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REPRESENTATIONS OF NASH FUNCTIONS

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Introduction.

The aim of this paper is to characterize Nash functions of m complex variables in term of rational functions of $m+1$ variables.

Using the notation introduced in Chapter I of the paper our main result (Theorem III.2.1) may be formulated as follows:

Let K be a compact, rationally convex subset of $\mathbb{C}^m$. A function

$$f\colon K \longrightarrow \mathbb{C}$$

extends to a Nash function in a neighborhood of K if and only if there is a rational function $R \in \mathbb{C}(z,w)$, holomorphic in neighborhood of $K \times T$ (where T denotes the unit circle in $\mathbb{C}$), such that

$$f(z) = \int_T R(z,w)dw \qquad \text{for} \qquad z \in K.$$

The paper is organized as follows:

Chapter I and II are of preparatory nature. In Chapter I we study the class of rationally convex compact sets. As this class is essential in our further considerations, we give detailed proves of all theorems that we shall use later.

The aim of Chapter II is to characterize Nash functions in terms of a special class of Nash functions – called simple Nash functions (Lemma II.3.2). This Lemma (in the case of m=1) was earlier obtained in [C–T]. In [D–L] similar result ("in local situation") was proved.

Chapter III contains main results of our paper.

Our result were inspired by [C–T] and [D–L]. We apply some methods used in these papers.

CHAPTER I
Rationally Convex Compact Sets

1. Rational Functions . In this section we present some basic properties of rational functions We shall need them in further sections of this paper.

Let us start with the definition of rational function on an algebraic subset V of $\mathbb{C}^m$.

DEFINITION 1. *The ring of rational functions of the set V, denoted by $\mathbb{C}(V)$, is the full ring of fraction of the coordinate ring R_V of the set V. An element of the ring $\mathbb{C}(V)$, is called a rational function on V.*

Let f be an arbitrary rational function on V. According to the definition there exist two regular functions P, Q on V such that:

1. Q is not a zero-divisor in the ring R_V (in other words Q is not identically equal 0 on any irreducible component of the algebraic set V),
2. $f = \dfrac{P}{Q}$.

DEFINITION 2. *A rational function $f = \dfrac{P}{Q}$ is said to be holomorphic at point $a \in V$ iff there exists a germ $g \in \mathcal{O}_a(V)$ of holomorphic function at the point a such that $g \cdot Q = P$.*

Let us notice that the germ g is uniquely determined (does not depend on the choice of regular functions P i Q). The set of point at which a rational function is holomorphic is an open and dense (in euclidean topology) subset of the set V.

The following theorem yields more precise characterization that set.

THEOREM 1. *Let f be a rational function of V. The set of all points at which the function f is not holomorphic, is a nowhere-dense algebraic subset of V.*

Proof. There exist regular functions $P, Q \in R_V$ such that $f = \dfrac{P}{Q}$ and the function Q does not vanish at any irreducible component of the set V. The set

$$X_0 := \{(x, w) \in V \times \mathbb{P}^1(\mathbb{C}) \colon Q(x) \neq 0 \quad \text{and} \quad w \cdot Q(x) = P(x)\}$$

is a constructible subset and the set $X := \overline{X_0}$ is an algebraic subset of $V \times \mathbb{P}^1(\mathbb{C})$. Moreover $X_1 := X \cap (\mathbb{C}^m \times \mathbb{C})$ is an algebraic subset of $\mathbb{C}^m \times \mathbb{C}$.

Let us assume that the function f is holomorphic at a point $a \in V$. There exists a germ $g \in \mathcal{O}_a(V)$ of holomorphic function such that $g \cdot Q = P$. In this situation

$$X \cap (\{a\} \times \mathbb{P}^1) = \{(a, g(a))\}.$$

Choose an arbitrary holomorphic germ $g_1 \in \mathcal{O}_a(\mathbb{C}^m)$ such that $g_1|_V = g$. The holomorphic germ $h \in \mathcal{O}_{(a,f(a))}(\mathbb{C}^m \times \mathbb{C})$ defined by the formula

$$h(z_1, \ldots, z_m, z_{m+1}) := z_{m+1} - g_1(z_1, \ldots, z_m),$$

is an element of the ideal of the germ of the analytic set X_1 at the point $(a, g(a))$.

Using the Serre Lemma (on polynomial generators) ([Ł], VII.15.3., p.337) we conclude that there exist polynomials $P_1, \ldots, P_k \in I(X_1)$ $(I(X_1)$ denotes the ideal of the algebraic set $X_1)$ and germs of holomorphic functions $g_1, \ldots, g_k \in \mathcal{O}_{(a,f(a))}(\mathbb{C}^m \times \mathbb{C})$ such that $g = g_1 P_1 + \ldots, g_k P_k$. Differentiating the above equality we observe that for at least one index $i = 1, \ldots, k$ we have $\frac{\partial P_i}{\partial z_{m+1}}(a, f(a)) \neq 0$.

Denoting by W the set of all points at which the function f is not holomorphic we state that

$$\begin{aligned} W_1 := \Big\{a \in V \colon \exists a_{m+1} \in \mathbb{C} \colon (a, a_{m+1}) \in X \\ \text{and} \quad \forall F \in I(X_1) \quad \frac{\partial F}{\partial z_{m+1}}(a, a_{m+1}) = 0\Big\} \\ \cup \Big\{a \in V \colon (a, \infty) \in X\Big\} \cup \Big\{a \in V \colon \#\left(X \cap (\{a\} \times \mathbb{P}^1)\right) \geq 2\Big\} \subset W. \end{aligned}$$

We shall prove that

$$W_1 = W.$$

Suppose, on the contrary, that $a \in W \setminus W_1$.

From the definition of W_1 we have

$$X \cap (\{a\}) \times \mathbb{P}^1 = \{(a, a_{m+1})\}, \qquad a_{m+1} \in \mathbb{C}.$$

Moreover there is a polynomial $F \in I(X_1)$ such that $\frac{\partial F}{\partial z_{m+1}}(a, a_{m+1}) \neq 0$. By the implicit function theorem there exist an open neighborhood U of $a \in \mathbb{C}^m$, a real number $r > 0$ and a holomorphic function $\phi \colon U \longrightarrow a_{m+1} + \Delta(r)$ (where $\Delta(r) := \{z \in \mathbb{C} \colon |z| < r\}$) such that $F^{-1}(0) \cap \left(U \times \left(a_{m+1} + \bar{\Delta}(r)\right)\right) = \phi$.

As the natural projection

$$\pi: X \ni (x_1, \ldots, x_m, x_{m+1}) \mapsto (x_1, \ldots, x_m) \in V$$

is a proper mapping we may assume (if necessary – after suitable decreasing of U and r) that

$$X \cap (U \times \mathbb{C}) \subset \phi|_{(V \times U)}.$$

From the latest equality we can deduce that for any point $z \in V \times U$ such that $Q(z) \neq 0$ we have $\dfrac{P(z)}{Q(z)} = \phi(z)$.

