{\circ}{\mathscr{E}}{}_1^*\to\mathscr{E}_2$ *such that* $H_2=H_1+V$ *as forms on* $D(H_1)\times D(H_2)$. *Finally, denote* E_j^c *the continuous component of the spectral measure of* E_j. *Then the following relative wave operators exist (hence are complete) :*

$$(3.2)\qquad W_1^{\pm}=\text{s-lim}_{t\to\pm\infty}e^{iH_2t}e^{-iH_1t}E_1^c(J);\qquad W_2^{\pm}=\text{s-lim}_{t\to\pm\infty}e^{iH_1t}e^{-iH_2t}E_2^c(J).$$

4. Pseudo-resolvents with a Spectral Gap

The theorems 3.1 and 3.2, as we stated them, do not seem to give optimal results for N-body Schrödinger hamiltonians. In fact, in this case $\mathscr{H}=L^2(\mathbb{R}^n)$ and one tries to take as conjugate operator the generator of dilations $A=\frac{1}{2}(PQ+QP)$, where $P=-i\nabla$ is the momentum and Q is the position observable (multiplication by $x\in\mathbb{R}^n$). The hamiltonian has the form $H=\frac{1}{2}P^2+V(Q)$ where V is a real distribution on $\mathbb{R}^n$ such that V(Q) (the operator of multiplication by V) is a continuous operator $\mathscr{H}^1(\mathbb{R}^n)\to\mathscr{H}^{-1}(\mathbb{R}^n)$ (usual Sobolev spaces). A natural choice for the form-domain of H is $\mathscr{G}=\mathscr{H}^1(\mathbb{R}^n)$. Then $[iH,A]=P^2-QV'(Q)=2H-(2V(Q)+QV'(Q))$ (where $V'=\nabla V$) as sesquilinear forms on $\mathscr{S}(\mathbb{R}^n)$. Clearly $H\in C^1(A;\mathscr{H}^1,\mathscr{H}^{-1})$ if and only if $QV'(Q)\in B(\mathscr{H}^1,\mathscr{H}^{-1})$. But this condition is, locally, stronger than needed (although it covers many examples in which the sum defining H exists only in form sense, so the usual Mourre theory does not apply). Our purpose now is to overcome this problem, in particular to recover the results of [BGM] from theorem 3.1. Observe that, if H is a N-body hamiltonian with short and long range interactions, then H is lower semibounded, so it has a spectral gap. We shall now study operators with spectral gaps but which are very singular: they need not be densely defined and we shall not require that their domains or form-domains be invariant under the group W_α. In particular, N-body Schrödinger hamiltonians with hard-core interactions are covered by this formalism (cf.joint work with A.Soffer).

Let $\mathscr{H}$ be a Hilbert space and $W_\alpha=e^{iA\alpha}$ a strongly continuous unitary group in $\mathscr{H}$, so A is a densely defined, self-adjoint operator in $\mathscr{H}$. We denote $D(A;\mathscr{H})$ the domain of A equipped with the graph-norm. Then $D(A;\mathscr{H})$ is a Hilbert space continuously and densely embedded in $\mathscr{H}$, hence we may define by real interpolation the Banach space:

$$(4.1)\qquad \mathscr{F}=(\mathscr{H},D(A;\mathscr{H}))_{1/2,1}\quad .$$

Then $D(A;\mathscr{H})\subset\mathscr{F}\subset\mathscr{H}$ continuously and densely. After the identification $\mathscr{H}\cong\mathscr{H}^*$, we get $\mathscr{F}\subset\mathscr{H}\subset\mathscr{F}^*$ continuously, in particular $B(\mathscr{H})\subset B(\mathscr{F},\mathscr{F}^*)$ continuously.

Let $\{R(z) | z \in \mathbb{C}\setminus\mathbb{R}\}$ be a *self-adjoint pseudo-resolvent* in $\mathcal{H}$, i.e. a family of bounded operators such that $R(z_1)-R(z_2)=(z_2-z_1)R(z_1)R(z_2)$ and $R(z^*)=R(z)^*$. It is known (see [HP]) that the closure of the image of R(z) is a subspace $\mathcal{H}_0$ of $\mathcal{H}$ independent of z, and there is a self-adjoint, densely defined in $\mathcal{H}_0$ operator H such that $R(z)|_{\mathcal{H}_0}=(z-H)^{-1}$ and $R(z)|_{\mathcal{H}\ominus\mathcal{H}_0}=0$ (formally, think that $H=\infty$ on $\mathcal{H}\ominus\mathcal{H}_0$). It is clear that R(z) is a holomorphic function of $z\in\mathbb{C}\setminus\mathbb{R}$. We shall say that *the pseudo-resolvent* $\{R(z)\}$ *has a spectral gap at the point* $\lambda_0\in\mathbb{R}$ if this function extends to an holomorphic function on a neighbourhood of λ_0. Of course, this is equivalent with saying that λ_0 is in the resolvent set of the operator H in $\mathcal{H}_0$.

LEMMA 4.1: *If the operator* $R(z_0)$ *is of class* $\mathscr{C}^1(A;\mathcal{H})$ *for some* z_0 *in the domain of holomorphy of* $\{R(z)\}$, *then* R(z) *will be of class* $\mathscr{C}^1(A;\mathcal{H})$ *for all* z *in this domain.*

Proof: The hypothesis means, according to (2.8):

(4.2) $\int_0^1 \|[A_\varepsilon,[A_\varepsilon,R(z_0)]]\|_{B(\mathcal{H})}\, d\varepsilon < \infty$.

Then this will be true if A_ε is replaced by $A_{-\varepsilon}$ too. Since $(A_\varepsilon)^*=A_{-\varepsilon}$ and $R(z_0)^*=R(z_0^*)$, it will follow that $R(z_0^*)\in\mathscr{C}^1(A;\mathcal{H})$. Hence, by an analytic continuation argument, it is enough to show that $R(z)\in\mathscr{C}^1(A;\mathcal{H})$ for z near z_0. If $|z-z_0|\,\|R(z_0)\|<1$, then $R(z)=R(z_0)[1+(z-z_0)R(z_0)]^{-1}$. So it is enough to prove two things: (i) if $S\in B(\mathcal{H})$ is bijective and $S\in\mathscr{C}^1(A;\mathcal{H})$, then $S^{-1}\in\mathscr{C}^1(A;\mathcal{H})$; (ii) if $S,T\in B(\mathcal{H})$ are of class $\mathscr{C}^1(A;\mathcal{H})$, then $ST\in\mathscr{C}^1(A;\mathcal{H})$. But:

(4.3) $[A_\varepsilon,[A_\varepsilon,S^{-1}]] = 2S^{-1}[A_\varepsilon,S]S^{-1}[A_\varepsilon,S]S^{-1}-S^{-1}[A_\varepsilon,[A_\varepsilon,S]]S^{-1}$

(4.4) $[A_\varepsilon,[A_\varepsilon,ST]] = 2[A_\varepsilon,S][A_\varepsilon,T] + [A_\varepsilon,[A_\varepsilon,S]]T + S[A_\varepsilon,[A_\varepsilon,T]]$.

