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ON THE CLASSIFICATION OF 2-GERBES AND 2-STACKS

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Contents

0 Introduction

There has in recent years been a great renewal of interest in certain aspects of category theory, due to the realization that this theory provided a powerful tool for dealing with such diverse subjects as knot theory and quantum field theory ([Ca], [Mo-Se]). More recently, this in turn has prompted the search for applications of higher categorical structures. The first such higher categorical structure is that of a 2-category (or a slight variant of it known as a bicategory). To state it briefly, a 2-category $\mathscr{C}$ consists of a set of objects $\mathcal{O}$, and for each pair of objects X, $Y \in \mathcal{O}$, of a category $\mathscr{A}r(X,Y)$ of arrows from X to Y satisfying appropriate axioms. For any positive integer $n \geq 2$, the notion of an n-category is defined iteratively, by attaching to every pair of elements X, Y in the set of objects $\mathcal{O}$ an $(n-1)$-category of arrows. Typical examples of such structures are respectively given by the 2-category of (small) categories, and the 3-category of small 2-categories. Recent applications of the notion of a 2-category include Kapranov and Voevodsky's interpretation [K-V] of the Zamolodchikov tetrahedral equations (which are higher analogs of the well known Yang-Baxter equations), and Fischer's work on higher knot theory [Fi].

The aim of the present text is to investigate the sheaf-theoretical and cohomological structures associated to these higher categories. In the case of ordinary categories, the sheaf-theoretical structure which first comes to mind is the naive notion of a "sheaf of categories" $\mathscr{C}$ on a space X. Such a structure consists of a sheaf of objects $\mathcal{O}$ and a sheaf of arrows $\mathscr{A}$ on X, together with source and target maps $s,t\colon \mathcal{A} \longrightarrow \mathcal{O}$, and an identity map $i\colon \mathcal{O} \longrightarrow \mathcal{A}$ satifying the requisite axioms. This defines, functorially in the open set U of X, a category $\mathcal{C}_U$, whose objects and arrows are respectively the sections of $\mathcal{O}$ and of $\mathcal{A}$ above U. While the given sheaf conditions on $\mathcal{A}$ and $\mathcal{O}$ do provide gluing conditions for both objects and arrows in the categories $\mathcal{C}_U$, the gluing axioms for objects obtained in this manner are too restrictive, and quite unnatural from a category-theory point of view. For this reason, it has proved necessary to introduce the concept of a stack on X. This is defined to be a sheaf of categories endowed with a strengthened gluing axiom for objects. Stacks are fairly familiar, as they play an important role in algebraic geometry, where they provide the most appropriate framework for the theory of moduli spaces ([De-Mu], [L-M]).

The corresponding sheaf-theoretic notions which may be built from 2- and, more generally, from n–categories, (and which are known respectively as 2- and n-stacks) are generally considered to be much more exotic. Their importance was emphasized by Grothendieck in his text which examined the relationship between homotopy theory and topos theory [Gr]. It was also observed by Deligne that an understanding of n–stacks would be necessary if one was to interpret geometrically the higher Chern class terms appearing in the Riemann-Roch formula, along the lines of his discussion in [Del 4] for the terms involving the first Chern class. More recently, Brylinski and McLaughlin [Br-M] have interpreted certain degree 4 characteristic classes for a Lie group in a similar geometric manner, and explored the implications of their construction in conformal field theory. Various sorts of higher level stacks have also appeared elsewhere in the litterature, in a variety of contexts ([Fr], [Ka]).

A drawback in considering n-stacks for $n \geq 3$ is the fact that the presently available explicit definitions of higher level categories are very complicated (we refer to [G-P-S], and [Le] for a discussion of the appropriate axioms in the case of tricategories). No such obstacle exists, however, in the case of 2–stacks. The definition of a 2-category is well understood, and its constituents can readily be represented by diagrams. While we will at times discuss higher stacks, the main aim of the present work is to provide a complete set of cohomological invariants for a 2-stack whose arrows and the 2-arrows are invertible in an appropriate sense. We will refer to such 2-stacks as 2-stacks in 2–groupoids. In the case of ordinary stacks (or 1-stacks), the analogous problem of determining such a set of invariants rapidly reduces to the problem of defining non-abelian degree 2 sheaf cohomology, and of describing in geometric terms those objects which such a cohomology set classifies. These geometric objects were defined by Giraud in [Gi] under the name of gerbes on X, and have been useful in a variety of situations ([De-Mi], [Bry]). At about the same time as these gerbes were being defined by Giraud, Dedecker introduced, mainly in the more restrictive context of group cohomology, certain explicit 2-cocycles with values in a non-abelian group G [Ded 1]. An important feature of Dedecker's theory is the fact that the coefficients of his cohomology theory are not really determined by the groups G themselves, but rather by certain length one complexes of groups $G \longrightarrow \Pi$ satisfying some additional conditions, and which are known as crossed modules. The relationship between the geometric approach to H^2 of Giraud and the cocyclic approach of Dedecker was first discussed, in the abelian case, in [Gi], where it was shown how to associate to a so-called abelian gerbe an ordinary abelian-valued Čech 2-cocycle. In the general (non-abelian) situation, the relation between these two aspects of the theory was worked out in [Br 2], [Br 4]. As in Dedecker's theory, the non-abelian G-valued degree 2 cocycles which are associated to a gerbe on X are to be interpreted as degree 1 cocycles taking their values in appropriate crossed modules.

The question which concerns us here is the corresponding classification problem for 2-stacks. As in the case of 1-stacks, this problem rapidly reduces to a problem in non-abelian cohomology. This consists in defining non-abelian degree 3 cohomology sets, and in determining the geometric objects which these sets classify. We give here a complete solution of this problem in a very general, sheaf theoretic, context. A first attempt at an explicit cocyclic description of a non-abelian H^3 goes back, in a cohomology of groups situation, to [Ded 2] (where the coefficients for the theory were however chosen in an overly restrictive manner). On the geometric side, it had been noticed by Duskin [Du 1] that the geometric objects which degree 3 cohomology classifies are fibered 2-categories satisfying appropriate conditions. We gave a definition of such objects in [Br 3] 4.1, where we called them 2-gerbes. We also observed there that one could associate a particular class of 2-gerbes, which we called the class of $\mathcal{G}$–2-gerbes, to any given stack of monoidal group-like groupoids (or *gr*–stack) $\mathcal{G}$ on a space X.

While it is possible to give a cohomological description of the full set of equivalence classes of arbitrary 2-gerbes on X, the set of equivalence classes of these $\mathcal{G}$-2-gerbes on X has a particularly pleasant interpretation. Once more, this is to be interpreted as an H^1, but now with values in a somewhat complicated coefficient object. To be a little more precise, let us say that the appropriate coefficient object for such a theory is the "crossed module of *gr*-stacks" defined by the inner conjugation functor $i: \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G})$ from the given *gr*–stack $\mathcal{G}$ to the *gr*–stack $\mathcal{E}q(\mathcal{G})$ of its self-equivalences. More restrictive coefficients for a theory of the non-abelian H^3 are provided by the crossed squares of Loday [Lo], or by the length 2 non-abelian complexes of groups defined by D. Conduché in [Co]. A first illustration of such a theory of the non-abelian H^3 is provided by the problem of classifying extensions of *gr*–categories and of *gr*–stacks. This was solved in [Br 3], where it was shown that the classes of extensions of the discrete *gr*–stack $\underline{K}$ associated to a sheaf of groups K on X by a *gr*–stack $\mathcal{G}$ on X are classified by the cohomology set $H^1(BK, \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G}))$ associated to the classifying space BK of K. It follows from the previous discussion that such extensions therefore correspond to $\mathcal{G}$-2-gerbes on BK. We refer to *op. cit.*, for an explicit description of the non-abelian 3-cocycles associated to such extensions, which generalizes Schreier's well-known description of ordinary group extensions in terms of non-abelian 2-cocycles.