Since the set $\{z \in V \times U : Q(z) \neq 0\}$ is dense in $V \times U$ we have

$$P(z) = Q(z) \cdot \phi(z) \qquad \text{for any} \qquad z \in V \times U,$$

and this means that the rational function f is holomorphic at the point a. We obtain a contradiction which proves that $W = W_1$.

Let us notice that $W = W_1$ is an algebraically constructible set ([K] Th.III.11.1.; [Ł], VII.Ɛ.3.— the Chevalley Theorem), and hence — since it is closed — an algebraic set (cf. [Ł], VII.8.3.,. p. 291—295). The proof is completed □

Let Ω be an open subset of $\mathbb{C}^m$. We shall denote by $\mathcal{R}(\Omega)$ the space of all holomorphic functions on Ω which are restrictions to the set Ω of rational functions. Let us notice that a function $f: \Omega \to \mathbb{C}$ belongs to $\mathcal{R}(\Omega)$ if and only if there exist polynomials $P, Q: \mathbb{C}^m \to \mathbb{C}$ such that $Q^{-1}(0) \cap \Omega = \emptyset$ and $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in \Omega$. If the polynomials P, Q are relatively prime then their are uniquely determined (up to a constant factor).

Let K be a fixed compact subset of $\mathbb{C}^m$. Denote

$$\begin{aligned}\mathcal{O}(K) := \{f: K \to \mathbb{C}: \quad & \text{there exist an open neighborhood } V \text{ of } K \\ & \text{and a function } \tilde{f} \in \mathcal{O}(V) \text{ such that } f = \tilde{f}|_K\}.\end{aligned}$$

An extension of a function from the class $\mathcal{O}(K)$ to an open neighborhood of K is not uniquely determined.

In the same way as $\mathcal{O}(K)$ we define the class $\mathcal{R}(K)$.

Let us observe that a function $f: K \to \mathbb{C}$ belongs to the class $\mathcal{R}(K)$ if and only if there exist polynomials $P, Q: \mathbb{C}^m \to \mathbb{C}$ such that $Q^{-1}(0) \cap K = \emptyset$ and $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in K$. Polynomials P and Q are not (in general) uniquely determined (even up to a constant factor).

The sets $\mathcal{O}(K)$ and $\mathcal{R}(K)$ have (with natural operations) the structure of ring.

Let $R_1, \ldots, R_k$ be rational functions of m complex variables and let $\rho_1, \ldots, \rho_k > 0$ be real numbers. We shall denote by

$$\{z \in \mathbb{C}^m : |R_i(z)| < \rho_i \quad \text{for} \quad i = 1, \ldots, m\}, \tag{1}$$

the set of all points of $\mathbb{C}^m$, at which all functions R_i are holomorphic and moreover all inequalities hold. The sets of the form (1) are open; we shall call them *rational polyhedra.*

2. Rationally convex compact sets . In this section we collect basic information concerning the class of rationally convex compact sets.

DEFINITION 1. *A compact subset K of the space $\mathbb{C}^m$ is said to be rationally convex iff for each point $z_0 \notin K$ the following two equivalent conditions hold:*

(i) There exists a rational function $R \in \mathcal{R}(K \cup \{z_0\})$ such that

$$|R(z_0)| > \|R\|_K, \tag{2}$$

(ii) There exists a polynomial $P: \mathbb{C}^m \to \mathbb{C}$ such that $P(z_0) = 0$ and $P(z) \neq 0$ for each point $z \in K$.

Proof of equivalence of conditions (i) *and* (ii).

Let us assume that for a fixed point $z_0 \notin K$ condition (i) holds. Let $R \in \mathcal{R}(K \cup \{z_0\})$ be any rational functions fulfilling inequality (2). Since $R \in \mathcal{R}(K \cup \{z_0\})$ we can find polynomials $Q_1, Q_2: \mathbb{C}^m \to \mathbb{C}$ such that $Q_2(z) \neq 0$ for each point $z \in K \cup \{z_0\}$ and $R(z) = \dfrac{Q_1(z)}{Q_2(z)}$ for $z \in K$. Then the polynomial $P(z) := Q_1(z) - R(z_0) \cdot Q_2(z)$ satisfies condition (ii).

Now, we shall prove the converse implication. Let $z_0 \notin K$ and let $P: \mathbb{C}^m \to \mathbb{C}$ be any polynomial such that $P(z_0) = 0$ and $P(z) \neq 0$ for each point $z \in K$. Put $c := \min\{|P(z)| : z \in K\}$. Since P does not vanish on K we have $c > 0$. If we put $R(z) := \dfrac{1}{3P(z) - c}$ we get $R \in \mathcal{R}(K \cup \{z_0\})$ and $|R(z_0)| = \frac{1}{c} > \frac{1}{2c} \geq \|R\|_K$. $\square$

The following remark proves that the class of rationally convex sets is very large

REMARK 1.

(1) Any compact set $K \subset \mathbb{C}$ is rationally convex.

(2) Any polynomially convex compact set is rationally convex.

(3) If the sum of two disjoint, compact sets is rationally convex then both summands are rationally convex too.

(4) Intersection of any number of rationally convex compact sets is rationally convex.

(5) Product of two rationally convex compact sets is rationally convex.

(6) If $f: \mathbb{C}^n \to \mathbb{C}^m$ is a proper holomorphic map, K is a rationally convex compact subset of $\mathbb{C}^m$ then the inverse image $f^{-1}(K)$ is rationally convex.

Proof. (1), (2), (4) are obvious. (3) is a simple consequence of Theorem I.3.1., we shall prove it later.

(5) Assume that K_1 and K_2 are rationally convex compact subset respectively of $\mathbb{C}^m$ and $\mathbb{C}^n$. Assume that $(w_0, z_0) \notin K_1 \times K_2$. If $w_0 \notin K_1$ then there exists polynomial $P: \mathbb{C}^n \to \mathbb{C}$ such that $P(w_0) = 0$ and $P(w) \neq 0$ for each point $w \in K_1$.

The polynomial $P_1: \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}$ defined by $P_1(w, z) = P(w)$ satisfies conditions $P_1(w_0, z_0) = 0$ and $P_1(w, z) \neq 0$ for each $(w, z) \in K_1 \times K_2$. The case of $z_0 \notin K_2$ is similar.

(6) Since f is proper the set $L := f^{-1}(K)$ is compact. Fix any point $z_0 \notin L$. Then $f(z_0) \notin K$ and consequently there exists a polynomial $P_1: \mathbb{C}^m \to \mathbb{C}$ such that $P_1(f(z_0)) = 0$ and P_1 does not vanish at any point of K.