It remains to observe that $\|[A_\varepsilon,S]\|\leq$const. if $S\in\mathscr{C}^1(A;\mathcal{H})$, because this implies $S\in C^1(A;\mathcal{H})$ and we may use (e) of lemma 2.4. ■

If the assertions of lemma 4.1 are true, we shall say that *the pseudo-resolvent* $\{R(z)\}$ *is of class* $\mathscr{C}^1(A)$. In the applications it is sometimes useful to be able to express this property directly in terms of the self-adjoint operator H. The next criterion is efficient in the N-body case.

PROPOSITION 4.2: *Assume that* $\{R(z)\}$ *is the resolvent of a self-adjoint, densely defined operator* H *in* $\mathcal{H}$ *with domain invariant under* W. *Denote* $\mathcal{G}$ *the domain of* H *equipped with graph-norm and identify* $\mathcal{G}\subset\mathcal{H}\subset\mathcal{G}^*$. *Then the pseudo-resolvent* $\{R(z)\}$ *is of class* $\mathscr{C}^1(A)$ *if and only if* $H\in\mathscr{C}^1(A;\mathcal{G},\mathcal{G}^*)$.

Proof: Assume $H\in\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)$. Let $z_0\in\mathbb{C}$ not in the spectrum of H Then $H-z_0\equiv S$ is an isomorphism of $\mathscr{G}$ onto $\mathscr{H}$ and $\mathscr{H}$ onto $\mathscr{G}^*$. Since A_ε is a bounded operator in each $\mathscr{G}^s$, it is easy to show that (4.3) is valid. We have to prove (4.2). The last term in (4.3) is integrable because it is bounded by $c\|[A_\varepsilon,[A_\varepsilon,H]]\|_{1,-1}$. The first term in the r.h.s. of (4.3) has norm in $B(\mathscr{H})$ bounded by

$$c\|[A_\varepsilon,H]\|_{1/2,-1}\,\|[A_\varepsilon,H]\|_{1,-1/2}=c\|[A_\varepsilon,H]\|^2_{1,-1/2}\le c'\,\varepsilon^{-2}\|W_\varepsilon HW^*_\varepsilon-H\|^2_{1,-1/2}\,.$$

Hence it is enough to prove that the last expression is integrable. We use the identity $2(\mathscr{W}_\varepsilon-1)=(\mathscr{W}_{2\varepsilon}-1)-(\mathscr{W}_\varepsilon-1)^2$ in order to obtain for $0<\varepsilon<1$:

$$2\|W_\varepsilon HW^*_\varepsilon-H\|_{1,-1/2}\le\|W_{2\varepsilon}HW^*_{2\varepsilon}-H\|_{1,-1/2}+c\varepsilon^2\|[A_\varepsilon[A_\varepsilon,H]]\|_{1,-1/2}. \tag{4.5}$$

Hence

$$2\Big[\int_0^1\varepsilon^{-2}\,\|W_\varepsilon HW^*_\varepsilon-H\|^2_{1,-1/2}\,d\varepsilon\Big]^{1/2}\le\Big[\int_0^1\varepsilon^{-2}\,\|W_{2\varepsilon}HW^*_{2\varepsilon}-H\|^2_{1,-1/2}\,d\varepsilon\Big]^{1/2}$$
$$+\;c\Big[\int_0^1\varepsilon^2\,\|[A_\varepsilon[A_\varepsilon,H]]\,\|^2_{1,-1/2}\,d\varepsilon\Big]^{1/2}. \tag{4.6}$$

In the first integral of the r.h.s. make the change of variable $2\varepsilon=\tau$; the contribution of the integral over $\tau\in(1,2)$ is finite, whereas the integral over $\tau\in(0,1)$ is $2^{-1/2}$ times the l.h.s. of (4.6). So, it is enough to prove that the last term above is finite. But we have, by complex interpolation:

$$\varepsilon^2\|[A_\varepsilon,[A_\varepsilon,H]]\|^2_{1,-1/2}\le\varepsilon^2\|[A_\varepsilon,[A_\varepsilon,H]]\|_{1,0}\,\|[A_\varepsilon,[A_\varepsilon,H]]\|_{1,-1}$$
$$\le c\|[A_\varepsilon,[A_\varepsilon,H]]\|_{1,-1}\,, \tag{4.7}$$

which finishes the proof of (4.2). In order to prove the converse ($S^{-1}\in\mathscr{C}^1(A;\mathscr{H})\Rightarrow S\in\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)$), a similar argument is applied to (4.3) with S replaced by S^{-1}. ■

Remark: There is a variant of this proposition for the case when W leaves invariant only the form-domain of H, i.e. the space $\mathscr{G}^{1/2}$. In order to be able to use this in applications, one needs some informations about D(H), which can be obtained by more refined methods if H is, say, an elliptic operator (see [GT]; observe that $\mathscr{G}$, the domain of H, could be a rather pathological space even if $\mathscr{G}^{1/2}$, its form-domain, is quite simple).

The next result is an easy corollary of theorem 3.1.

PROPOSITION 4.3: *Let* $\{R(z)\}$ *be a self-adjoint pseudo-resolvent of class* $\mathscr{C}^1(A)$. *Assume that* $\{R(z)\}$ *has a spectral gap at some point* $\lambda_0\in\mathbb{R}$ *and let* J *be an open subset of* $\mathbb{R}$ *such that* λ_0 *does not belong to its closure. Finally, suppose that* A *is conjugated to* $R(\lambda_0)$ *on* $\tilde{J}=\{(\lambda_0-\lambda)^{-1}\mid\lambda\in J\}$. *Then there is* $J_0\subset J$, *with* $J\setminus J_0$ *a finite set* *such that the holomorphic function* $\mathbb{C}^{\pm}\ni z\mapsto R(z)\in B(\mathcal{F},\mathcal{F}^*)$ *extends to a weak**-*continuous function on* $\mathbb{C}^{\pm}\cup J_0$. *If* A *is strictly conjugated to* $R(\lambda_0)$ *on* J, *then* $J_0=J$.

Remark: $J\setminus J_0$ coincides with the set of eigenvalues in J of the self-adjoint (non-densely defined in general) operator H; these eigenvalues are of finite multiplicity and the associated eigenvectors belong to the range of R(z) (which is independent of z). If the domain of H is invariant under W, proposition 3.3 of [BG3] shows how to verify the fact that A is conjugated to $R(\lambda_0)$.