Let us now describe in some detail the contents of the present text. We begin by examining the gluing conditions embodied in the concepts of an n–stack. We then review (§2) the definition of a gerbe on a space X, and give an alternate description of such a gerbe in terms of 2-cocycles. While we already dealt with these questions in [Br 2] and [Br 4] §5, we have found it necessary to return to this topic here, since it paves the way for our subsequent study of 2-gerbes. We have chosen to carry out this discussion in terms of traditional covers of the space X by open sets, instead of working in the more general context of Grothendieck topologies. We hope that this choice of a somewhat more limited framework will make the theory accessible to a wider readership, even though it is with Grothendieck topologies that one often has to deal, both in the context of algebraic geometry and in that of topos theory. Let us however emphasize that the entire discussion carried out here remains valid, without change, in the wider context. Indeed, in order for our results to be directly applicable to the general situation, we have not restricted ourselves, as previous authors, to the Čech cohomology case, a framework which would have been quite adequate (under a paracompactness assumption on X) in dealing with the cohomology of ordinary topological spaces. We work here instead, despite the numerous complications which this entails, with the more general (derived functor) cohomology, which must be described in terms of hypercovers of the space X, rather than in terms of ordinary open covers of X. The "calculus of cocycles" which is then required is already quite intricate at the level of degree 2 cohomology considered in this preliminary section. We have however chosen to work it out in some detail, since we have found this to be quite enlightening. This section ends with three propositions (2.11-2.14) which clarify the relationship between the three related notions of a gerbe, a G–gerbe and an abelian G–gerbe.

Our study of 2-gerbes begins with an examination of the various related conditions by which these 2-gerbes may be defined (§ 3). The two subsequent paragraphs contain the main results of the present work. There we examine, both in the Čech (proposition 4.6 and theorem 5.6) and in the more general hypercover case (proposition 4.10 and theorem 5.9), the manner in which non-abelian 3-cocycles relate to locally trivialized 2-gerbes on a space X. This discussion generalizes the study of extensions of gr–stacks carried out in [Br 3] in several respects. First of all, we are no longer examining here the classification problem for 2-gerbes on a classifying space BK of a group K, but rather for 2-gerbes defined on an arbitrary space X. Secondly, it is now arbitrary 2-gerbes which are under discussion, and no longer the more restricted class of $\mathcal{G}$-2-gerbes associated to a given gr–stack $\mathcal{G}$ on X, as in the classification problem for extensions of gr–stacks. Finally, the present discussion is much less abstract than that of *op. cit.*, and while the construction of a 2-gerbe presented here is not essentially different from the one which was carried out there, it does yield a much more explicit method for building a 2-gerbe out of the associated cohomological data. The price to pay for such explicitness is the intricacy of the corresponding calculus of cocycles, and the reader is advised to begin by studying the much simpler Čech case, in which the main features of the theory are already apparent. We nevertheless hope that the concrete description of an arbitrary 2-gerbe which is presented here will, despite its complexity, be of some use in transfering the incipient theory of n-stacks from the realm of abstract nonsense to that of "applicable mathematics".

The subsequent sections are devoted to various applications of the theory of the non-abelian H^3. In § 6, we carry out the construction of the 2-gerbe of realizations of a given lien L on a space X. This 2-gerbe represents an abelian degree 3 cohomology class, which embodies the obstruction to the existence of a gerbe $\mathcal{C}$ on X whose lien is isomorphic to the given lien L. It represents a vast generalization of the classical MacLane invariant describing, for any pair of groups G and H, the obstruction to the realization of an H-valued abstract kernel $\varphi: G \longrightarrow Out(H)$ by a group extension of G by H. A cocyclic description of this obstruction had been obtained by Giraud in [Gi] VI §2, and a more explicit one, in terms of hypercover cocycles, was given by Duskin in [Du 1] II §6. Here we interpret Giraud's construction in fully geometric terms, by introducing the 2-gerbe on X which embodies the obstruction 3-cocycle in question.

Another topic discussed here (§7), in preparation for the classification of 2-stacks, is the problem of representing in geometric terms the cohomology class associated to an extension of groups

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1. \tag{0.1}$$

in a topos. As we have already said, it is known in the abstract case since Schreier's time that such extensions are classified by the cohomology set which we would now denote by $H^1(BK, G \longrightarrow Aut(G))$. It is not, however, all that easy to describe in geometric terms the "Schreier gerbe" on BK which such an extension represents. The most efficient way of describing the cocycle associated to the extension (0.1) is to consider the fibration

$$BG \longrightarrow BH \longrightarrow BK. \tag{0.2}$$

induced by the extension. The invariant for this fibration (which is somewhat exotic, as it has both a non simply-connected base BK and non simply-connected fiber BG) encompasses the full structure of the extension (0.1). Indeed, the extension (0.1) can essentially be retrieved from the fibration (0.2) by applying the loop functor. However, such a description of a group extension is somewhat unsatisfactory, since it is topological rather than geometrical. A first possible geometric approach to a description of the cohomological data defined by the group extension (0.1) occurs in [Gi] VIII 7.2, under the name of "extension of the topos BK". Here we prefer to pass from the various elements of the fibration (0.2) to their associated stacks. This means, in more concrete terms, that we replace the fibration (0.2) by the morphism

$$Tors(H) \longrightarrow Tors(K) \tag{0.3}$$

between the stacks of torsors under the groups H and K induced by the given projection of H on K. We view the morphism (0.3) as describing a gerbe $Tors(H)$ defined "above $Tors(K)$". The difficulty here is that $Tors(H)$ is a stack, not a space, and in order to treat such a question properly, it would have been necessary to develop a theory of Grothendieck topologies for categories, with appropriate open sets covering both the set of objects and the set of morphisms in the category, a topic which would have taken us too far afield. Instead, we have preferred to pull back the morphism (0.3) by the canonical map from the classifying space BK to its associated stack $Tors\,(K)$, thereby obtaining an appropriately defined gerbe on BK. We describe the latter gerbe quite explicitly, as a twisted version of the trivial gerbe $Tors\,(G)$ constructed from it by what topologists would call the "Borel construction" associated to the action of K on $Tors\,(G)$ defined by the extension (0.1).

We also begin in §7 our promised classification of 2-stacks, by examining the special case of 2-stacks with a unique object in each fibre, in other words, of stacks endowed with a group law, or *gr*–stacks, on X. This is a topic with which we already dealt with in [Br 3], where we viewed both the classification of *gr*-stacks, and the classification of group extensions, as two special cases of the problem of classifying extensions of *gr*–stacks. Just as we did above in the case of group extensions, we interpret here in geometric terms the cohomological invariants which classify such *gr*–stacks. A related question is that of the classification of *gr*-stacks $\mathscr{C}$ on X, for which the multiplication law posesses some extra commutativity property, such as being braided, or symmetric monoidal (the latter commutativity condition is also sometimes refered to as Picard). Many aspects of such a classification in the *gr*- or Picard cases were certainly known to Grothendieck, even though no specific reference directly attributable to him is available. The simpler cases of a *gr*- or Picard category are treated in detail by his student Mrs Sinh in her thesis [Si] (*see* also [Saa]). In the case of *gr*–categories, this problem is essentially equivalent to the more classical problem of classifying equivalence classes of crossed modules (discussed for example in [K.Br] IV §5 or in [MacL 2]). The intermediate case of braided categories has more recently been studied by Joyal and Street in [J-S], (*see* also [Br 3]). In the context of stacks, rather than of categories, the main reference for such questions remains the text of Deligne ([SGA 4] exposé XVIII), which however only deals with the fully abelian situation of a so-called strict Picard stack. We discuss all the different cases here in a unified setting. Once more, our emphasis in this discussion is on the construction of those geometric objects which represent the cohomological invariants under consideration, and our preferred method for constructing them is an appropriate generalization of the Borel construction. In order to carry out this discussion in sufficient generality, we have preferred to consider general Grothendieck topologies. We have allowed ourselves to speak freely, from this point on, of the topos associated to a given site, and not simply, as heretofore, of the category of sheaves on an ordinary topological space X. Possible references for the definition of Grothendieck topologies and of their associated topoi include [Ar], [SGA 4], [Gi], [Mi]. It must, however, be emphasized that only the most elementary aspects of this theory are required here.

We deal in §8 with the general classification problem for 2-stacks in groupoids on a space X. The key to such a classification is a generalization to the context of stacks of the upside-down Postnikov decomposition for a topological space X with three successive non-trivial homotopy groups (algebraic topologists call such spaces 3-stage Postnikov systems). By "upside-down Postnikov decomposition" we mean the reconstruction of the space X from an Eilenberg-MacLane space by an inductive system of fibrations whose base is at each stage is an Eilenberg-MacLane space (whereas in the ordinary Postnikov decomposition, one prefers to consider the reconstruction of X by a projective system of fibrations whose successive fibers are Eilenberg-MacLane spaces). The analysis of this decomposition rapidly reduces here to the examination of a specific associated 2-gerbe, so that the main ingredients for dealing with this topic are now at hand. It must be emphasized here that this Postnikov decomposition is much richer, and much more elaborate, in the sheaf-theoretical context under discussion here, than in the setting of ordinary algebraic topology.