Put $c := \min\{|P_1(z)| : z \in K\} > 0$. Since $P_1 \circ f \in \mathcal{O}(\mathbb{C}^n)$ there exists a polynomial $P: \mathbb{C}^n \to \mathbb{C}$ such that $P(z_0) = P_1(f(z_0)) = 0$ and $\|P - P_1 \circ f\|_{f^{-1}(K)} < c$. Polynomial P does not vanish at any point of the set L, and this finishes the proof. □

LEMMA 1. *Any rationally convex compact set has a fundamental system of neighborhood consisting of rational polyhedra.*

Proof. Let Ω be an open neighborhood of a compact, rationally convex set K. Since K is compact there exists a rational number ρ such that $K \subset \Delta^m(\rho) := \{z \in \mathbb{C}^m : |z_i| < \rho \quad \text{for} \quad i = 1, \ldots, m\}$. From the rational convexity of K it follows that for each $\zeta \in \bar{\Delta}^m(\rho) \setminus \Omega$ there exists a rational function $R_\zeta \in \mathbb{C}(z)$ such that $R_\zeta|_{K \cup \{\zeta\}} \in \mathcal{R}(K \cup \{\zeta\})$ and $|R_\zeta(\zeta)| > 1 > \|R_\zeta\|_K$.

The collection of open sets

$$\{\Omega_\zeta := \{z \in \mathbb{C}^m : |R_\zeta(z)| > 1\}\}_{\zeta \in \bar{\Delta}^m(\rho) \setminus \Omega}$$

gives an open covering of the compact set $\bar{\Delta}^m(\rho) \setminus \Omega$. Let $\Omega_{\zeta_1}, \ldots, \Omega_{\zeta_r}$ be a finite subcovering. In this situation we get

$$K \subset \{z \in \mathbb{C}^m : |\frac{z_i}{\rho}| < 1, i = 1, \ldots, m \text{ and } |R_{\zeta_j}(z)| < 1, j = 1, \ldots, r\} \subset \Omega,$$

which proves the lemma. □

3. Rational approximations. We start this section with the following version of the Runge Theorem (cf. [Fu],Tw.I.2.1.; [G-R], Th.VII.A.6.)

THEOREM 1. *Let K be a compact, rationally convex subset of $\mathbb{C}^m$. For every holomorphic function $f \in \mathcal{O}(K)$ and every real number $\epsilon > 0$ there exists a rational function $g \in \mathcal{R}(K)$ such that $\|f - g\|_K < \epsilon$.*

Proof. By the definition of $\mathcal{O}(K)$ there exist an open neighborhood Ω of K and a function $\tilde{f} \in \mathcal{O}(\Omega)$ such that $\tilde{f}|_K = f$. By Lemma I.2.1. there exist $R_1, \ldots, R_s \in \mathbb{C}(z)$ such that $K \subset \{z \in \mathbb{C}^m : |R_i(z)| < 1, i = 1, \ldots, s\} \subset\subset \Omega$. The set K is compact so there exists a rational number $\delta > 0$ such that $K \subset \{z \in \mathbb{C}^m : |R_1(z)| < 1 - \delta, \ldots, |R_s(z)| < 1 - \delta\}$. Since the rational functions $R_1, \ldots, R_s$ belong to the class $\mathcal{R}(K)$ there exist polynomials $P_1, \ldots, P_s, Q \in \mathbb{C}[z]$ such that Q does not vanish at any point of K and $R_i = \dfrac{P_i}{Q}$, $i = 1, \ldots, s$. Let

$$\Omega_1 := \{(z, w) \in \mathbb{C}^m \times \mathbb{C} : |w \cdot Q(z) - 1| < \delta, |P_i(z) \cdot w| < 1 - \delta, \quad i = 1, \ldots, s\}.$$

Then for $(z, w) \in \Omega_1$ we have $|\dfrac{P_i(z)}{Q(z)}| < 1$ and consequently $z \in \Omega$. We can define a function $f_1 \in \mathcal{O}(\Omega_1)$ by the formula $f_1(z, w) := \tilde{f}(z)$. The set Ω_1 is a polynomial polyhedron, and the set $\tilde{K} := \left\{\left(z, \frac{1}{Q(z)}\right) : z \in K\right\}$ is its compact subset so by the Runge Theorem for polynomial polyhedra ([G-R], Th.I.G.8.) there exists a polynomial $R \in \mathbb{C}[z, w]$ such that $\|R - f_1\|_{\tilde{K}} < \epsilon$. Defining $g(z) := R\left(z, \frac{1}{Q(z)}\right)$ we can get $g \in \mathcal{R}(K)$ and $\|f - g\|_K < \epsilon$. □

Proof of Remark 2.1(3). Let K_1 and K_2 be two disjoint compact sets such that the sum $K_1 \cup K_2$ is rationally convex. We shall prove that K_1 is rationally convex. Let $z_0 \notin K_1$.

If $z_0 \notin K_2$ then $z_0 \notin K_1 \cup K_2$ hence there exists polynomial $P \in \mathbb{C}[z]$ such that $P(z_0) = 0$ and P does not vanish on $K_1 \cup K_2$, and therefore P does not vanish on K_1.

If $z_0 \in K_2$ then consider the function

$$f : K_1 \cup K_2 \ni z \mapsto f(z) := \begin{cases} 1 & \text{if } z \in K_2. \\ 0 & \text{if } z \in K_1. \end{cases}$$

Since sets K_1 i K_2 are compact and disjoint we have $f \in \mathcal{O}(K_1 \cup K_2)$. Therefore by Theorem 1. there exists rational function $R \in \mathcal{R}(K_1 \cup K_2)$ such that $\|f - R\|_{K_1 \cup K_2} < \frac{1}{2}$. But in this situation $R_1 := R|_{K_1 \cup \{z_0\}} \in \mathcal{R}(K_1 \cup \{z_0\})$ and $|R(z_0)| > \frac{1}{2} > \|R\|_{K_1}$. This proves the remark. □

COROLLARY 1. *If K is a compact, rationally convex subset of $\mathbb{C}^m$ and if $f \in \mathcal{O}(K)$, then $f = \{(z, f(z)) : z \in K\} \subset \mathbb{C}^{m+1}$ is rationally convex, too.*

Proof. Let $(z_0, w_0) \notin f$. We shall consider two cases

(1) $z_0 \notin K$.

Then there exists a polynomial $P \in \mathbb{C}[z]$ such that $P(z_0) = 0$ and P does not vanish on K. The polynomial $P_1 \in \mathbb{C}[z, w]$ defined by the formula $P_1(z, w) = P(z)$ does not vanish on f and $P_1(z_0, w_0) = 0$.

(2) $z_0 \in K$, $w_0 \neq f(z_0)$.

Let $\epsilon := |w_0 - f(z_0)| > 0$. By Theorem 1. there exists a rational function $R \in \mathcal{R}(K)$ such that $\|R - f\|_K < \frac{\epsilon}{3}$. Let $R_1(z, w) := w - R(z)$. We have $R_1 \in \mathcal{R}(f \cup \{(z_0, w_0)\})$; moreover $|R_1(w_0, z_0)| = |w_0 - R(z_0)| > \frac{2}{3}\epsilon > \frac{\epsilon}{3} > \|f - R\|_K = \|R_1\|_f$.