Proof: Observe first that $R(\lambda_0)$ is a bounded, self-adjoint operator.A number $\mu\in\tilde{J}$ is an eigenvalue of $R(\lambda_0)$ if and only if $\lambda_0-\mu^{-1}$ is an eigenvalue of H (in $\mathcal{H}_0$; observe that $0\notin\tilde{J}$) the multiplicities being the same. We apply theorem 3.1 with $\mathcal{G}=\mathcal{H}$ and H replaced by $R(\lambda_0)$; hence $\mathcal{E}=\mathcal{F}$. Then remark that for non-real z we have

$$R(z) = (z-\lambda_0)^{-1}R(\lambda_0)[R(\lambda_0)+(z-\lambda_0)^{-1}]^{-1}.$$

In fact, for $|z-\lambda_0|\ \|R(\lambda_0)\|<1$ this follows from the equation defining the notion of pseudo-resolvent and for arbitrary z it remains true by holomorphy. Finally, use the fact that $z\mapsto(\lambda_0-z)^{-1}$ is a homeomorphism of $\mathbb{C}^{\pm}\cup J_0$ onto $\mathbb{C}^{\pm}\cup(\tilde{J}\setminus\{$eigenvalues of $R(\lambda_0)\})$. ■

The space $\mathcal{F}$ in which the limiting absorption principle has been proved is too small for several important applications. In order to improve it, we follow [PSS] and use the formula

$$R(z) = R(\lambda_0) + (\lambda_0-z)R(\lambda_0)^2 + (\lambda_0-z)^2R(\lambda_0)R(z)R(\lambda_0) \tag{4.8}$$

obtained after an iteration from $R(z)=R(\lambda_0)+(\lambda_0-z)R(\lambda_0)R(z)$ (sometimes the form $R(z)=R(\lambda_0)+(\lambda_0-z)R(\lambda_0)^{1/2}R(z)R(\lambda_0)^{1/2}$, with $R(\lambda_0)^{1/2}$ conveniently defined, is of simpler use). As an example, we state the following general form of the limiting absorption principle:

PROPOSITION 4.4: *Assume that the conditions of Proposition* 4.3 *are fulfilled. Let* $\mathcal{K},\mathcal{K}_1$ *be Hilbert spaces such that* $\mathcal{K}_1\subset\mathcal{K}$ *and* $\mathcal{H}\subset\mathcal{K}$ *continuously and densely. Identify* $\mathcal{K}^*\subset\mathcal{H}\cong\mathcal{H}^*\subset\mathcal{K}$ *and assume that* $R(\lambda_0)$ *extends to a continuous operator* $\mathcal{K}\to\mathcal{K}^*$ *with the property* $R(\lambda_0)\mathcal{K}_1\subset D(A;\mathcal{H})$. *Denote* $\mathcal{K}_{1/2,1}=(\mathcal{K},\mathcal{K}_1)_{1/2,1}$ *(real interpolation) and observe that* $\mathcal{K}_{1/2,1}\subset\mathcal{K}$ *continuously and densely, so that* $\mathcal{K}^*\subset\mathcal{K}^*_{1/2,1}$ *and* $B(\mathcal{K},\mathcal{K}^*)\subset B(\mathcal{K}_{1/2,1},\mathcal{K}^*_{1/2,1})$ *continuously. Then* :

(i) $R(z)\in B(\mathcal{K},\mathcal{K}^*)$ *for each* $z\in\mathbb{C}^{\pm}$ *and the function* $\mathbb{C}^{\pm}\ni z\mapsto R(z)\in B(\mathcal{K},\mathcal{K}^*)$ *is holomorphic* ;

(ii) *When considered with values in* $B(\mathcal{K}_{1/2,1},\mathcal{K}^*_{1/2,1})$, *the preceding application extends to a weak*-continuous function on* $\mathbb{C}^{\pm}\cup J_0$.

Proof: The assertion (i) follows trivially from (4.8). Closed graph theorem implies $R(\lambda_0)\in B(\mathcal{K}_1,D(A;\mathcal{H}))$. Since $R(\lambda_0)\in B(\mathcal{K},\mathcal{H})$ also, we get $R(\lambda_0):\mathcal{K}_{1/2,1}\to\mathcal{F}$ continuously by interpolation. Then taking adjoints and using the symmetry of $R(\lambda_0)$ we obtain $R(\lambda_0):\mathcal{F}^*\to\mathcal{K}^*_{1/2,1}$. Hence (4.8) and proposition 4.3 imply (ii).■

Let us consider, as an example, a situation which covers the N-body Schrödinger hamiltonians with very singular (even hard-core) interactions. Let $\mathcal{H}=L^2(\mathbb{R}^n)$, $\mathcal{K}=\mathcal{H}^{-1}(\mathbb{R}^n)$, $\mathcal{K}^*=\mathcal{H}^1(\mathbb{R}^n)$. We take $A=\frac{1}{2}(PQ+QP)$ the generator of dilations. If $\mathcal{H}^s_t=\{u\in\mathcal{S}^*(\mathbb{R}^n) \mid \langle P\rangle^s\langle Q\rangle^t u\in\mathcal{H}\}$ are the usual weighted Sobolev spaces, we take $\mathcal{K}_1=\mathcal{H}^{-1}_1$. The spaces $\mathcal{K}_{1/2,1}$ can be explicitely described as follows (see [BG2]). Let $\theta,\eta\in C_0^\infty(\mathbb{R}^n)$ be such that $\theta(x)>0$ if $2^{-1}<|x|<2$ and $\theta(x)=0$ otherwise; $\eta(x)>0$ if $|x|<2$ and $\eta(x)=0$ otherwise. For any $s,t\in\mathbb{R}$ and $1\le p\le\infty$ let $\mathcal{H}^s_{t,p}$ be the Banach space of all temperate distributions u such that:

$$\|\langle P\rangle^s\eta(Q)u\|_{\mathcal{H}} + \left[\int_1^\infty \|\langle P\rangle^s r^t\,\theta(r^{-1}Q)\,u\|^p_{\mathcal{H}}\, r^{-1}dr\,\right]^{1/p} < \infty.$$

Then $\mathcal{K}_{1/2,1}=\mathcal{H}^{-1}_{1/2,1}$ and $\mathcal{K}^*_{1/2,1}=\mathcal{H}^1_{-1/2,\infty}$.

If $\{R(z)\}$ is a pseudo-resolvent in $\mathcal{H}$ such that $R(\lambda_0)\in B(\mathcal{H}^{-1},\mathcal{H}^{+1})$, in order to get the results of proposition 4.4 we have to ask $R(\lambda_0)\mathcal{H}^{-1}_1\subset D(A;\mathcal{H})$. For this $PQR(\lambda_0)\mathcal{H}^{-1}_1\subset\mathcal{H}$ would be enough and this condition is a consequence of $[Q_j,R(\lambda_0)]\mathcal{H}^{-1}\subset\mathcal{H}^1$.