This text comes to a conclusion with a somewhat informal discussion of commutativity laws on 2-categories and 2-stacks. This is really a chapter in the classification theory of n-stacks, with $n \geq 3$. We allow ourselves here a more informal style than in the earlier parts of our text, and in particular do not worry about how a complete set of axioms for an n–category is defined (in fact, this was also the case for some arguments involving the Borel construction for 3-stacks in §7). Let us, however, emphasize that the results which we obtain here are quite precise, despite the apparent informality of our approach, since they are solidly based on a detailed knowledge of the cohomology of the corresponding Eilenberg-MacLane spaces. Just as there exists a hierarchy of group laws on a stack (or a category), which begins with gr–stacks, and continues through braided stacks to Picard ones, we show here that there exists a similar hierarchy for 2-stacks. This begins with 2-gr–stacks, in other words group-like 2-stacks endowed with a coherently associative group law, and continues through the 2-stack version of (a slight modification of) Kapranov and Voevodsky's braided 2-categories [K-V], and then through intermediate objects which we call strongly braided 2-stacks, before ending with the appropriately defined Picard 2-stacks. While we do not discuss higher level generalizations of these group laws, it is apparent from an examination of the cohomology of the corresponding Eilenberg-MacLane spaces, that for arbitrary integers n, such a hierarchy for n–gr–stacks will go through $n+1$ distinct stages before reaching the stable level of fully commutative Picard n–stacks.

1. Gluing conditions in fibered categories and 2-categories

1.1 We begin by reviewing the theory of fibered categories from the present perspective. A fibered category $\mathscr{C}$ over a space X is the category-theory version of the notion of "presheaf of categories" on X. It is defined by giving oneself for any open set $U \subset X$ a category $\mathscr{C}_U$ and for every inclusion of open sets $f: U_2 \hookrightarrow U_1$ an inverse image functor

$$f^*: \mathscr{C}_{U_1} \longrightarrow \mathscr{C}_{U_2} , \tag{1.1.1}$$

which may be taken to be the identity whenever $f = 1_U$. One is also given, for every pair of composable inclusions of open sets $f: U_2 \hookrightarrow U_1$ and $g: U_3 \hookrightarrow U_2$, a natural transformation

$$\varphi_{f,g} : (fg)^* \Rightarrow g^* \circ f^* . \tag{1.1.2}$$

Finally, if one has another composable inclusion $h: U_4 \hookrightarrow U_3$, the composite transformations $\psi_{f,g,h}$ and $\chi_{f,g,h}$ defined respectively by

$$(fgh)^* \Rightarrow h^* \circ (fg)^* \Rightarrow h^* \circ (g^* \circ f^*) \tag{1.1.3}$$

and

$$(fgh)^* \Rightarrow (gh)^* \circ f^* \Rightarrow (h^* \circ g^*) \circ f^*$$

are required to coincide. We will often refer to the inverse image of an object $x \in \mathscr{C}_U$ by an inclusion $i: V \hookrightarrow U$ as the restriction $x_{|V}$ of x above V.

A functor $F: \mathscr{C} \longrightarrow \mathscr{D}$ between fibred categories consists in a family of functors $F_U : \mathscr{C}_U \longrightarrow \mathscr{D}_U$ indexed by the open sets U, together with, for every morphism $f: U_2 \longrightarrow U_1$, a natural transformations

$$\varphi_f : f^* \circ F_{U_1} \longrightarrow F_{U_2} \circ f^*. \tag{1.1.4}$$

This is required to be compatible, for any pair of composable morphisms f and g, with the natural transformations (1.1.2) for $\mathscr{C}$ and for $\mathscr{D}$. Such a functor F is said to be Cartesian. Finally, a 2-arrow $\Psi: F \Rightarrow G$ between a pair of Cartesian functors F and G from $\mathscr{C}$ to $\mathscr{D}$ is defined by a family of natural transformations $\Psi_U : F_U \Rightarrow G_U$ which are compatible with the restriction functors (1.1.1). We refer to [SGA1] exposé VI for a more intrinsic definition of fibered categories, Cartesian functors and 2-arrows.

Remark: We have so far indexed the constituents $\mathscr{C}_U$ of our fibered category $\mathscr{C}$ by open sets $i: U \hookrightarrow X$. These may be viewed as objects of a category $\mathscr{S}_X$, whose morphisms $(i: U \hookrightarrow X) \longrightarrow (j: V \hookrightarrow X)$ are the inclusions $k: U \hookrightarrow V$ such that $i = jk$. The definition of a fibered category still makes sense in a more general context, in which the indexing category $\mathscr{S}_X$ is replaced by an appropriate subcategory $\mathscr{S}$ of the category of all spaces above X, whose objects are therefore maps $U \longrightarrow X$ which are no longer necessarily injective, and which are to be viewed as generalized open sets of X. When the objects of $\mathscr{S}$ are provided with an appropriate notion of a covering, the category $\mathscr{S}$ is said to be a site, and this defines a Grothendieck topology on X. The gluing conditions which we are about to examine also make sense for categories fibered over an arbitrary site $\mathscr{S}$. We will however generally restrict ourselves in the sequel to sites defined by the open sets in a topological space, and refer to [Ar], [SGA 4], [Mi] for further details in the general context.

We now introduce the notion of a stack on a space X. This may be thought of as the fully sheafified version of the more naive concept of a "sheaf of categories" on X. To be specific, we consider the following gluing law on objects in a fibered category, known in algebraic geometry as the effective descent condition. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of an open set U of X, and suppose that we are given a family of objects $x_\alpha \in \mathscr{C}_{U_\alpha}$, and a family of isomorphisms

$$\varphi_{\alpha\beta} : x_\beta \,|\, U_{\alpha\beta} \longrightarrow x_\alpha \,|\, U_{\alpha\beta} \tag{1.1.5}$$

in $\mathscr{C}_{U_{\alpha\beta}}$, satisfying the compatibility (or descent) condition $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ in $\mathscr{C}_{U_{\alpha\beta\gamma}}$ and such that $\varphi_{\alpha\alpha} = 1_{x_\alpha}$ in $\mathscr{C}_{U_\alpha}$. This descent condition is said to be effective if there exists an object $x \in \mathscr{C}_U$, together with a family of isomorphisms $\psi_\alpha : x_{|U_\alpha} \longrightarrow x_\alpha$, whose restrictions to the various open sets $U_{\alpha\beta}$ are compatible with the transformations (1.1.2), (1.1.5). The requirement that every descent condition in $\mathscr{C}$ be effective may be viewed as a gluing condition on the objects of $\mathscr{C}$. A gluing condition on morphisms in $\mathscr{C}$ is also given as part of the axioms for a stack. This simply states that for any pair of objects x and y in a fiber category $\mathscr{C}_U$, and any open cover (U_α) of U, the ordinary sheaf (of sets) axiom for the presheaf $Ar(x,y)$ of arrows from x to y, given by exactness of the sequence

$$Ar(x,y) \longrightarrow Ar(x_\alpha, y_\alpha) \rightrightarrows Ar(x_{\alpha\beta}, y_{\alpha\beta}), \tag{1.1.6}$$

is satisfied, once the identifications (1.1.2) have been performed for the objects x and y. Note that a more restrictive descent condition on objects in $\mathscr{C}$ would be the requirement that an object $x \in \mathscr{C}_U$ be associated only to those collection of objects $x_\alpha \in \mathscr{C}_{U_\alpha}$ for which (1.1.5) is replaced by the more restrictive descent condition

$$x_\beta \,|\, U_{\alpha\beta} = x_\alpha \,|\, U_{\alpha\beta}. \tag{1.1.7}$$

A fibered category $\mathscr{C}$ over X satisfying the gluing condition (1.1.6) on morphisms and the restrictive gluing condition (1.1.7) on objects is called a sheaf of categories [Gi] II 2.2.1. Any fibered category $\mathscr{C}$ can be replaced by an equivalent one, in which the natural transformation (1.1.2) is replaced by the identity transformation. Once this very ungeometric modification has been performed on a sheaf of categories $\mathscr{C}$, the functor $U \longmapsto ob(\mathscr{C}_U)$ becomes a sheaf of sets on X, and for every pair of objects x and $y \in \mathscr{C}_U$, the presheaf $V \longmapsto Ar(x_V, y_V)$ is a sheaf on U. Such a sheaf of categories is described by its nerve $N\mathscr{C}$, which is then a simplicial sheaf on X.