Combining cases (1) and (2) we finish the proof. □

COROLLARY 2. *Let K be a compact, rationally convex subset of $\mathbb{C}^m$ and let V be an algebraic subset of $\mathbb{C}^m$ such that $V \cap K = \emptyset$. Then there exists a polynomial $P: \mathbb{C}^m \to \mathbb{C}$ such that $P = 0$ on V and P does not vanish at any point of K.*

Proof. By Lemma I.2.1. there exists a holomorphically convex open neighborhood Ω of K, disjoint with V. Since the set V is algebraic, there exist polynomials $P_1, \ldots, P_k \in \mathbb{C}[z]$ such that $V = \{z \in \mathbb{C}^m : P_i(z) = 0, i = 1, \ldots, k\}$. Since $V \cap \Omega = \emptyset$, functions $P_1|_\Omega, \ldots, P_k|_\Omega$ has no common zeros. By the B-Cartan Theorem ([G-R], Th. VIII.A.14., Cor. VIII.A.16.) there exist holomorphic functions $g_1, \ldots, g_k \in \mathcal{O}(\Omega)$ such that $g_1P_1 + \cdots + g_kP_k = 1$. Let $\epsilon := (\|P_1\|_K + \cdots + \|P_k\|_K)^{-1}$. By Theorem 1. there exist rational functions $R_1, \ldots, R_k \in \mathcal{R}(K)$ such that $\|g_i - R_i\| < \frac{\epsilon}{2}$. In this situation for any $z \in K$ we get $|R_1(z)P_1(z) + \cdots + R_k(z)P_k(z)| > \frac{1}{2}$.

Since $R_i \in \mathcal{R}(K)$, there exist polynomials $S_1, \ldots, S_k, Q \in \mathbb{C}[z]$ such that Q does not vanish at any point of K and $R_i(z) = \dfrac{S_i(z)}{Q(z)}$ for $z \in K$. Taking $P(z) := S_1(z)P_1(z) + \cdots + S_k(z)P_k(z)$ we get $P = 0$ on V and P does not vanish at any point of K. □

COROLLARY 3. *Let X be an algebraic subset of C^m and let $K \subset X$ be a rationally convex compact set. Then for any function f rational on X, holomorphic at points of the set K there exist polynomials $P, Q \in \mathbb{C}[z]$ such that Q does not vanish at any point of K and $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in K$.*

Proof. Let $V \subset X$ be the set of all point at which function f is not holomorphic. Theorem I.1.1. shows that V is an algebraic set disjoint with K. By Corollary 2. there exists a polynomial $Q_1 \in \mathbb{C}[z]$ such that $Q_1 = 0$ on

V and Q_1 does not vanish on K. The set $\tilde{X} := \{(z,w) \in \mathbb{C}^m \times \mathbb{C}: z \in X \text{ and } Q_1(z) \cdot w = 1\}$ is an algebraic subset of $\mathbb{C}^m \times C$. The function $\tilde{f}: \tilde{X} \ni (z,w) \mapsto f(z) \in \mathbb{C}$ is rational and holomorphic on the affine algebraic set $\tilde{X}$ hence by the Serre Theorem ([L], VII.16.3, p.342) there exists a polynomial $P_1 \in \mathbb{C}[z,w]$ such that $\tilde{f} = P_1|_{\tilde{X}}$. Therefore $f(z) = P_1\left(z, \frac{1}{Q_1(z)}\right)$ for $z \in K$ and consequently

$$\begin{aligned} P(z) :=& P_1\left(z, \frac{1}{Q_1(z)}\right) \cdot (Q_1(z))^{\deg_w P_1}, \\ Q(z) :=& (Q_1(z))^{\deg_w P_1}. \end{aligned}$$

satisfy the assertion. □

4. The rational hull of a compact set. Let K be a compact subset of $\mathbb{C}^m$. Following Remark 2.1. we can formulate

DEFINITION 1. *The rational hull of K, denoted by $\tilde{K}$, is defined to be the smallest rationally convex compact set containing K.*

LEMMA 1.

(i) For each compact set K the map

$$\pi : \mathcal{R}(\tilde{K}) \ni R \mapsto R|_K \in \mathcal{R}(K)$$

is an isomorphism.

(ii) For any two compact sets K_1 and K_2 we have

$$\widetilde{K_1 \times K_2} = \tilde{K}_1 \times \tilde{K}_2.$$

Proof. (i) We first prove that the map π is surjectiv. If $R_1 \in \mathcal{R}(K)$ then there exist polynomials $P, Q \in \mathbb{C}(z)$ such that Q does not vanish on K and $R_1(z) = \dfrac{P(z)}{Q(z)}$ for $z \in K$. Then $c := \min\{|Q(z)|: z \in K\} > 0$, and moreover

$$K_1 := \left\{z \in \tilde{K}: |Q(z)| \geq \frac{c}{2}\right\}$$

is a rationally convex compact set. (If $z_0 \notin K_1$, then $z_0 \notin \tilde{K}$ or $|P(z_0)| > \frac{c}{2}$. In the first case, since $\tilde{K}$ is rationally convex, there exists a polynomial P_1 such that $P_1(z_0) = 0$ and P_1 does not vanish on $\tilde{K}$, and consequently on K_1. In the second case the polynomial $P_2(z) := Q(z) - Q(z_0)$ does not vanish on K_1 and $P_2(z_0) = 0$.)

By the definition of rational convexity $\tilde{K} \subset K_1$ hence Q does not vanish on $\tilde{K}$. Consequently $R := \left(\dfrac{P}{Q}\right)\Big|_{\tilde{K}} \in \mathcal{R}(\tilde{K})$ and $R|_K = R_1$. This proves the surjectivity.

Now, we shall show that π is injective. To prove this we take an arbitrary rational function $R \in \mathcal{R}(\tilde{K})$ such that $R|_K = 0$. There exist polynomials $P, Q \in \mathbb{C}(z)$ such that Q does not vanish on $\tilde{K}$ and $R(z) = \dfrac{P(z)}{Q(z)}$ for $z \in \tilde{K}$, therefore we have $P|_K = 0$. Let us notice that the set $K_1 := \left\{z \in \tilde{K}: P(z) = 0\right\}$ is rationally convex.

(Let $z_0 \notin K_1$. If $z_0 \notin \tilde{K}$ then by the rational convexity of the set $\tilde{K}$ there exists a polynomial $P_1 \in \mathbb{C}[z]$ such that $P_1(z_0) = 0$ and P_1 does not vanish on $\tilde{K}$, and consequently on K_1. If $z_0 \in \tilde{K}$ then $P(z_0) \neq 0$ and polynomial P does not vanish at any point of the set K_1.)

From the definition of rational hull we conclude that $\tilde{K} \subset K_1$, hence that $P|_{\tilde{K}} = 0$, and consequently that $R = 0$.

(ii) Inclusion $\widetilde{K_1 \times K_2} \subset \tilde{K}_1 \times \tilde{K}_2$ is a srtaightforward consequence of the definition of rational hull and Remark I.2.1(5).