COROLLARY 4.5: *Let* $\{R(z)\}$ *be a self-adjoint pseudo-resolvent on the Hilbert space* $\mathcal{H}=L^2(\mathbb{R}^n)$. *Assume that* $\{R(z)\}$ *has a spectral gap at* $\lambda_0\in\mathbb{R}$ *and that* $R(\lambda_0)$ *and* $[Q_j,R(\lambda_0)]$ *belong to* $B(\mathcal{H}^{-1},\mathcal{H}^1)$ $(j=1,\dots,n)$. *Moreover, assume that a closed countable set* $\tau(H)\subset\mathbb{R}$ *is given such that* $A=\frac{1}{2}(PQ+QP)$ *is locally conjugated to* $R(\lambda_0)$ *on* $\{(\lambda_0-\lambda)^{-1} \mid \lambda\notin\tau(H)\}$ *and that* $\{R(z)\}$ *is of class* $\mathcal{C}^1(A)$. *Then there is a closed countable set* $c(H)\subset\mathbb{R}$ *such that the holomorphic function* $\mathbb{C}^{\pm}\ni z\mapsto R(z)\in B(\mathcal{H}^{-1}_{1/2,1},\mathcal{H}^1_{-1/2,\infty})$ *extends to a weak*-continuous function on* $\mathbb{C}^{\pm}\cup(\mathbb{R}\setminus c(H))$.

If one uses the main idea of the proof of theorem 3.2 in the preceding context (the fact that the Banach space $\mathcal{H}^{-1}_{1/2,1}$ is of cotype 2; see [BG2]) one immediately obtains a very precise criterion for the existence and the completeness of the wave operators. We state it only for densely defined operators, although the general case is very similar.

COROLLARY 4.6: *Let* H_1, H_2 *be two self-adjoint, bounded from below densely defined operators in* $\mathcal{H}=L^2(\mathbb{R}^n)$ *with* $\mathcal{H}^1$ *as form-domain and such that* $[Q_j,H_k]\in B(\mathcal{H}^1,\mathcal{H}^{-1})$ *if* $j=1,\dots,n$; $k=1,2$. *Assume that* $H_1-H_2:\mathcal{H}^1\to\mathcal{H}^{-1}$ *extends to a bounded operator from the closure of* $\mathcal{H}^1$ *in* $\mathcal{H}^1_{-1/2,\infty}$ *into* $\mathcal{H}^{-1}_{1/2,1}$. *Finally, assume that for some* $\lambda_0\in\mathbb{R}$ *the operators* $(\lambda_0-H_1)^{-1}$ *and* $(\lambda_0-H_2)^{-1}$ *are of class* $\mathcal{C}^1(A)$, $A=\frac{1}{2}(PQ+QP)$, *and that* A *is locally conjugated to them outside a closed countable set. Then* H_1, H_2 *have no singularly continuous spectrum and the wave operators* $\text{s-lim}_{t\to\pm\infty}e^{iH_2t}e^{-iH_1t}E_1^c$ *exist and have* $E_2^c\mathcal{H}$ *as range* (E_k^c *is the projection on the subspace of continuity of* H_k).

5. Examples . Optimality of the Results .

The results of the preceding section are corollaries of the theorems 3.1 and 3.2 and are formulated in a form suited to N-body type hamiltonians. In this section we shall consider other situations and obtain results which demonstrate not only the power of the theorem 3.1 but also its fineness (especially in connection with the $\mathcal{C}^1(A)$ assumption). We first prove a very precise division theorem (only the one-dimensional case is treated because of lack of space).

PROPOSITION 5.1: *Let* $h:\mathbb{R}\to\mathbb{R}$ *be such that* $\int_0^1 \varepsilon^{-2}\omega_2(\varepsilon)d\varepsilon<\infty$, *where* $\omega_2(\varepsilon)=\sup_{x\in\mathbb{R}}|h(x+\varepsilon)-2h(x)+h(x-\varepsilon)|$ *is the second modulus of continuity of* h. *Then* h *is of class* C^1. *Assume that* h *is a homeomorphism and that* h′ *is bounded. Then for each* $\lambda\in\mathbb{R}$ *the limits* $\lim_{\varepsilon\to 0}(h(x)-\lambda\mp i\varepsilon)^{-1}\equiv(h(x)-\lambda\mp io)^{-1}$ *exist in the sense of distributions. Moreover, the operator of multiplication by the distribution* $(h(x)-\lambda\mp io)^{-1}$ *belongs to* $B(\mathcal{H}^{1/2,1}(\mathbb{R}),\mathcal{H}^{-1/2,\infty}(\mathbb{R}))$ *and depend* $*$*-weakly continuously on* λ. *In particular, the Besov space* $\mathcal{H}^{1/2,1}(\mathbb{R})$ *consists of continuous functions and the distribution* $\mathrm{VP}h(x)^{-1}$ *belongs to the Besov space* $\mathcal{H}^{-1/2,\infty}(\mathbb{R})$.

Proof: Let us mention first that $\mathcal{H}^{s,p}(\mathbb{R})$ are the Besov spaces denoted $B^{s}{}_{2}^{p}(\mathbb{R})$ in [T]. In the Hilbert space $\mathcal{H}=L^2(\mathbb{R})$ we consider the tranlation group $(W_\alpha u)(x)=u(x-\alpha)$. Then $W_\alpha=e^{-i\alpha P}$ and we take $A=-P=i\frac{d}{dx}$, $H=h(Q)$ the operator of multiplication by h in $\mathcal{H}$ (we assume, without loss of generality, that $h'(x)>0$ for all $x\in\mathbb{R}$). We have to take $\mathcal{G}=D(|H|^{1/2})=\{u\in\mathcal{H}\mid(1+|h(Q)|)^{1/2}u\in\mathcal{H}\}$. Since h is Lipschitz, $\mathcal{G}$ is invariant under W. Observe that

$$\|W_\varepsilon H W_\varepsilon^* - 2H + W_{-\varepsilon} H W_{-\varepsilon}^*\|_{B(\mathscr{H})} = \omega_2(\varepsilon),$$

so that $H\in\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)$. A remark made after definition 2.6 implies that the function $\alpha\mapsto W_\alpha h(Q)W_\alpha^*=h(Q-\alpha)\in B(\mathscr{G},\mathscr{G}^*)$ is norm-C^1. In particular h is of class C^1 (for another proof of this fact, see theorem 3.3, p.87 of [Sh]). Then $[iH,A]=h'(Q)$ which easily implies the Mourre estimate (if $I\subset\mathbb{R}$ is compact, $h^{-1}(I)$ is also compact and the inf of h' on compact sets is strictly positive). Finally, observe that

$$\mathscr{E}=(\mathscr{G}^*,D(A;\mathscr{G}^*))_{-1/2,1}\supset(\mathscr{H},D(A;\mathscr{H}))_{1/2,1}=(\mathscr{H},\mathscr{H}^1)_{1/2,1}=\mathscr{H}^{1/2,1}(\mathbb{R})$$

and $(\mathscr{H}^{1/2,1})^*=\mathscr{H}^{-1/2,\infty}$. Taking $h(x)=x$ we see that $\mathscr{H}^{1/2,1}(\mathbb{R})\subset C^0(\mathbb{R})$. ■