A fibered category satisfying both the gluing condition on morphisms and the effective descent condition (1.1.5) on objects is called a stack on X [*see* Gi, De-Mu, L-M, Bry][1]. When each of the categories $\mathscr{C}_U$ is in addition a groupoid (in other words a category in which every arrow is invertible), we say that $\mathscr{C}$ is a stack in groupoids. We will henceforth restrict ourselves to such stacks and will stretch the usual terminology slightly by referring to a sheaf of groupoids as a prestack (in groupoids). In particular, a sheaf of sets F on X determines a stack $\underline{F}$ on X, whose fiber category $\underline{F}_U$ is the discrete category with the set $\Gamma(U,F)$ of sections of F above the open set U as objects. We will henceforth call this the discrete stack defined by F. By a construction modeled on the corresponding construction for sheaves, there corresponds to any prestack $\mathscr{C}$ on X an associated stack $\mathscr{C}^{\sim}$, together with a Cartesian functor

$$i: \mathscr{C} \longrightarrow \mathscr{C}^{\sim} \tag{1.1.8}$$

which is fully faithful and universal for cartesian functors from $\mathscr{C}$ into stacks ([L-M] lemma 2.2). Following [Del 4] 3.1, we observe that the associated stack $\mathscr{C}^{\sim}$ is characterized by the property that every object of $\mathscr{C}^{\sim}$ is locally contained in the essential image of i.

Example: Let G be a sheaf of groups on X, and consider the corresponding sheaf $G[1]$ of groupoids on X, whose sheaf of objects is the trivial sheaf on X and with G as sheaf of automorphisms of the trivial object. We identify this

[1] However, what we call a stack is referred to in [Bry] as a "sheaf of categories".

groupoid with its nerve, which is the classifying simplicial sheaf BG on X. The associated stack is the stack $Tors(G)$ of right G–torsors (or right G-principal homogeneous spaces) on X, whose value above an open set $U \subset X$ is the groupoid $\mathcal{G}_U = Tors(G_U)$ of $G_{|U}$-torsors on U. The associated stack functor (1.1.8) therefore specializes in this situation to the functor

$$i: G[1] \longrightarrow Tors(G) \tag{1.1.9}$$

which we can also write as

$$i: BG \longrightarrow Tors(G). \tag{1.1.10}$$

1.2 We will now introduce group laws on stacks. Let $\mathcal{C}$ be a prestack on X. We begin by supposing that $\mathcal{C}$ is endowed with a strictly associative Cartesian functor

$$m: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \tag{1.2.1}$$

which has unit objects and inverses. In that case its nerve is a simplicial group $N\mathcal{C}$ on X. As explained in [Br 3] 1.1, it is characterized in the homotopy category by its associated Moore complex MG, which is a left crossed module on X, in other words a length 1 complex of sheaves of groups on X

$$\delta: H \longrightarrow K, \tag{1.2.2}$$

endowed with a left action of K on H satisfying the two well-known axioms

$$\delta({}^{k}h) = khk^{-1} \tag{1.2.3}$$

$${}^{\delta(h')}k = h'k(h')^{-1}.$$

The reverse construction associates to any homomorphism $\delta: H \longrightarrow K$ of sheaves of groups on X the presheaf of groupoids $\mathcal{C}$ whose objects in a fiber category $\mathcal{C}_U$ are the sections above U of K, and for which a section h of H above U describes an arrow $h: 1 \longrightarrow \delta(h)$ sourced at the neutral element $1 \in K$. More generally, an arbitrary arrow in $\mathscr{C}_U$ is determined as follows by an element (h,k) in $H \times K$:

$$(h,k)\colon k \longrightarrow \delta(h)k. \tag{1.2.4}$$

The composite arrow

$$k \xrightarrow{(h,k)} \delta(h)k \xrightarrow{(h',\delta(h)k)} \delta(h'h)k$$

is defined by the rule

$$(h',\delta(h)k) \circ (h,k) = (h'h,\, k). \tag{1.2.5}$$

We will refer to the stack associated to this prestack as the stack $(H \longrightarrow K)$ associated to the homomorphism δ. Observe that this definition did not involve the full group law on K, but only the action of H on the underlying set of K defined by δ, and that this construction therefore actually associates a stack $(H \longmapsto \mathscr{X})^{\sim}$ to any sheaf of groups H acting on the left on a sheaf of sets $\mathscr{X}$. Its objects above an open set U are pairs (P,s), where P is an H-torsor on U, and s a section of the associated fiber space $P \wedge^H \mathscr{X}$.

Let us now suppose that the homomorphism $\delta\colon H \longrightarrow K$ (1.2.2) is endowed with a crossed module structure. In that case the group law m on the associated stack $\mathscr{C} = (H \longrightarrow K)^{\sim}$ is defined on objects by the group law in K, and on arrows by the semi-direct product rule

$$(h,k) \otimes (h',k') = (h\,{}^{k}h',\, kk') \tag{1.2.6}$$

determined by the given action of K on H. In particular, we see by comparing formulas (1.2.5) and (1.2.6) when $k = k' = 1$, that the restriction of the tensor law (1.2.6) to pairs of arrows sourced at 1 coincides with the given group law in H, and also with the composition law

$$1 \xrightarrow{h'} \delta(h') \xrightarrow{(h,\delta(h'))} \delta(hh')$$

for the pair of composable arrows determined by h and h'.

A prestack (*resp.*, a stack) in groupoids $\mathcal{C}$ on X, endowed with a monoidal (but no longer strictly associative) functor (1.2.1), with identity objects and with inverses will be called a gr-prestack (*resp.*, a gr–stack) on X. Other possible terminologies, derived from category theory, would be rigid AU ⊗-stack [Saa], or compact closed monoidal stack in groupoids. We refer to [K-L], [Saa], [Del 4] or [Br 3] for the precise definition of gr–stacks $\mathcal{C}$ and of morphisms and natural transformations between them. We will henceforth refer to a functor such as (1.2.1) on a gr-stack $\mathcal{C}$ as a group law on $\mathcal{C}$, irrespective of whether it is strictly associative or simply associative up to a coherent isomorphism (a more common, but in our opinion less suggestive, terminology for such functors is that of tensor functor on $\mathcal{C}$). The stack associated to a crossed module $(H \longrightarrow K)$ consists of pairs (P, s), where P is an H–torsor, and s is a section of the induced K-torsor $\delta_*(P)$. We refer to [Br 2] 4.5 for the description of the group law on this stack $(H \longrightarrow K)^{\sim}$.

An important example of groupoid endowed with a group law is the groupoid $Eq\,(BG)$ of self-equivalences of BG, the group law being given by composition of equivalences. This group law is most pleasantly analyzed in the categorical setting, by considering the groupoid of functors (and natural transformations) from the prestack $G[1]$ defined by the sheaf of groups G to itself. Such a functor sends the unique object e of $G[1]$ to itself, so that it is entirely determined by its action on arrows, in other words by an automorphism u of G. Similarly, a natural transformation $u_1 \Rightarrow u_2$ between such a pair of functors, respectively defined by automorphisms u_1 and u_2 of G, is determined by the corresponding arrow $g\colon u_1(e) \longrightarrow u_2(e)$ in $G[1]$. The naturality property commands that, for any arrow $\gamma : e \longrightarrow e$ in$G[1]$, the diagram

$$\begin{array}{ccc} u_1(e) & \xrightarrow{g} & u_2(e) \\ {\scriptstyle u_1(\gamma)}\downarrow & & \downarrow{\scriptstyle u_2(\gamma)} \\ u_1(e) & \xrightarrow[g]{} & u_2(e) \end{array} \tag{1.2.7}$$

commutes, in other words that the automorphisms u_1 and u_2 be related by

$$u_2 = i_g \circ u_1 \,, \tag{1.2.8}$$

where i is the inner conjugation homomorphism from the sheaf of groups G to its sheaf of automorphisms:

$$\begin{aligned} i\colon G &\longrightarrow Aut(G) \\ g &\longmapsto i_g : g_1 \longmapsto g g_1 g^{-1} \,. \end{aligned} \tag{1.2.9}$$

The manner in which these arrows and natural transformations compose is described by the following lemma.

Lemma 1.3: *Let G be a sheaf of groups on X. The gr–prestack $Eq(BG)$ is determined by the crossed module $i\colon G \longrightarrow Aut(G)$ defined by the inner conjugation map* (1.2.9), *for the obvious left action of $Aut(G)$ on G.*

More generally, a functor $H[1] \longrightarrow G[1]$ between two such groupoids BH and BG is determined by a homomorphism $u\colon H \longrightarrow G$, and it follows from the corresponding diagram (1.2.7) that a natural transformation $u_1 \Rightarrow u_2$ between two such functor is given by an element $\gamma \in G$ such that the formula analogous to (1.2.8) is satisfied. The prestack $Eq(BH, BG)$ of such equivalences may therefore be described as the prestack defined by the action $(G \longmapsto Isom(H, G))$ of G on the sheaf $Isom(H, G)$ defined by the rule analogous to (1.2.8).