The opposite inclusion follows immediately from the obvious equality $\{z_0\} \times \tilde{K}_2 = \widetilde{\{z_0\} \times K_2}$. $\square$

LEMMA 2. *If $\Omega \subset \mathbb{C}^m$ is a rational polyhedron, and if K is a compact subset of Ω then $\tilde{K} \subset \Omega$.*

Proof. Let
$$\Omega = \{z \in \mathbb{C}^m: |R_1(z)| < 1, \ldots, |R_s(z)| < 1\}$$
where $R_1, \ldots, R_s \in \mathbb{C}(z)$. There exist relatively prime polynomials P_i, Q_i such that $R_i = \dfrac{P_i}{Q_i}$ $(i = 1, \ldots, s)$. Since $Q_i(z) \neq 0$ for any point $z \in K$, we have (as in the proof of Lemma 1.) $Q_i(z) \neq 0$ $(i = 1, \ldots, m)$ for any point $z \in \tilde{K}$. Define $r_i := \|R_i\|_K$, $(i = 1, \ldots, s)$. The set
$$K_1 := \{z \in \tilde{K}: |R_i(z)| \leq r_i\}$$
is compact.

Now, we shall prove that the set K_1 is rationally convex. In order to do this let us choose an arbitrary point $z_0 \notin K_1$. If $z_0 \notin \tilde{K}$ then from the rational convexity of $\tilde{K}$ it follows that there exists a rational function $R \in \mathcal{R}\left(\tilde{K} \cup \{z_0\}\right)$ such that $|R(z_0)| > \|R\|_{\tilde{K}}$. If $z_0 \in \tilde{K}$ then by the definition of the set K_1 there exists an index $i_0 \in \{1, \ldots, s\}$ such that $|R_{i_0}(z_0)| > r_{i_0}$. In this situation the function $R_{i_0}|_{K_1} \cup \{z_0\} \in \mathcal{R}(K_1 \cup \{z_0\})$ satisfies the condition (ii) of the definition of rational convexity.

Let us observe that $K \subset K_1$ hence that by the definition of rational convexity $\tilde{K} \subset K_1 \subset \Omega$, which completes the proof. $\square$

CHAPTER II
Nash Functions

1. Definitions and basic properties. In this section we shall present some basic properites of Nash functions. We shall start with the following definition

DEFINITION 1. *Let Ω be an open subset of $\mathbb{C}^m$ and let $z_0 \in \Omega$. We say that a holomorphic function $f: \Omega \to \mathbb{C}$ is a Nash function at z_0 if there exist open neighborhood U of z_0, $U \subset \Omega$, a polynomial $P: \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$, $P \not\equiv 0$, such that $P(z, f(z)) = 0$ for $z \in U$. A holomorphic function defined on Ω is said to be a Nash function iff it is a Nash function at every point of Ω. The family of all Nash function defined on Ω we denote by $\mathcal{N}(\Omega)$.*

The restrictions of rational functions which are holomorphic on a fixed open set Ω give examples of Nash functions on Ω. Therefore we have the following inclusions

$$\mathcal{R}(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{O}(\Omega).$$

Assuming that Ω is connected, we can give a more precise description of the class of Nash functions on Ω ([T], Remark 1.2 p. 228)

REMARK 1. *Let D be a connected, open subset of $\mathbb{C}^m$, and let x_0 be a fixed point of D. If f is a holomorphic function defined on D then the following conditions are equivalent:*

(1) f is a Nash function at x_0,

(2) $f \in \mathcal{N}(D)$,

(3) there exists a proper algebraic subset X of $\mathbb{C}^m \times \mathbb{C}$ such that $f = \{(z, f(z)) : z \in D\} \subset X$,

(4) there exists a unique irreducible algebraic hypersurface $X \subset \mathbb{C}^m \times \mathbb{C}$ such that $f \subset X$,

(5) there exists an irreducible polynomial $P: \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$, unique up to scalars, such that $P(x, f(x)) = 0$ for $x \in D$.

Moreover, it can be seen that X in (4) is equal to the Zariski closure $\bar{f}^{\mathrm{Z}}$ of f in $\mathbb{C}^m \times \mathbb{C}$.

DEFINITION 2. *Let Ω be an open subset of $\mathbb{C}^m$ and let z_0 be a fixed point of Ω. A holomorphic mapping $F\colon \Omega \to \mathbb{C}^n$ is said to be a Nash mapping at z_0 iff all components of F a Nash functions at z_0. A mapping F is said to be a Nash mapping if it is a Nash mapping at every point of Ω.*

In this paper we shall use only the simplest properties of Nash functions. For more deteils (and proofs of quoted theorem) we refer the reader to [T].

THEOREM 1. ([T], Th. 1.10.) *The composition of Nash mapping is a Nash mapping too.*

THEOREM 2. ([T], Cor. 1.11.) *If Ω is an open subset of $\mathbb{C}^m$ then $\mathcal{N}(\Omega)$ is a subring of the $\mathcal{O}(\Omega)$ of holomorphic on Ω.*

THEOREM 3. ([T], Cor. 1.12.) *If Ω is an open subset of $\mathbb{C}^m$, $f \in \mathcal{N}(\Omega)$, then*

$$\frac{\partial f}{\partial x_i} \in \mathcal{N}(\Omega) \quad \text{for } i = 1, \dots, m.$$

2. Simple Nash functions . Let D be $\mathbb{C}^m$ and let $g \in \mathcal{N}(D)$. The set $X_g := \bar{g}^{\mathbf{z}} \cap (D \times \mathbb{C})$ is an analytic subset of $D \times \mathbb{C}$, of pure dimension m. Since g is an irreducible m-dimensional analytic subset of $D \times \mathbb{C}$ contained in X_g, it is an irreducible component of X_g. Let Y_g be the union of the other components of X_g.

DEFINITION 1. *A function $g \in \mathcal{N}(D)$ is said to be a simple Nash function if $g \cap Y_g = \emptyset$. We denote by $\mathcal{SN}(D)$ the family of all simple Nasha functions on D.*

Since $g = \{(z, g(z)) : z \in D\}$ is a complex manifold, we see that $g \cap Y_g = \emptyset$ if and only if each point of g is a regular point of the algebraic set $\bar{g}^{\mathbf{z}}$, and so

$$\mathcal{SN}(D) = \{g \in \mathcal{N}(D) : g \subset \operatorname{Reg}(\bar{g}^{\mathbf{z}})\}.$$

LEMMA 1. *Let D be an open connected subset of $\mathbb{C}^m$, $R \in \mathcal{R}(D)$ and $g \in \mathcal{N}(D)$. If $F_R\colon D \times \mathbb{C} \ni (z, w) \mapsto (z, w + R(z)) \in D \times \mathbb{C}$, then*

$$X_{g+R} = F_R(X_g) \qquad \text{and} \qquad Y_{g+R} = F_R(Y_g).$$

Moreover, if $g \in \mathcal{SN}(D)$ then $g + R \in \mathcal{SN}(D)$.