This proposition allows us to make some comments concerning the degree of optimality of theorem 3.1. Two different questions have to be considered: 1) is the space $\mathscr{E}$ optimal, i.e. is it, in some sense, the largest space, in which the L.A.P. holds? 2) Is the regularity assumption $H\in\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)$ optimal, or could it be replaced by $H\in C^1_n(A;\mathscr{G},\mathscr{G}^*)$? Let us discuss these questions in the setting of proposition 5.1. Example 2, page 50, of [P] shows that the best (i.e. smallest) local Besov space $\mathscr{H}^{s,p}_{loc}(\mathbb{R})$ which could contain the distributions $(x\pm io)^{-1}$ is obtained for $s=-1/2$, $p=\infty$ (because the imaginary part of $\mp\pi^{-1}(x\pm io)^{-1}$ is the Dirac measure at zero) and we have proved that in fact they do belong to this space. So in the scale of Besov spaces our space $\mathscr{E}$ gives the optimal result in this example. However, as explained at the end of section 4 of [BGM2], there is a Banach space $\mathscr{K}$ such that $\mathscr{H}^{1/2,1}\subset\mathscr{K}\subset\mathscr{H}$ strictly and the L.A.P. is valid in $B(\mathscr{K},\mathscr{K}^*)$ (but this space is not comparable with $\mathscr{H}^{1/2}$). Let us pass now to the second question. Consider a C^1-diffeomorphism $h:\mathbb{R}\to\mathbb{R}$ and $\lambda\in\mathbb{R}$. Even if the distribution $(h(x)-\lambda+io)^{-1}$ exists, then it does not belong to $\mathscr{H}^{-1/2,\infty}_{loc}$ in general, because the derivative of h could be any (positive) continuous function and the space $\mathscr{H}^{-1/2,1}_{loc}$ is not stable under multiplication by continuous functions (otherwise it would be just $C^0(\mathbb{R})$) (the derivative of h appears when the action on test functions of the distribution $(h(x)-\lambda+io)^{-1}$ is calculated). But something much worse can happen. Using an example due to Lusin (see §13, ch.VIII in [Be]) it is easy to construct a C^1-diffeomorphism h with absolutely continuous derivative such that for every rational number $\lambda\in[0,2\pi]$ the limit of $(h(x)-\lambda+i\varepsilon)^{-1}$ as $\varepsilon\to+0$ does not exist in $\mathscr{D}'(\mathbb{R})$, i.e. in distribution sense (or one can use theorem 5.2 from [Ga] in order to construct a strictly positive, bounded, uniformly continuous function g with Hilbert transform equal to infinity on a dense set and then define h by $h'(x) = (g(h(x))^{-1})$. Finally, let us mention that a condition essentially weaker than $\int_0^1 \varepsilon^{-2}\omega_2(\varepsilon)d\varepsilon < \infty$

cannot force the uniform continuity of h′, i.e. the modulus of continuity of h′ can be made of order $\int_0^\varepsilon t^{-2}\omega_2(t)dt$ (see page 88 of [Sh]).

The next proposition is a remark which concerns the generality of the locally conjugate operator method. We mention it because of the obvious connection with proposition 5.1 and because the construction we make explains the terminology "locally conjugate operator".

PROPOSITION 5.2: *Assume that a self-adjoint operator* H *has a purely absolutely continuous spectrum of constant multiplicity on an open interval* $I\subset\mathbb{R}$. *Then there is an operator* A *which is strictly conjugate to* H *on any compact subset of* I (*and the derivative of the function* $\alpha\mapsto W_\alpha HW_\alpha^*$ *is a* $B(\mathscr{H})$*-valued* C^∞ *function*).

Proof: The assumption we made on H means that there is a Hilbert space $\mathscr{K}$ such that HE(I) is unitarily equivalent to the operator Q of multiplication by the variable x in the Hilbert space $\mathscr{H}_0=L^2(I,dx;\mathscr{K})$ of square-integrable $\mathscr{K}$-valued functions on I. Let $F:I\to\mathbb{R}$ be a bounded function of class C^∞ with all derivatives bounded, with $F(x)>0$ for $x\in I$ and such that $\int_a^c F(x)^{-1}dx=\int_c^b F(x)^{-1}dx = \infty$ (where a<c<b and I=(a,b)). Then $A_0=-1/2(F(Q)P+PF(Q))$ is a self-adjoint operator in $\mathscr{H}_0$ such that $[iQ,A_0]=F(Q)$ is strictly positive on each compact subset of I. We take A equal to $U^{-1}A_0U$ on $E(I)\mathscr{H}$ (U is the unitary operator $E(I)\mathscr{H}\to\mathscr{H}_0$ which transforms HE(I) in Q) and equal to zero on $E(\mathbb{R}\setminus I)\mathscr{H}$. Observe that if we take F(x)=0 for $x\notin I$, we shall have [iH,A]=F(H). ■

We shall now give a simple example of a hard-core type situation, in which neither the domain nor the form-domain of the hamiltonian are invariant under W, but the conjugate operator method can be used if one works directly with the resolvent. In $\mathscr{H}=L^2(\mathbb{R})$ let $H_0=P^2=-\frac{d^2}{dx^2}$ and $R_0=(H_0+1)^{-1}$. We would like to study the operator $H_\infty=H_0+V_\infty$ where, formally, $V_\infty(x)=+\infty$ if x<0 and $V_\infty(x)=0$ if x>0. Rigorously, this operator is the limit in the norm-resolvent sense as $\kappa\to+\infty$ of $H_\kappa=H_0+\kappa(1-E)$ where E is the operator of multiplication by the characteristic function of $(0,\infty)$. Let $\phi(x)=2^{-1/2}E(x)e^{-x}$. Then $R_\infty=\lim_{\kappa\to\infty}(H_\kappa+1)^{-1}=ER_0E-\phi\otimes\phi$ where $\phi\otimes\phi$ is the rank one operator which sends u into $\phi\langle\phi|u\rangle$. We shall calculate the order of regularity of R_∞ with respect to the translation group (we do this because the result is simpler; in fact the dilation group must be used in order to have an example relevant for the N-body case; however, if the point zero, where the potential becomes infinite, is replaced by an arbitrary non-zero point, the order of regularity of R_∞ with respect to the translation or the dilation group are obviously the same). If $T_\alpha=e^{iP\alpha}$, then $T_\alpha R_\infty T_{-\alpha}=E_\alpha R_0E_\alpha-\phi_\alpha\otimes\phi_\alpha$ where E_α is the operator of multiplication by the characteristic function of $(-\alpha,\infty)$ and