Applying the "associated stack" functor (1.1.9) to both terms of lemma 1.3, we deduce from it the following result.

Lemma 1.4: *The gr–stack $Eq(Tors(G))$ of self–equivalences of the stack $Tors(G)$ is the stack $(G \longrightarrow Aut\, G)^{\sim}$ associated to the crossed module i.*

Similarly the stack $Eq(Tors(H), Tors(G))$ of equivalences between the stacks $Tors(H)$ and $Tors(G)$ is the stack $(G \longmapsto Isom(H, G))^{\sim}$ associated to the prestack defined above. The first part of the following lemma spells out, for future reference, a weak form of this assertion. The other two parts of the lemma then follow immediately, once the rule for composing equivalences has been explicitly determined.

Lemma 1.5: *i) Let μ, ν: $H \longrightarrow G$ be a pair of homorphisms between two sheaves of groups on a space X, and let μ_*, ν_*: $Tors(H) \longrightarrow Tors(G)$ the induced functors between the corresponding categories of torsors. A natural transformation $\mu_* \Rightarrow \nu_*$ is determined by the choice of a section g of G on X such that*

(1.5.1) $$i_g \circ \mu = \nu.$$

ii) Consider homomorphisms ρ: $G \longrightarrow F$ (resp., σ: $K \longrightarrow H$), and let ρ_* (resp., σ_*) be the induced functors between the corresponding categories of torsors. The two induced natural transformations

$$\rho_* \mu_* \Rightarrow \rho_* \nu_* : Tors(H) \longrightarrow Tors(F)$$

and

$$\mu_* \sigma_* \Rightarrow \nu_* \sigma_* : Tors(K) \longrightarrow Tors(G)$$

are respectively described by the section $\rho(g)$ of F and the section g of G.

iii) Let $\lambda: H \longrightarrow G$ be a group homomorphism, and let g' be a section of G, which determines, as in (1.5.1), *a natural transformation $\lambda_* \Rightarrow \mu_*$.*

The composite transformation $\lambda_* \Rightarrow \mu_* \Rightarrow \nu_*$ *is described by the section* gg' *of* G.

Another description of the full stack $Eq\,(Tors\,(H),\ Tors\,(G))$ is given by the following proposition, which in view of lemmas 1.4 and 1.5 is essentially the "Morita theorem" of [Gi] IV proposition 5.2.5 *iii*). Recall that, for any pair of sheaves of groups H and G on a space X, an (H,G)–bitorsor on X is a right G–torsor P on X, endowed with a left H–action which commutes with the G-action, and for which P is also a left H––torsor [Gi], [Br 2]. These (H,G)-bitorsors form a stack on X, and the contracted product of bitorsors

$$
\begin{aligned}
(1.5.2) \qquad Bitors\,(K,H) \times Bitors\,(H,G) &\longrightarrow Bitors\,(K,G) \\
(Q\,,P) &\longmapsto (Q \wedge^H P)
\end{aligned}
$$

is coherently associative. Setting $G = H = K$, the contracted product (1.5.2) defines a group structure on the stack $Bitors\,(G)$ of $(G,\ G)$–bitorsors (we will henceforth refer to these as G–bitorsors).

Proposition 1.6: *i*) *The stack* $Bitors\,(H,G)$ *is associated to the prestack* $(G \longmapsto Isom\,(H,G))$.

ii) *The gr–stack* $Bitors\,(G)$ is the *gr–stack opposite* (*i.e., with opposite group law*) *to the gr stack* $Eq\,(Tors\,(G))$.

Proof: Part *i*) is proved by observing that the functor

$$
(1.6.1) \qquad \wedge: (G \longmapsto Isom\,(H,G)) \longrightarrow Bitors\,(H,\ G)
$$

which associates to a section u of $Isom\,(H,G)$ the trivial right G–torsor, with left action of h on the trivial section s defined by the rule

$$
h\,s =: s\,(u(g))
$$

is fully faithful, and locally essentially surjective, so that it satisfies the universal condition characterizing the associated stack of a prestack. The compatibility of the pairing of the prestacks $Eq(BH, BG)$ defined by composition of equivalences with the contracted product (1.5.2) is then easily verified. In view of lemma 1.4, this specializes to the assertion *ii)* when $G=H$.

The crossed module $(G \longrightarrow Aut(G))$ introduced in (1.2.9) may be analyzed by two sorts of dévissage, which respectively yield the long exact sequences (4.3.17) and (5.2.4) of [Br 2]. These can best be visualized by the following diagram, in which ZG is the center of the group G and $Out(G)$ its sheaf of outer automorphisms.

$$\begin{array}{ccccc} & & ZG[1] & & \\ & & \downarrow & & \\ Aut(G) & \longrightarrow & (G \longrightarrow Aut(G)) & \longrightarrow & G[1] \\ & & \downarrow & & \\ & & Out(G) & & \end{array} \tag{1.6.2}$$

The right hand horizontal map in this diagram is not a morphism of crossed modules, but instead lives in the homotopy category (since the groupoid $G[1]$ is not endowed with a group structure unless G is abelian). It is of some interest to describe geometrically the various terms in the induced diagrams of associated stacks (each of these except $Tors(G)$ is in fact a *gr*–stack, and the maps between them respect the group structures):

$$\begin{array}{ccccc} & & Tors(ZG) & & \\ & & \downarrow & & \\ Aut(G) & \longrightarrow & Eq(Tors(G)) & \longrightarrow & Tors(G) \\ & & \downarrow & & \\ & & Out(G) & & \end{array} \tag{1.6.3}$$

Here, the right hand horizontal map is the "evaluation at the trivial G–torsor T" map

(1.6.4)
$$Eq\,(Tors\,(G)) \longrightarrow Tors\,(G)$$
$$v \longmapsto v\,(T)$$

The left hand map

(1.6.5)
$$Aut\,(G) \longrightarrow Eq\,(Tors\,(G))$$
$$u \longmapsto u_*$$

associates to an automorphism u of G the "change of structure group" self-equivalence u_*, It therefore defines an equivalence between the discrete stack $Aut\,(G)$ and stack $Eq.(Tors\,(G))$ of pointed self-equivalences of $Tors\,(G)$, in other words of those self equivalences which send the trivial torsor T to itself. Similarly, the bottom vertical map is best described by introducing the notion of the lien associated to a (trivial) gerbe $Tors\,(G)$, for which we refer to 2.10 below. The vertical map in question can then best be described as the map of gr–stacks

$$\Psi\colon Eq\,(Tors\,(G)) \longrightarrow Aut\,(lien\,(G))$$

induced by the *lien* functor. The following lemma is an immediate consequence of this interpretation, and of the exactness of the vertical short exact sequence of crossed modules (1.6.2).

Lemma 1.7: *Let G be a sheaf of groups on X,* and *$L = lien\,(G)$ the associated lien. The stack $Eq_L(Tors\,(G))$ of self equivalences of $Tors\,(G)$ which induce the identity map on L (in other words the stack of L-morphisms in the terminology of* [Gi] *IV* 2.2.7) *is equivalent to the stack $Tors\,(ZG)$.*

The functor from $Tors\,(ZG)$ to $Eq\,(Tors\,(G))$ which describes the top vertical map in (1.6.3) is most easily described as the pushout map

(1.7.1)
$$\Xi\colon Tors\,(ZG) \longrightarrow Bitors\,(G)$$
$$Q \longmapsto Q \wedge^{ZG} G\,,$$

determined by the inclusion of ZG in G . The left action of G on $\Xi(Q)$ is given by the formula

$$g'(q,g) = (q,\, g'g).$$

Suppose now that $G = A$ is an abelian group. In that case the inner conjugation map (1.2.9) is trivial, so that the crossed module $A \longrightarrow Aut(A)$ is split. It does not, however, become completetely trivial as a crossed module, since there remains a non-trivial action of $Aut(A)$ on A. The two composite diagonal maps implicit in diagram (1.6.3) respectively become the identity map on $Aut(A)$ and on $A[1]$, so that the vertical maps now determine splittings of the horizontal ones. Passing to the associated stacks, we begin by observing that the equivalence which lemma 1.7 determines is now simply defined by translation in the gr-stack $\mathcal{A} = Tors(A)$:

(1.7.2)
$$\begin{aligned} \Xi: Tors(A) &\longrightarrow Eq_L(Tors(A)) \\ Q &\longrightarrow (P \longmapsto P \wedge^A Q) \end{aligned}$$