Proof. It follows from Remark II.1.1. that there exists a polynomial $P\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that $(z, w) \in \bar{g}^{\mathbf{z}}$ if and only if $P(z, w) = 0$. Since $R \in \mathcal{R}(D)$, it follows that there exist polynomials $Q_1, Q_2 \in \mathbb{C}[z]$ such that Q_2 does not vanish on D and $R(z) = \dfrac{Q_1(z)}{Q_2(z)}$ for $z \in D$. The function $P_1(z,w) := P(z, w - R(z)) \cdot (Q_2(z))^{\deg P}$ is a polynomial and moreover $P(z, g(z) + R(z)) = 0$ for $z \in D$. Therefore

$$X_{g+R} \subset \{(z,w) \in D \times \mathbb{C} : P_1(z,w) = 0\} = F_R(X_g).$$

Now, fix R and g. Suppose on the contrary that $X_{g+R} \subsetneqq F_R(X_g)$. Then $X_g = X_{(g+R)+(-R)} \subset F_{-R}(X_{g+R}) \subsetneqq F_{-R}(F_R(X_g)) = X_g$ which is impossible, and, so $X_{g+R} = F_R(X_g)$. The mapping F_R is a biholomorphism, so it maps irreducible components of X_g onto irreducible components of X_{g+R}. And consequently from $F_R(g) = g + R$ we see that $Y_{g+R} = F_R(Y_g)$.

If $g \in {}^{\circ}\mathcal{N}(D)$ then, by definition, $g \cap Y_g = \emptyset$. We have $(g + R) \cap Y_{g+R} = F_R(g) \cap F_R(Y_g) = F_R(g \cap Y_g) = \emptyset$, hence $g + R \in {}^{\circ}\mathcal{N}(D)$, and the proof is complete. $\square$

LEMMA 2. *Let K be a compact, rationally convex subset of $\mathbb{C}^m$, D an open, connected neighborhood of K, and let $f \in \mathcal{N}(D)$, $g \in {}^{\circ}\mathcal{N}(D)$. If $P_1, Q_1 : \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ are polynomials such that $\dfrac{P_1(z,g(z))}{Q_1(z,g(z))} = f(z)$ on a dense subset of D then there exist polynomials $P, Q : \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that for every $z \in K$:*

1) $Q(z, g(z)) \neq 0,$

2) $f(z) = \dfrac{P(z,g(z))}{Q(z,g(z))}.$

Proof. Let X be the Zariski closure of g in $\mathbb{C}^m \times \mathbb{C}$. Consider the rational function on X defined by $\dfrac{P_1}{Q_1}$. This function is holomorphic at each point of the set g. The Lemma follows from Corollaries I.3.1. and I.3.3. $\square$

The lemma is no longer true if it is only assumed that g is a Nash function ($g \in \mathcal{N}(D)$).

EXAMPLE 1. Let $D = B(0, \frac{1}{4})$, and let $f(z) := 1 - \sqrt{1-4z}$ where $\sqrt{1-4z}$ is a holomorphic branch of the square root of $1 - 4z$ in D such that $f(0) = 0$. Define $P_1(z,w) := w$, $Q_1(z,w) := z$, $K := \bar{B}(0, \frac{1}{5})$ and $g(z) := z \cdot f(z)$ for $z \in D$. Then

$$\frac{P_1(z,g(z))}{Q_1(z,g(z))} = \frac{z \cdot f(z)}{z} = f(z),$$

in $D \setminus \{0\}$.

Suposse that there exist polynomials P, Q such that

$$f(z) = \frac{P(z, g(z))}{Q(z, g(z))},$$

for $z \in K$. Without loss of generality we can assume that $Q(0,0) = 1$. Since $P(0,0) = 0$, there exist polynomials $R_1, R_2, R_3, R_4 \in \mathbb{C}[z]$ such that

$$P(z, g(z)) = zR_1(z) + z\left(1 - \sqrt{1-4z}\right) R_2(z)$$

and

$$Q(z, g(z)) = 1 + zR_3(z) + z\left(1 - \sqrt{1-4z}\right) R_4(z)$$

$z \in \bar{B}\left(0, \frac{1}{5}\right)$. Consequently we get

$$\left(1 - \sqrt{1-4z}\right) = \frac{zR_1(z) + z\left(1 - \sqrt{1-4z}\right) R_2(z)}{1 + zR_3(z) + z\left(1 - \sqrt{1-4z}\right) R_4(z)}, \quad \text{for} \quad z \in \bar{B}\left(0, \frac{1}{5}\right).$$

From the above equation we get

$$\left(1 - \sqrt{1-4z}\right)\left(1 + zR_3(z) + 2zR_4(z) - R_2(z)\right) = zR_1(z) + 4z^2R_4(z),$$

and so $1 - \sqrt{1-4z} \in \mathbb{C}(z)$ which is impossible.

3. Resolution of singularities of a graph of Nash function. The aim of this section is to give a special characterization of Nash function.

LEMMA 1. *Let E be a rationally convex compact set, and let D and G be open, connected subset of $\mathbb{C}^m$ such that $\emptyset \neq G \subset E \subset D$. If $f \in \mathcal{N}(D)$, then exist a function $g \in \mathcal{N}(G)$ and polynomials $P, Q\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that $f(z) = \dfrac{P(z, g(z))}{Q(z, g(z))}$ on a dense subset of G.*

Proof. Let us denote by X the Zariski closure $\bar{f}^{\mathbf{Z}}$ of f in $\mathbb{C}^m \times \mathbb{C}$. By Remark II.1.1. there exists a unique (up to constant) irreducible polynomial $W\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$, $W \neq 0$ such that $X = \{(z, w) \in \mathbb{C}^m \times \mathbb{C}\colon W(z, w) = 0\}$. Let $W(z, w) = \sum_{i=0}^{k} a_i(z) w^i$ $(a_k \neq 0)$.

Observe that without loss of generality, we can assume that $a_k = 1$. (Otherwise, instead of f we consider function $f_1 = a_k \cdot f$, which has the desired property. If we have the representation for the function f_1 then the one for f we shall get by dividing by a_k.) Under this assumption the hypersurface X satisfies the following additional property: the resriction $\pi_X := \pi|_X$ to X of the natural projection $\pi\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ is a proper mapping.

Let the couple $(\tilde{X}, \tau)$ consisting of algebraic set $\tilde{X} \subset \mathbb{C}^r$ and proper, regular and birational mapping $\tau: \tilde{X} \to X$ be a normalisation of X as in ([Ł], VII.16.5., p. 347, Prop. 4.) Since the mapping $\pi_X \circ \tau$ is proper, it follows that $F := \tau^{-1}\left(\pi_X^{-1}(E)\right)$ is a compact, rationally convex set. Using the notation of the previous section we see that f is a irreducible component of $X_f = X \cap (D \times \mathbb{C})$.

As

$$\tau|_{\tau^{-1}(X_f)}: \tau^{-1}(X_f) \to X_f$$

is a normalisation of X_f, it follows that ([Ł], Prop. 1, p. 255) there exists an open–closed subset H of $\tau^{-1}(X_f)$ such that

$$\tau|_H: H \to f$$

is a biholomorphism.

The set $F_0 := H \cap F$ is an open–closed subset of F, so defining

$$\chi: F \ni x \mapsto \chi(x) = \begin{cases} 0, & \text{if } x \in F_0, \\ 1, & \text{if } x \notin F_0, \end{cases}$$

we get $\chi \in \mathcal{O}(F)$. By Theorem I.3.1. there exists $\chi_1 \in \mathcal{R}(F)$ such that $\|\chi - \chi_1\|_F < \frac{1}{3}$. Let χ_2 be a rational function on $\tilde{X}$ such that $\chi_2|_F = \chi_1$.