$\phi_\alpha = T_\alpha\phi$. Calculating the derivative at $\alpha=0$ one gets $[iP,R_\infty]=2\phi\otimes\phi$, hence R_∞ is of class $C^1(P;\mathscr{H})$.Then $T_\alpha[iP,R_\infty]T_{-\alpha}=2\phi_\alpha\otimes\phi_\alpha$ and $\|\phi_\alpha-\phi\|_{\mathscr{H}}=2^{-1/2}|1-e^{-\alpha}|^{1/2}\sim\alpha^{1/2}$. To conclude, R_∞ is of class $C^{3/2}(P;\mathscr{H})$ and not more.

We mention now another explicitly soluble example in which the conjugate operator method works but the domain of the hamiltonian is not invariant under the group. Let δ be Dirac measure at zero on $\mathbb{R}$. Let $\mathscr{H}$ and H_0 as above and $H=H_0+g\delta$ with $g\in\mathbb{R}\setminus\{0\}$ (form-sum). The form-domain of H is $\mathscr{H}^1(\mathbb{R})$, but the functions in the domain of H have to verify $u'(+0)-u'(-0)=gu(0)$, so that the domain is not invariant under the dilation group W. If $g<0$, then H has a bound state of energy $-g^2/4$, if $g>0$ then H has no bound states and it always has a purely absolutely continuous spectrum equal to $[0,\infty)$.The form-domain of H is obviously invariant under W and $W_\alpha H W_\alpha^*=e^{-2\alpha}P^2+e^\alpha g\delta$ as forms on $\mathscr{H}^1$ (because δ is homogeneous of degree $-n$ in $\mathbb{R}^n$; or use $\langle u|Hu\rangle=\int |u'(x)|^2dx+g|u(0)|^2$). Hence H is of class $C^\infty(A;\mathscr{H}^1,\mathscr{H}^{-1})$ and $[iH,A]=2H-3g\delta$. Since $\delta:\mathscr{H}^1\to\mathscr{H}^{-1}$ is a continuous operator of rank one, A will be conjugate (strictly if $g<0$) to H on (ε,∞) and $-A$ will be conjugate (strictly if $g>0$) to H on $(-\infty,-\varepsilon)$ for each $\varepsilon>0$. Hence we get all spectral properties of H from theorem 3.1.

Our final topic is an improvement of the perturbative method of verifying Mourre estimate presented in proposition 7.6 of [BG2]. This allows one to treat locally very singular potentials. We begin with the following simple remark:

LEMMA 5.3: *Let* H,H_0 *be self-adjoint ,not necessarily densely defined, operators, in some Hilbert space* $\mathscr{H}$. *If* $(H-z)^{-m}-(H_0-z)^{-m}$ *is compact for some fixed* $m\geq 1$ *and for all* $z\in\mathbb{C}\setminus\mathbb{R}$, *then* $f(H)-f(H_0)$ *is compact for each* $f:\mathbb{R}\to\mathbb{C}$ *continuous and convergent to zero at infinity. In this case* H *and* H_0 *have the same essential spectrum.*

Proof: Let $R(z)=(H-z)^{-1}$, $R_0(z)=(H_0-z)^{-1}$ the associated pseudo-resolvents. If $f=g^{(m-1)}$ for some $g\in C_0^\infty(\mathbb{R})$, formula (6) from [BG1] gives:

$$f(H)=\Sigma_{k=0}^{n-1}\frac{(-1)^{m-1}(m-1)!}{\pi k!}\int_{\mathbb{R}} g^{(k)}(\lambda)\mathrm{Im}[i^k R^m(\lambda+i)]d\lambda +$$

$$+\frac{(-1)^{m-1}(m-1)!}{\pi(n-1)!}\int_0^1\varepsilon^{n-1}d\varepsilon\int_{\mathbb{R}} g^{(n)}(\lambda)\ \mathrm{Im}[i^n R^m(\lambda+i\varepsilon)]d\lambda.$$

Here $n\geq m+1$ in order to have norm-convergent integrals. A similar formula for $f(H_0)$ shows that $f(H)-f(H_0)$ is compact for such f. Let

$$C_\infty(\mathbb{R})=\{\varphi:\mathbb{R}\to\mathbb{C}\ |\ \varphi \text{ continuous and } \varphi(x)\to 0 \text{ if } |x|\to\infty\}$$

with the sup norm. Since $C_\infty(\mathbb{R}) \ni \varphi \mapsto \varphi(H)-\varphi(H_0) \in B(\mathscr{H})$ is norm-continuous, it is enough to show that $\mathscr{N}=\{g^{(m-1)} \mid g \in C_0^\infty(\mathbb{R})\}$ is a dense subspace of $C_\infty(\mathbb{R})$, or of $C_0^\infty(\mathbb{R})$ equipped with the sup norm. But $f \in \mathscr{N}$ if and only if

$$\int f(x)dx = \int xf(x)dx = \ldots = \int x^{m-2} f(x)dx = 0$$

and $f \mapsto \int x^j f(x)dx$ are linear functionals on C_0^∞ which are not continuous for the sup norm, so the intersection $(j=0,\ldots,m-2)$ of their kernels is dense for this norm. Since λ does not belong to the essential spectrum of H if and only if there is $f \in C_0^\infty(\mathbb{R})$ with $f(\lambda) \neq 0$ and $f(H)$=compact, the last assertion is trivial. ■

The assumption of Lemma 5.3 is easy to verify and allows quite singular perturbations H of H_0 (see the discussion in section 8 of [Pe]). In the next proposition we shall say that a pseudo-resolvent $\{R(z)\}$ is of class $C_n^1(A)$ if $R(z)$ is of class $C_n^1(A;\mathscr{H})$ for some z in the domain of holomorphy; the proof of lemma 4.1 shows that this will remain true for all such z.

PROPOSITION 5.4: *Let* $\{R_0(z)\}$, $\{R(z)\}$ *be two self-adjoint pseudo-resolvents which are of class* $C_n^1(A)$ *for some self-adjoint, densely defined operator* A. *Assume that* $R(z)-R_0(z)$ *is compact for some* z *and that one of them has a spectral gap, so that they have a common spectral gap at some point* $\lambda_0 \in \mathbb{R}$. *Then* A *is conjugated to* $R(\lambda_0)$ *at some point* $\lambda \in \mathbb{R}$ *if and only if it is conjugated to* $R_0(\lambda_0)$ *at* λ.