(note that the equivalence $P \longmapsto P \wedge^A Q$ does indeed induce as required the identity map at the *lien* level: this is a statement which can be checked locally, so that one may suppose that Q is trivial, in which case it is obvious). On the other hand, it may be verified either directly, or by invoking the Dold-Kan theorem on the equivalence between the categories of chain complexes and of simplicial abelian groups, that those self-equivalences of $\mathcal{A}$ which lie in the image of (1.6.5) are the self-equivalences of the stack $\mathcal{A}$ which preserve the group structure of $\mathcal{A}$. We denote the stack of these self-equivalences by $\mathcal{E}q(\mathcal{A})$, in order to distinguish it from the larger stack $Eq(\mathcal{A})$ of arbitrary self-equivalences of $\mathcal{A}$. Taking into account the equivalence $Aut(A) \longrightarrow \mathcal{E}q(\mathcal{A})$ just described, we see that the diagram (1.6.3) collapses in the abelian case to the diagram

(1.7.3)
$$\mathcal{E}q(\mathcal{A}) \longrightarrow Eq(\mathcal{A}) \longrightarrow \mathcal{A}$$

Here the right hand map in (1.6.3) is split by the translation map (1.7.2). The opposite diagram

$$\mathcal{A} \longrightarrow Eq(\mathcal{A}) \longrightarrow \mathcal{E}q(\mathcal{A})$$

can be viewed as yielding a "semi-direct product"decomposition of the

gr–stack $Eq(\mathcal{A})$ associated to the split crossed module $(A \longrightarrow Aut(A))$ in terms of the gr–stack $\mathcal{E}q\ (\mathcal{A})$ of pointed self-equivalences of $\mathcal{A}$, which is associated to $Aut(A)$, and the abelian gr–stack of translations by $\mathcal{A}$, which is associated to $A[1]$.

1.8 There exist various natural commutativity conditions on the group law of a gr-stack, which may be expressed at the crossed module level. Recall that, according to the terminology of [J-S], a gr–category $\mathcal{C}$ is said to be braided whenever the group law in $\mathcal{C}$ is endowed with a commutativity isomorphism

(1.8.1) $$s_{X,Y} : XY \longrightarrow YX$$

which is functorial in X and Y, and for which the two well-known hexagonal diagrams (2.4.6.2) and (2.4.6.3) of [Br 2] commute. It is called a Picard (or group-like symetric monoidal) category whenever the additional condition

(1.8.2) $$s_{Y,X} \circ s_{X,Y} = 1_{XY}$$

is satisfied for every pair of objects X and Y in $\mathcal{C}$. Finally, it is called strict Picard whenever the condition

(1.8.3) $$s_{X,X} = 1_X$$

is also satisfied for all objects X. A gr–stack endowed with a commutativity natural transformation s for the group law (1.8.1) will be said to be braided (*resp.*, Picard, *resp.*, strict Picard) whenever it satisfies the corresponding conditions. These conditions may be expressed as follows for a gr–stack associated to a crossed module $\delta : H \longrightarrow K$. The crossed module is said to be braided if there exists a lifting

(1.8.4) $$\begin{aligned} K \times K &\longrightarrow H \\ (k_1, k_2) &\longmapsto \{k_1, k_2\} \end{aligned}$$

of the commutator map on K, which satisfies the conditions given in [Co] (2.11)-(2.13). Recalling the recipe (1.2.4) by which a groupoid $\mathscr{C}$ is associated to a crossed module $H \longrightarrow K$, it is immediate that the braiding map (1.8.4) determines for each pair of objects $X, Y \in K = ob(\mathscr{C})$, an arrow (1.8.1)

$$s_{X,Y} = (\{Y,X\}, XY) : XY \longrightarrow YXY^{-1}X^{-1}XY = YX$$

in $\mathscr{C}$, and it is readily verified that the conditions (2.11) in *op. cit.*, correspond to the axioms which the braiding (1.8.1) must satisfy (in particular, the last two conditions (2.11) correspond, in view of the composition and multiplication rules (1.2.5) and (1.2.6) in $\mathscr{C}$ to each of the two hexagon laws for the braiding). It is worth pointing out here that these somewhat complicated conditions on a crossed module $H \longrightarrow K$ may also be expressed very compactly as the assertion that the trivially commuting diagram of crossed modules

$$\begin{array}{ccc} H & \xrightarrow{\delta} & K \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle 1} \\ K & \xrightarrow[1]{} & K \end{array} \tag{1.8.5}$$

is in fact a crossed square, in the sense of Loday [Lo] (*see* also [Br 3] 1.2). Indeed, one of the axioms for such a crossed square is existence of a map (1.8.4) between the appropriate vertices of the square, satisfying the requisite identities. We finally observe that the braiding on the prestack $\mathscr{C}$ obtained in this manner induces a braiding on the associated stack.

Similarly, the additional condition given in [Co] (3.11) defines a so-called stable crossed module. It corresponds to the identity (1.8.2) in the groupoid $\mathscr{C}$, so that the associated stack is then a Picard stack. Finally, when the pairing (1.8.4) also satisfies the condition

$$\{k,k\} = 1 \tag{1.8.6}$$

for all sections k of K, the associated stack is Picard strict since (1.8.6) translates to condition (1.8.3). For this reason, we will also say that such a stable crossed module is strict. The simplest example of such a strict Picard stack is the one associated to the trivial stable crossed module $A \longrightarrow 1$ defined by a sheaf of abelian group A. As seen in 1.1, this is simply the stack $\mathcal{A} = Tors(A)$ of A-torsors on X associated to the group A, but now endowed with the group law defined by the usual contracted product of torsors under an abelian group. A closely related example is that of the Picard stack associated to a complex of abelian groups $\delta: A_1 \longrightarrow A_0$. We have seen that this consists of the pairs (P,s) A_1-torsors P, together with trivialisations s of their pushouts to A_0. The group law on these pairs is now be seen to be abelian, and Deligne has shown in [SGA4] that every strict Picard stack is equivalent to one obtained in this manner. A related example is the following one.

Example 1.9: Let G be a reductive group, and denote by G^{sc} the simply connected cover of is derived group G^{der}. Consider the composed homomorphism

(1.9.1) $$\delta: G^{sc} \longrightarrow G^{der} \hookrightarrow G.$$

Deligne observes in [Del 2] 2.0.2 that since the adjoint group $G^{ad} = G/ZG$ of G is isomorphic to the adjoint group of G^{sc}, and since the commutator pairing in a group factors through its adjoint group, the commutator map of G lifts to a map

(1.9.2) $$G \times G \longrightarrow G^{ad} \times G^{ad} \simeq (G^{sc})^{ad} \times (G^{sc})^{ad} \longrightarrow G^{sc}.$$

Since the action of G on itself also factors through an action on G of G^{ad}, the same line of reasoning implies that the action of G on itself lifts to an action

(1.9.3) $$G \longrightarrow G^{ad} \simeq (G^{sc})^{ad} \longrightarrow Aut(G^{sc})$$

of G on G^{sc}. The latter defines a structure of crossed module on the complex

$\delta: G^{sc} \longrightarrow G$ (1.9.1), and this crossed module is in fact endowed with a strict Picard structure since the requisite conditions now follow from the standard relations satisfied by commutators. As observed in *op. cit.*, (2.4.7), the inclusion

$$(Z^{sc} \longrightarrow Z) \longrightarrow (G^{sc} \longrightarrow G) \tag{1.9.4}$$

from the abelian complex of centers of the groups G^{sc} and G to the crossed module $G^{sc} \longrightarrow G$ itself is a quasi-isomorphism. It therefore induces an explicit equivalence between the strict Picard stack associated to the stable complex $(G^{sc} \longrightarrow G)$ and one associated to a complex of abelian sheaves.

1.10 We will now discuss some of the corresponding sheaf-theoretic notions for 2-categories. We refer to [Be], [Hak], [K-V], [K-S] for the definition and the basic properties of 2-categories and 2-functors. The notion of fibered 2-category $\mathcal{C}$ over a space (or site) X has been defined in [Hak] I.3. An equivalent definition, in the spirit of the one reviewed above in the case of fibered categories, is the following. A fibered 2-category $\mathcal{C}$ on X consists of a family of 2-categories $\mathcal{C}_U$ indexed by the open sets U of X. For any inclusion $f: U_2 \hookrightarrow U_1$ of open sets, we are also given an inverse image 2-functor

$$\begin{aligned} f^*: \mathcal{C}_{U_1} &\longrightarrow \mathcal{C}_{U_2} \\ x &\longmapsto x_{|U_2} \end{aligned}$$

and for each pair of composable inclusions f and $g: U_3 \hookrightarrow U_2$, a natural transformation $\varphi_{f,g}$ (1.1.2). We no longer require, as for fibered 1-categories, that the two transformations $\psi_{f,g,h}$ and $\chi_{f,g,h}$ defined as in (1.1.3) coincide. Instead, we give ourselves a 3-arrow (also known as a modification)

$$\alpha_{f,g,h}: \psi_{f,g,h} \Rrightarrow \chi_{f,g,h} \tag{1.10.1}$$

between these two natural transformations. We require that it satisfies an additional coherence condition, asserting that for yet another composable inclusion of open sets $k: U_5 \hookrightarrow U_4$, the two possible methods for deriving from (1.10.1) a 3-arrow which compares the 2-arrows

$$(fghk)^* \Rightarrow (ghk)^* f^* \Rightarrow ((hk)^* g^*) f^* \Rightarrow k^* h^* g^* f^*$$

and

$$(fghk)^* \Rightarrow k^* (fgh)^* \Rightarrow k^* (h^* (fg)^*) \Rightarrow k^* h^* g^* f^*$$

in $\mathscr{C}_{U_5}$ must coincide[2]. A prime example of such a structure is the fibered 2-category *FIB*(X) of all (small) fibered categories over a space *X*.