$\Psi := (\pi_X \circ \tau, \chi_2)$ is a rational map between $\tilde{X}$, and an irreducible algebraic hypersurface $X_1 \subset \mathbb{C}^m \times \mathbb{C}$. Since for any $y \in H \cap \tau^{-1}(X \cap (G \times \mathbb{C})) \subset F_0$ we have $\Psi^{-1}(\Psi(y)) = \{y\}$, it follows from ([K], Th. III.11.1.) that the mapping Ψ is birational.

From the definition of χ_2 we see that for the holomorphic function

$$g: G \ni z \mapsto \chi_2\left((\tau|_H)^{-1}((z, f(z)))\right) \in \mathbb{C},$$

he have

$$g = \left(G \times B\left(0, \frac{1}{3}\right)\right) \cap X_1,$$

and consequently $g \in \mathcal{N}(G)$.

Consider the rational mapping between X_1 *and* X defined by

$$\Phi := \tau \circ \Psi^{-1}.$$

From the above constructions it follows that $\Phi(z, w) = (z, \phi(z, w))$, where ϕ is a rational function on X_1 such that

$$f(z) = \phi(z, g(z))$$

for $z \in G$.

There exist polynomials $P, Q\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that $\phi = \frac{P}{Q}$. Consequently, we get

$$f(z) = \frac{P(z, g(z))}{Q(z, g(z))}$$

on a dense subset of G (out of the zeros of the denominator), which is the desired conclusion. □

LEMMA 2. *Let E be a compact, rationally convex set and let D and G be connected open subset of $\mathbb{C}^m$ such that $\emptyset \neq G \subset E \subset D$. Let a be a fixed point of G. If $f \in \mathcal{N}(D)$ then there exist a function $g \in \mathcal{N}(G)$ and polynomials $P, Q\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that*

(1) $\quad g(a) = 0,$

(2) $\quad g(G) \subset U,$

(3) $\quad \bar{g}^{\mathbf{z}} \cap (G \times \bar{U}) = g|_G,$

(4) $\quad Q^{-1}(0) \cap (G \times \bar{U}) = \emptyset,$

(5) $\quad f(z) = \dfrac{P(z, g(z))}{Q(z, g(z))} \quad$ *for* $\quad z \in G.$

Proof. By Lemmas I.2.1. and I.4.2. there exist an open neighborhood Ω of E and a compact, rationally convex set E_1 such that $\Omega \subset E_1 \subset D$. Let D_1 be the component of Ω which contains G. Since, by Remark I.2.1., $D_1 \cap E$ is a rationally convex set, without loss of generality we can assume that $E \subset D_1$.

Lemma II.3.1. shows that there exists a function $g_1 \in {}^{\circ}\mathcal{N}(D_1)$ and polynomials $P_1, Q_1\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that $f(z) = \dfrac{P_1(z, g_1(z))}{Q_1(z, g_1(z))}$ on a dense subset of D_1, whereas from Lemma II.2.2. we see that there exist polynomials $P_2, Q_2\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that for $z \in E$ we have $Q_2(z, g_1(z)) \neq 0$ and $f(z) = \dfrac{P_2(z, g_1(z))}{Q_2(z, g_1(z))}$.

The set E is compact, $g_1 \cap Y_{g_1} = \emptyset$ and $g_1|_E \cap Q_2^{-1}(0) = \emptyset$, so there exists $d > 0$ such that $\operatorname{dist}(g_1|_E, Y_{g_1} \cup Q_2^{-1}(0)) \geq 2d$. By Theorem I.3.1. there exists a rational function $R \in \mathcal{R}(E)$ such that $R(a) = g_1(a)$ and $\|g_1 - R\|_E < d$. Taking

$$g := \frac{1}{d}(g_1 - R), \qquad \phi := \frac{P_2(z, dw + R(z))}{Q_2(z, dw + R(z))},$$

we get $g \in \mathcal{N}(G)$ and moreover:

(a) $g(a) = 0,$

(b) $|g(z)| < 1$ for $z \in G$,

(c) $|w| > 1$ for $(z, w) \in Y_g, \quad z \in G$.

But $\phi \in \mathcal{R}(E \times \bar{U})$ hence there exist polynomials $P, Q: \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ such that

(d) $Q^{-1}(0) \cap (G \times \bar{U}) = \emptyset$,

(e) $f(z) = \dfrac{P(z, g(z))}{Q(z, g(z))}$ dla $z \in G$.

The proof of the lemma is complete. □

CHAPTER III
Representations of Nash functions.

1. The operator S . Let K be a fixed compact subset of $\mathbb{C}^m$. We shall use the following notation

$$\mathcal{N}(K) := \Big\{ f\colon K \to \mathbb{C}\text{: there exists open neighborhood } V \text{ of } K$$
$$\text{and a function } \tilde{f} \in \mathcal{N}(V) \text{ such that } \tilde{f}|_K = f \Big\}.$$

In this section we shall consider the operator

$$S: \mathcal{O}(K \times T) \longrightarrow \mathcal{O}(K),$$

defined by $S(f) := a_0$, where $f(z, w) = \sum_{n \in \mathbb{Z}} a_n(z) w^n, \quad a_n \in \mathcal{O}(K)$. This operator admits the following integral representation:

$$S(f)(z) = \frac{1}{2\pi \mathrm{i}} \int_T \frac{f(z, w)}{w} dw, \ z \in K.$$

In the same way as in [C-T] (in the case of m=1) one can prove that for every compact connected subset K of $\mathbb{C}^m$ the following inclusion holds

$$S(\mathcal{R}(K \times T)) \subset \mathcal{N}(K).$$

In the next section we shall give detailed proof of this inclusion and investigate the converse one.

2. Representations of Nash functions . The main result of this section is

THEOREM 1. *If K is a compact, connected and rationally convex subset of $\mathbb{C}^m$ then*

$$S\left(\mathcal{R}(K\times T)\right)=\mathcal{N}(K).$$

Proof. Let us choose an arbitrary function $f\in\mathcal{R}(K\times T)$. There exist polynomials $P,Q\colon\mathbb{C}^m\times\mathbb{C}\to\mathbb{C}$ such that Q does not vanish at any point of $K\times T$ and $\dfrac{P(z)}{Q(z)}=f(z)$ for $z\in K\times T$ and an open, connected neighborhood D of K such that $Q^{-1}(0)\cap\left(\bar{D}\times T\right)=\emptyset$. Put $\tilde{f}(z):=\dfrac{P(z)}{Q(z)}$ for $z\in\bar{D}\times T$.

There exist a non-empty open subset D_1 of D and Nash functions $\Phi_1,\ldots,\Phi_k\in\mathcal{N}(D_1)$ with pair-wise disjoint graphs such that

$$\{(z,w)\in D_1\times U\colon Q(z,w)\cdot w=0\}=\Phi_1\cup\cdots\cup\Phi_k,$$

where U denotes unit disc in $\mathbb{C}$.