Proof: Write $R=R(\lambda_0)$, $R_0=R_0(\lambda_0)$. Since

$$[iA,R]-[iA,R_0]=\lim_{\varepsilon\to 0}\varepsilon^{-1}\left[W_\varepsilon(R-R_0)W_\varepsilon^*-(R-R_0)\right]$$

is norm limit of compact operators, it will also be compact. Let us write $S \sim T$ if $S-T$ is compact. Then $\varphi(R) \sim \varphi(R_0)$ for each continuous function φ. Hence $\varphi(R)[iA,R]\varphi(R) \sim \varphi(R_0)[iA,R_0]\varphi(R_0)$. From this the assertion of the proposition follows easily. ■

If $\{R_0(z)\}$ is of class $\mathscr{C}^1(A)$, then one may deduce that $\{R(z)\}$ has the same property by applying theorem 6.2 or 6.3 from [BG2] to the difference $R(z)-R_0(z)$ for some fixed z. Then theorems 3.1 and 3.2 will give a detailed spectral and scattering theory for H. For example, results like theorem 8.1 of [Pe] are easily obtained. Observe that one has to put conditions only on the difference of the resolvents of H and H_0 (as in Kato's criterion for the existence of wave operators), so H could be very singular with respect to H_0 (for example a differential operator of higher order). Remark that not only short-range, but also long-range singular perturbations are allowed. Moreover, the unperturbed operator H_0 can be quite complicated (e.g. a N-body hamiltonian), a situation in which usual Enss method (as presented in [Pe] for example) does not work.

Appendix: A Tauberian Estimate

We shall prove here an estimate which plays an important role in the applications we have in mind and which improves the tauberian theorem described in [BG2]. Below we denote $BC(\mathbb{R}^n)$ the C^*-algebra of bounded, continuous functions on $\mathbb{R}^n$ equipped with the norm $\|f\|_\infty=\sup\{|f(x)| \mid x\in\mathbb{R}^n\}$. $C_0^\infty(\mathbb{R}^n)$ is equipped with the usual Schwartz topology.

We shall consider a subalgebra $\mathcal{M}\subset BC(\mathbb{R}^n)$, which contains the constants, and which is equipped with a norm $|\cdot|_{\mathcal{M}}$ for which $\mathcal{M}$ is a Banach space with continuous multiplication (i.e. $\exists M<\infty$ such that $|fg|_{\mathcal{M}}\leq M|g|_{\mathcal{M}}$ for all f,g in $\mathcal{M}$). We assume that $C_0^\infty(\mathbb{R}^n)\subset\mathcal{M}\subset BC(\mathbb{R}^n)$ the embeddings being continuous. Let us denote $f^\sigma(x)=f(\sigma x)$ for each function f on $\mathbb{R}^n$ and each $\sigma\geq 0$. Our final assumption is that $\mathcal{M}$ is invariant under dilations, i.e. $f^\sigma\in\mathcal{M}$ if $f\in\mathcal{M}$ and $\sigma>0$, and that there are constants $0<M,N<\infty$ such that

$$\text{(A.1)} \qquad |f^\sigma|_{\mathcal{M}} \leq M<\sigma>^N|f|_{\mathcal{M}} \quad \text{for all } f\in\mathcal{M} \quad (<\sigma>=(1+\sigma^2)^{1/2}).$$

THEOREM: *Assume that* **E** *is a Banach space and that a continuous, unital homomorphism* $\mathcal{M}\ni f\longmapsto f(\Lambda)\in B(E)$ *is given. Denote* $f(\sigma\Lambda)\equiv f^\sigma(\Lambda)$. *Let* $\rho\in\mathcal{M}$ *and assume that there is a number* $\ell>N$ *such that for any function* $\theta\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ *we have* $|\rho^\tau\theta|_{\mathcal{M}} \leq c(\theta)\tau^\ell$ *if* $0<\tau<1$. *Let* $\xi:\mathbb{R}^n\to\mathbb{R}$ *be a function of class* C^∞ *and such that* $\xi(x)=0$ *(resp.* $\xi(x)=1$*) in a neighbourhood of zero (resp. of infinity). Denote* $\eta(x)=x\nabla\xi(x)$. *Then there is a constant* c *such that for all* $u\in E$ *and all* $0<\varepsilon<1$:

$$\text{(A.2)} \qquad \|\rho(\varepsilon\Lambda)u\| \leq c\|\xi(\varepsilon\Lambda)u\| + c\varepsilon^\ell \int_\varepsilon^1 \|\eta(\tau\Lambda)u\|\, \tau^{-1-\ell}\, d\tau + c\varepsilon^\ell\|u\|.$$

Remarks: Here Λ has to be interpreted as a symbol which helps to distinguish the function $f\in\mathcal{M}$ and the operator acting in **E** associated to it by the homomorphism. However, in applications Λ is in fact an operator or a finite set of operators in **E**. Observe that $\eta\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ so it belongs to $\mathcal{M}$, and $\xi-1\in C_0^\infty(\mathbb{R}^n)$, so that ξ belongs to $\mathcal{M}$ too. Hence all terms in (A.2) are well defined and (A.2) is an estimate of the rate of decay of $\|\rho(\varepsilon\Lambda)u\|$ as $\varepsilon\to 0$ in terms of the rate of decay of $\|\xi(\varepsilon\Lambda)u\|$ and $\|\eta(\varepsilon\Lambda)u\|$. The condition we put on ρ is satisfied if there are $\omega\in C^\infty(\mathbb{R}^n\setminus\{0\})$ and $\rho_0\in\mathcal{M}$ such that $\rho(x)=\omega(x)\rho_0(x)$ for $x\neq 0$ and $\omega(\tau x)=\tau^\ell\omega(x)$ for $\tau>0$ and $x\neq 0$. In fact, we shall then have:

$$|\rho^\tau\theta|_{\mathcal{M}} = |\omega^\tau\rho_0^\tau\theta|_{\mathcal{M}} = \tau^\ell|\omega\theta\rho_0^\tau|_{\mathcal{M}} \leq M\tau^\ell|\omega\theta|_{\mathcal{M}}|\rho_0^\tau|_{\mathcal{M}} \leq c\tau^\ell$$