Similarly, a (cartesian) 2-functor $F: \mathscr{C} \longrightarrow \mathscr{D}$ between a pair of fibered 2-categories over *X* consists of a collection of 2-functors $\mathscr{C}_U \longrightarrow \mathscr{D}_U$ for varying open sets *U*, together with natural transformations of 2-functors $\varphi_f : f^* \circ F_{U_1} \Rightarrow F_{U_2} \circ f^*$ for every inclusion $f: U_2 \hookrightarrow U_1$. For every pair of composable inclusions *f* and *g*, a 3-arrow $\alpha_{f,g}$ is also given, which compares the natural transformation from $(fg)^* \circ F_{U_1}$ to $F_{U_3} \circ g^* \circ f^*$ defined by φ_f and φ_g with that constructed from φ_{fg}. This 3-arrow is required to satisfy a coherence condition relating the two induced natural transformations associated to a triplet (f,g,h) of composable inclusions. We do not spell out this condition, nor do we carry out the routine task of defining[3] in a similar manner Cartesian natural transformations $u: F \Rightarrow G$ between Cartesian 2-functors, and Cartesian 3–arrows $\alpha: u \Rrightarrow v$ between pairs of such natural transformations. This give us a 3-category (2-*Fib*(X)), whose objects are the fibered 2-categories over *X*.

Let us now examine sheaf-like gluing conditions in fibered 2-categories. These conditions are quite analogous to those discussed above for 1-categories, but with an additional level of complication. Gluing

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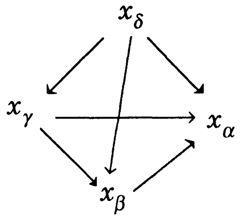
[2] Composition of 2-arrows is often referred to as pasting (*see* for example[KV], [KS] and references therein).

[3] These definitions require the introduction of two further open subsets U_4 and U_5 in U_3.

conditions are now required for objects, arrows and 2-arrows , and it is for objects that these conditions are the most elaborate. They were made explicit by J. Duskin in [Du 3], under the name of 2-descent, and go as follows. Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ once more denote an open cover of an open set U in the space X. and suppose that we are given a collection x_α of objects in the 2-categories $\mathcal{C}_{U_\alpha}$, a family of 1-arrows $\varphi_{\alpha\beta} : x_\beta \longrightarrow x_\alpha$ between the restrictions to $\mathcal{C}_{U_{\alpha\beta}}$ of the objects x_β and x_α and a family of 2-arrows

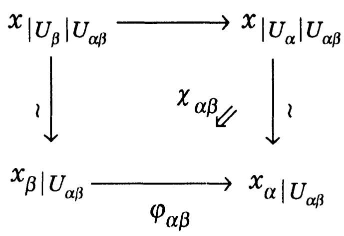
(1.10.2) $$\psi_{\alpha\beta\gamma} : \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} \Rightarrow \varphi_{\alpha\gamma}$$

in $\mathcal{C}_{U_{\alpha\beta\gamma}}$. We require that these 2-arrows satisfy the compatibility condition described by the commutativity of the following diagram of 2-arrows, which lives in the fiber category $\mathcal{C}_{U_{\alpha\beta\gamma\delta}}$.



(1.10.3)

Each face of this tetrahedron is determined by the corresponding 2-arrow (1.10.2), and these 2-arrows compose in the obvious manner. The data $(\varphi_{\alpha\beta}, \psi_{\alpha\beta\gamma})$, together with the compatibility condition (1.10.3) for the $\psi_{\alpha\beta\gamma}$ is called a set of 2-descent data for the collection of objects $(x_i)_{\iota \in I}$. We say that 2-descent is effective in $\mathcal{C}$ if, for each set of 2-descent data in $\mathcal{C}$, there exists an object $x \in \mathcal{C}_U$, for each $\alpha \in I$, an isomorphism $x_{|U_\alpha} \longrightarrow x_\alpha$ in $\mathcal{C}_{U_\alpha}$ and for each $\alpha, \beta \in I$, a family of 2-arrows $\chi_{\alpha\beta}$



satisfying the appropriate compatibility with the 2-arrow $\psi_{\alpha\beta\gamma}$ (1.10.2). The corresponding gluing conditions on 1- and 2-arrows in $\mathscr{C}$ are described, in their most compact form by the following assertion. For any pair of objects x and y in a fibre 2-category $\mathscr{C}_U$, the fibered category $\mathscr{A}r(x,y)$ on U, whose value on an open set V in U is the category $Ar(x_V,y_V)$ of 1-arrows in $\mathscr{C}_V$ between the restrictions x_V and y_V of x and y to $\mathscr{C}_V$, is a stack on U. A fibered 2-category $\mathscr{C}$ on X in which all such fibered categories of morphisms $\mathscr{A}r(x,y)$ are stacks will be called a 2-prestack. When, in addition, every set of 2-descent data is effective, $\mathscr{C}$ is called a 2-stack on X. For any 2-prestack $\mathscr{C}$ on X, one defines by the same sheafification method as for 1-stacks, an "associated 2-stack" 2-functor

$$a: \mathscr{C} \longrightarrow \mathscr{C}^{\sim},$$

which is universal for Cartesian 2-functors $b: \mathscr{C} \longrightarrow \mathscr{D}$ from $\mathscr{C}$ into 2-stacks. The latter property characterizes $\mathscr{C}^{\sim}$ up to a 2-equivalence. Indeed, suppose that such a Cartesian 2-functor $b: \mathscr{C} \longrightarrow \mathscr{D}$ satisfies the following two conditions:

i) b is fibrewise fully faithful, in other words, for any pair of objects x and y in a fiber 2-category $\mathscr{C}_U$, the induced map of stacks on U

$$\mathscr{A}r(x,y) \longrightarrow \mathscr{A}r(b(x), b(y))$$

is an equivalence.

ii) every object in $\mathscr{D}$ is locally isomorphic to one in the image of $\mathscr{C}$.

It then follows that $\mathscr{D}$ is an associated 2-stack of $\mathscr{C}$.

Examples 1.11:

i) The fibered 2-category $Stack_X$, whose fiber on an open set $U \subset X$ is the category $Stack(U)$ of stacks on U, is an example of a 2-stack on X. Indeed, suppose that we are given an open cover $\mathscr{U} = (U_i)_{i \in I}$ of an open set

U in X, and a family of stacks $\mathcal{C}_i$ on U_i, together with gluing Cartesian functors $\varphi_{ij}:\mathcal{C}_j \longrightarrow \mathcal{C}_i$ defined on the open sets U_{ij} and natural transformations (1.10.2) $\psi_{ijk}:\varphi_{ij}\circ\varphi_{jk}\Rightarrow\varphi_{ik}$ on U_{ijk} satisfying the tetrahedral compatibility condition (1.10.3). We can then define a 2-descended stack $\mathcal{C}$ on U as follows. An object of the fiber category $\mathcal{C}_U$ is defined to be a collection of objects $x_i \in \mathcal{C}_i$ $(i\in I)$, together with a family of arrows $f_{ij}:\varphi_{ij}(x_j)\longrightarrow x_i$ in $\mathcal{C}_i$ above U_{ij} for all $i,j\in I$, compatible in an obvious sense with the natural transformation ψ_{ijk}. Similarly, an arrow $m:x\longrightarrow y$ in $\mathcal{C}_U$ consists in a family of arrows $m_i:x_i\longrightarrow y_i$ in $\mathcal{C}_i$ which lie above U_i and which are compatible with the gluing 1-arrows f_{ij}. It may be verified that the fibered category $\mathcal{C}$ thus defined is a stack, so that the 2-descent condition is indeed satisfied in $Stack_X$. Furthermore, for any pair of stacks $\mathcal{C}$ and $\mathcal{D}$ on an open set U in X, the fibered category $\mathcal{H}om\,(\mathcal{C}\,,\mathcal{D})$ of Cartesian functors from $\mathcal{C}$ to $\mathcal{D}$ is a stack ([Gi] II 2.1.5), so that $Stack_X$ is indeed a 2-stack.