From the above equality and definition of S we get

$$S\left(\tilde{f}\right)(z)=\sum_{j=1}^{k}\frac{1}{N!}\frac{\partial^N}{\partial w^N}\left(\left(w-\Phi_j(z)\right)^{N+1}\frac{P(z,w)}{wQ(z,w)}\right)(z,\Phi_j(z)),\ \ z\in D_1$$

where N is sufficiently large integer. From Theorems II.1.1., II.1.2. and II.1.3. we get $S\left(\tilde{f}\right)|_{D_1}\in\mathcal{N}(D_1)$, hence, by Remark II.2.1, $S\left(\tilde{f}\right)|_{D}\in\mathcal{N}(D)$, and consequently $S(f)=S\left(\tilde{f}\right)|_K\in\mathcal{N}(K)$.

Now, let $g\in\mathcal{N}(K)$. There exists a connected, open neighborhood D of K and Nash function $\tilde{g}\in\mathcal{N}(D)$ such that $g=\tilde{g}|_K$. From Lemmas I.2.1. and I.4.2. it follows that there exist connected, open neighborhood G of K and rationally convex compact set E such that $G\subset E\subset D$. From Lemma II.3.2. we get

$$\tilde{g}(z)=\frac{P\left(z,h(z)\right)}{Q\left(z,h(z)\right)}\qquad\text{for}\qquad z\in G$$

(where P,Q,h satisfy assertion of this Lemma).

Let R be an irreducible polynomial describing the graph of h (cf. Remark II.1.1.). As $h(z)$ is the only zero ("with multiplicity one ") in U of the entire function

$$\mathbb{C}\ni w\mapsto R(z,w)\in\mathbb{C}$$

we have

$$\tilde{g}(z) = \frac{1}{2\pi i}\int_T \frac{P(z,w)}{Q(z,w)} \cdot \frac{R_w(z,w)}{R(z,w)} dw, \qquad z \in G.$$

Taking

$$F(z,w) := w \cdot \frac{P(z,w)}{Q(z,w)} \cdot \frac{R_w(z,w)}{R(z,w)}, \qquad (z,w) \in K \times T$$

we get $F \in \mathcal{R}(K \times T)$, $S(F) = g$ and consequently $g \in S(\mathcal{R}(K \times T))$, which proves the theorem. □

COROLLARY 1. *If K is compact and connected subset of $\mathbb{C}^m$, then*

$$S(\mathcal{R}(K \times T)) = \left\{ h|_K : h \in \mathcal{N}\left(\tilde{K}\right)\right\} \subset \mathcal{N}(K),$$

where $\tilde{K}$ is the rational hull of K.

Proof. Let $g \in S(\mathcal{R}(K \times T))$. There exists a rational function $R \in \mathcal{R}(K \times T)$ such that $S(R) = g$. Lemma I.3.1. shows that $R = R_1|_{K\times T}$ for a rational function $R_1 \in \mathcal{R}\left(\widetilde{K \times T}\right) = \mathcal{R}\left(\tilde{K} \times T\right)$.

Define $h := S(R_1)$, Theorem 1. gives $h \in \mathcal{N}(\tilde{K})$ and $g = h|_K$.

In order to prove the converse inclusion assume that $g \in \mathcal{N}(K)$ and there exists a Nash function $h \in \mathcal{N}(\tilde{K})$ such that $h|_K = g$. Theorem 1. gives $h = S(R_1)$ for a rational function $R_1 \in \mathcal{R}\left(\tilde{K} \times T\right)$. Taking $R := R_1|_{K\times T}$ we get $R \in \mathcal{R}(K \times T)$ and $g = S(R)$, and the prove is complete. □

The following example proves that Theorem 1. is no longer true when we drop the assumption that K is rationally closed.

EXAMPLE 1. Let

$$K := \left\{(z,w) \in \mathbb{C}^2 : |z| \le |w| \le 1\right\} \cup \left\{ (1 + \exp(it), 0) : t \in [-\pi, \frac{5}{6}\pi] \right\}$$

and let $g(z,w) = \sqrt{\frac{3}{2} - z}$ for $(z,w) \in K$. Since $g^2(z,w) + (z - \frac{3}{2}) = 0$, we have $f \in \mathcal{N}(K)$.

Observe that $\bar{U}^2 \subset \tilde{K}$. (If $(z_0, w_0) \in \bar{U}^2 \setminus \tilde{K}$ then, by Definition I.2.1., there exists a polynomial P such that $P(z_0, w_0) = 0$ and P does not vanish on $\tilde{K}$. In this situation the function $\frac{1}{P}$ is holomorphic in neighborhood of K, and so has a unique extension to $\bar{U}^2$ — which contradicts $P(z_0, w_0) = 0$.)

Then

$$\{(1+\exp(it),0) : t \in [0,2\pi]\} \subset \tilde{K},$$

hence there is no function $\tilde{g} \in \mathcal{O}\left(\tilde{K}\right)$ such that $\tilde{g}|_K = g$, and it follows from Collorary 1. that $g \notin S(\mathcal{R}(K \times T))$.

Theorem 1 is no longer true if we replace compact set K by an open set and define S in the same way as before

EXAMPLE 2. Let $F := \{z = x + iy \in \mathbb{C} \colon y = \exp x - 1, \quad x \geq 0\}$. Then $\Omega := \mathbb{C} \setminus F$ is a Runge domain biholomorphic with the unit disc.

Let $f(z) = \sqrt{z}$ be a holomorphic branch of the square root in Ω. We shall prove that f is not image under the operator S of any rational function, which is holomorphic in a neighborhood of $\Omega \times T$. Assume that R is the desired rational function, holomorphic in a neighborhood of $\Omega \times T$. In this situation we have $R = \dfrac{P}{Q}$ where P, Q are relatively prime polynomials. Consequently $Q^{-1}(0) \cap (\Omega \times T) = \emptyset$, so $A := \Pi_1 (\{(z,w) \colon Q(z,w) = 0, |w| = 1\}) \subset F$ (where $\Pi_1 \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is defined by $\Pi_1(z,w) = z$).

From the Tarski—Seidenberg Theorem we deduce that A is a semialgebraic subset of F, so it is finite. But $S(R) \in \mathcal{O}(\mathbb{C} \setminus A)$, and hence $S(R) \neq f$.

The following example proves that $\mathcal{R}(K \times T)$ in Theorem 1, cannot be replaced by $\mathcal{N}(K \times T)$.

EXAMPLE 3. Let

$$f(z,w) = \left(1 - \frac{z}{2w}\right)^{-\frac{1}{2}} \left(1 - \frac{w}{2}\right)^{-\frac{1}{2}} \qquad (z,w) \in \mathcal{N}(\bar{U} \times T).$$

Then $f \in \mathcal{N}\left(\bar{U} \times T\right)$, but

$$S(f)(z) = \sum_{n \in \mathbb{N}} \binom{2n}{n}^2 64^{-n} z^n$$

is a transcendental function (cf. [B], p. 159 and [Bo]).

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