for $0<\tau\leq 1$, because $\omega\theta\in C_0^\infty\subset\mathcal{M}$. Observe that ρ has a zero of finite order ℓ at zero in this example, while ξ and η have zeroes of infinite order: this explains why we call (A.2) a "tauberian estimate". Let us mention that in all the applications $\|\rho(\varepsilon\Lambda)\|$ is a constant independent of ε. For example, if Λ is an unbounded self-adjoint operator in a Hilbert space $\mathbf{E}$, then $\|\rho(\varepsilon\Lambda)\|$=const. while, if supp θ is included in $0<a\leq|x|\leq b<\infty$, then $\|\rho(\tau\Lambda)\theta(\Lambda)\| \leq \sup_x|\rho(\tau x)\theta(x)| \leq c\sup\{|\rho(\lambda)| \mid a\tau\leq|\lambda|\leq b\tau\}$. Finally, let us observe that if $\zeta\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ and $\zeta(x)=1$ on supp η, then $\|\eta(\tau\Lambda)u\|=\|\eta(\tau\Lambda)\zeta(\tau\Lambda)u\|\leq \|\eta(\tau\Lambda)\|\,\|\zeta(\tau\Lambda)u\|\leq C\|\zeta(\tau\Lambda)u\|$ for $\tau\leq 1$, hence the precise form of η is irrelevant. Moreover, if ξ_1 is a function with properties similar to ξ, then there is $\mu>0$ such that $\xi_1(\mu x)=1$ for $x\in$ supp ξ, hence $\|\xi(\varepsilon\Lambda)u\|=\|\xi(\varepsilon\Lambda)\xi_1(\varepsilon\mu\Lambda)u\|\leq c\|\xi_1(\varepsilon\mu\Lambda)u\|$ for $\varepsilon\leq 1$, so the precise form of ξ is also irrelevant.

Proof of the theorem : Observe first that for $0<a<b<\infty$ and $x\neq 0$ we have $\xi(bx)-\xi(ax)=\int_a^b \eta(tx)t^{-1}dt$. In particular $1=\xi(x)+\int_1^\infty \eta(tx)\, t^{-1}dt$ if $x\neq 0$, which implies

(A.3) $\quad \rho^\varepsilon(x) = \rho^\varepsilon(x)\xi^\varepsilon(x)+\int_1^\infty \rho^\varepsilon(x)\eta^{\varepsilon t}(x)\, t^{-1}dt \qquad (x\neq 0).$

The application $\sigma\longmapsto\eta^\sigma\in C_0^\infty(\mathbb{R}^n)$ is continuous on $(0,\infty)$, hence $t\longmapsto\eta^{\varepsilon t}\in\mathcal{M}$ has the same property. Moreover, for $t\geq 1$:

(A.4) $\quad |\rho^\varepsilon\eta^{\varepsilon t}|_{\mathcal{M}} = |(\rho t^{-1}\eta)^{\varepsilon t}|_{\mathcal{M}} \leq M\langle\varepsilon t\rangle^N|\rho t^{-1}\eta|_{\mathcal{M}} \leq c(\varepsilon)t^{N-\ell},$

because $\eta\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$.

Hence $\int_1^\infty |\rho^\varepsilon\eta^{\varepsilon t}|_{\mathcal{M}}\, t^{-1}dt<\infty$, so that the integral $\int_1^\infty \rho^\varepsilon\eta^{\varepsilon t}\, t^{-1}dt$ exists in $\mathcal{M}$ (in norm). Using (A.3) we obtain:

(A.5) $\quad \rho^\varepsilon = \rho^\varepsilon\xi^\varepsilon + \int_1^\infty \rho^\varepsilon\eta^{\varepsilon t}\, t^{-1}dt$

equality in $\mathcal{M}$ (in fact, since all the terms are in $\mathcal{M}$ and $\mathcal{M}$ consists of continuous functions, it is enough to show that the values at each $x\neq 0$ of the right and left side are equal, which is assured by (A.3)). The continuity of the homomorphism $f\longmapsto f(\Lambda)$ implies now:

(A.6) $\quad \rho(\varepsilon\Lambda) = \rho(\varepsilon\Lambda)\xi(\varepsilon\Lambda) + \int_1^\infty \rho(\varepsilon\Lambda)\eta(\varepsilon t\Lambda)\, t^{-1}dt$

(the integral exists in norm in $B(\mathbf{E})$ due to (A.4)).

Consider now some $u\in\mathbf{E}$ and let us apply (A.6) to it. Since $\|\rho(\varepsilon\Lambda)\|\leq c|\rho^\varepsilon|_{\mathcal{M}}\leq$ const. for $0<\varepsilon\leq 1$, we get:

(A.7) $\quad \|\rho(\varepsilon\Lambda)u\| \leq c\|\xi(\varepsilon\Lambda)u\| + \int_1^\infty \|\rho(\varepsilon\Lambda)\eta(\varepsilon t\Lambda)u\|\, t^{-1}dt =$

$$= c\|\xi(\varepsilon\Lambda)u\| + \int_\varepsilon^1 \|\rho(\varepsilon\Lambda)\eta(\sigma\Lambda)u\|\, \sigma^{-1}d\sigma + \int_1^\infty \|\rho(\varepsilon\Lambda)\eta(\sigma\Lambda)u\|\, \sigma^{-1}d\sigma.$$

In order to estimate the first integral above, let $\theta \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ be such that $\theta\eta=\eta$. Then

$$\|\rho(\varepsilon\Lambda)\eta(\sigma\Lambda)u\| = \|\rho(\varepsilon\Lambda)\theta(\sigma\Lambda)\eta(\sigma\Lambda)u\| \leq \|(\rho^\varepsilon\theta^\sigma)(\Lambda)\| \, \|\eta(\sigma\Lambda)u\| \leq$$

$$\leq c|\rho^\varepsilon\theta^\sigma|_{\mathcal{M}}\|\eta(\sigma\Lambda)u\| = c|(\rho^{\varepsilon\sigma^{-1}}\theta)^\sigma|_{\mathcal{M}} \, \|\eta(\sigma\Lambda)u\|$$

$$\leq cM<\sigma>^N|\rho^{\varepsilon\sigma^{-1}}\theta|_{\mathcal{M}} \, \|\eta(\sigma\Lambda)u\| \leq c_1<\sigma>^N(\varepsilon/\sigma)^\ell\|\eta(\sigma\Lambda)u\|.$$

If we use this estimate in the first integral from the last member of (A.7), we obtain the second term from the right-hand side of (A.2). Finally, we estimate the last integral from (A.7) using (observe that $\eta \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $\sigma \geq 1$):

$$\|\rho(\varepsilon\Lambda)\eta(\sigma\Lambda)u\| \leq c|\rho^\varepsilon\eta^\sigma|_{\mathcal{M}}\|u\| = c|(\rho^{\varepsilon\sigma^{-1}}\eta)^\sigma|_{\mathcal{M}} \, \|u\| \leq$$

$$\leq c_1\sigma^N|\rho^{\varepsilon\sigma^{-1}}\eta|_{\mathcal{M}} \, \|u\| \leq c_2\sigma^{N-\ell}\varepsilon^\ell\|u\|.$$

Since $\ell > N$, we shall obtain the last term of (A.2). ■

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