ii) A slightly more general notion than that of a 2-category is that of bicategory (or lax 2-category). Here, composition of 1-arrows is no longer strictly associative, but simply associative up to a given coherent 2-arrow (*see* [Be], [K-V] definition 2.7, or [Ke]) The whole discussion 1.10 remains valid in the context of bicategories. That such generality is not gratuitous is illustrated by the example of the fibered bicategory $\mathcal{G}[1]$ associated to a *gr*–stack $\mathcal{G}$ on X. Its fibre over each open set $U\subset X$ has a single object e, and $\mathcal{A}r(e,e)_U$ is defined to be the category $\mathcal{G}_U$. Composition of arrows in $\mathcal{G}[1]$ is defined by the group law in $\mathcal{G}$ and is therefore not strictly associative, unless this happens to be the case for the group law in $\mathcal{G}$. Since $\mathcal{G}$ is a group-like groupoid, 1-arrows in $\mathcal{G}[1]$ are invertible up to 2-arrows, and 2-arrows in $\mathcal{G}[1]$ are strictly invertible, so that $\mathcal{G}[1]$ is fibered in 2-groupoids (or rather bi-groupoids), in a sense which will be made explicit below in § 3.1. Furthermore, since $\mathcal{G}$ is a stack on X, the fibered 2-category $\mathcal{G}[1]$ is a 2-prestack on X. Consider the fibered 2-category $Tors\,(\mathcal{G})$, whose fiber on an open set U in X is the 2-category $Tors(\mathcal{G}_{|U})$ of right $\mathcal{G}$-torsors on U (defined in [Br 2] 6.1). This is a sub-2-stack of the 2-stack $Stack_X$, and the 2-functor

$$a: \mathcal{G}[1] \longrightarrow Tors(\mathcal{G}), \tag{1.11.1}$$

which sends the unique object of each fibre of $\mathcal{G}[1]$ to the trivial $\mathcal{G}$-torsor is fully faithful. Since every $\mathcal{G}$-torsor is locally trivial, it follows that $Tors(\mathcal{G})$ is the associated 2-stack of the 2-prestack $\mathcal{G}[1]$. It is worth pointing out that even when the group law in $\mathcal{G}$ is not strictly associative, so that $\mathcal{G}[1]$ is simply a fibered bicategory, the associated 2-stack $Tors(\mathcal{G})$ is a fibered 2-category.

Remark 1.12: Let us end this discussion of gluing laws in fibered categories and 2-categories by carrying it one step further. The collection of all 2-stacks on X defines a sub-3-category 2-*Stack*(X) of the 3-category 2-*Fib*(X). We may even introduce the fibered 3-category 2–$Stack_X$ on X, whose fibre on an open set $U \subset X$ is the 3-category 2-*Stack*(U). In fact, one can say that 2–$Stack_X$ is a 3-stack on X, once one has made precise the appropriate 3-descent condition on objects of this fibered 3-category. These are difficult to visualise, since they involve a commutative diagram of 3-arrows, modeled on the standard simplex $\Delta(4)$, and whose five faces are therefore tetrahedral 3-arrows (such a condition is illustrated in [St] diagram O_4 p. 290, with a convention for the orientation of 1- and 2-arrows which is opposite to ours).

2. Gerbes and 2-cocycles, revisited

We now define the geometric object which corresponds to a degree two cohomology class.

Definition 2.1: *A gerbe $\mathcal{G}$ on X is a stack in groupoids $\mathcal{G}$ on X which is locally non–empty and locally connected. A morphism of gerbes (resp. a natural transformation between such morphisms) is a (Cartesian) functor between the underlying stacks (resp., a natural transformation between these cartesian functors).*

The first condition on $\mathcal{G}$ asserts that there exists a covering $\mathcal{U} = (U_i)_{i \in I}$ of X for which the set of objects of the category $\mathcal{G}_{U_i}$ is non-empty. The connectedness condition is the requirement that, for every pair of objects x and y in $\mathcal{G}_U$, there exists an open cover $\mathcal{U} = (U_i)_{i \in I}$ of U such that the set of arrows from $x_{|U_i}$ to $y_{|U_i}$ is non-empty for all i.

The simplest example of a gerbe is the stack $Tors(G)$ on X associated to a sheaf of groups G on X. It is easy to see that this stack is in fact a gerbe. It is (globally) non-empty, since its fiber on X always contains a distinguished object, the trivial G–bundle on X. It is also locally connected, since any G–torsor is locally isomorphic to the trivial G–torsor. Conversely, a gerbe on X whose fiber $\mathcal{G}_X$ above X is non-empty is called a neutral gerbe, and the choice of an object $x \in \mathcal{G}_X$ determines an equivalence

$$\begin{aligned} \Phi_x : \mathcal{G} &\longrightarrow Tors(\underline{Aut}(x)) \\ g &\longmapsto \underline{Isom}(x,g) \end{aligned} \tag{2.1.1}$$

between $\mathcal{G}$ and a gerbe of torsors. We refer to this equivalence as the neutralization of $\mathcal{G}$ defined by the (global) object x. An arrow $f: y \longrightarrow x$ in $\mathcal{G}$ induces an isomorphism

$$\begin{aligned} \lambda: \underline{Aut}(x) &\longrightarrow \underline{Aut}(y) \\ u &\longmapsto f^{-1} u f \end{aligned} \tag{2.1.2}$$

for which the diagram

$$\begin{array}{ccc} & \mathcal{G} & \\ {}^{\Phi_x}\swarrow & & \searrow^{\Phi_y} \\ Tors(\underline{Aut}(x)) & \xrightarrow[\lambda]{} & Tors(\underline{Aut}(y)) \end{array} \tag{2.1.3}$$

is essentially commutative. The horizontal arrow λ, which is induced by (2.1.2), sends the torsor $\underline{Isom}(x,g)$ to the torsor $\underline{Isom}(y,g)$. It may be viewed as associating to a section $u: x \longrightarrow g$ of $\underline{Isom}(x,g)$ the composite section $u \circ f : y \longrightarrow g$ of $\underline{Isom}(y,g)$.

2.2 We now review the classification of gerbes in terms of cocycles given in [Br 4] §5 (*see* also [Br 2]). Let us recall that, in order to describe a torsor in cocyclic terms, one must begin by choosing a set of local trivializations of the torsor. In the gerbe case, the situation is more complicated, since a gerbe is an object fibered in categories, not in sets as in the torsor situation. Before one can hope for a cocyclic description, one therefore has to make two successive sets of choices, each of which reflects one of the two conditions occuring in definition of a gerbe. Let us begin by fixing an open cover $\mathcal{U} = (U_i)_{i \in I}$ of X, and a family of sheaves of groups G_i on the open sets U_i.

Definition 2.3: *A gerbe $\mathcal{G}$ is said to be relevant to the family of groups $\{G_i\}_{i\in I}$, or is simply called a $\{G_i\}_{i\in I}$ - gerbe, if there exists, for each $i\in I$, an object $x_i \in \mathcal{G}_{U_i}$ and an isomorphism of sheaves of groups on U_i*

$$\eta_i : G_i \longrightarrow \underline{Aut}(x_i) \tag{2.3.1}$$

between G_i and the sheaf of automorphisms of the object x_i. The choice of such a collection of objects x_i and of the corresponding isomorphisms (2.3.1) *will be called a labeling of the gerbe $\mathcal{G}$.*

The first axiom for gerbes states that any gerbe is relevant to some family of groups. Of particular interest is the special case in which the groups G_i are simply the restrictions to the open cover U_i of X of a fixed sheaf of groups G on X. In that case, a $\{G_i\}_{i\in I}$–gerbe is simply called a G–gerbe.

We now make our second set of choices. This consists in selecting, for each pair $i,j\in I$, an arrow

$$\varphi_{ij} : x_j|_{U_{ij}} \longrightarrow x_i|_{U_{ij}} \tag{2.3.2}$$

in $\mathcal{G}_{U_{ij}}$ (with the proviso that

$$\varphi_{ii} = 1_{x_i} \tag{2.3.3}$$

for all $i\in I$). The second gerbe axiom does not, however, guarantee the existence of such arrows φ_{ij}. Instead, since both the source and target of (2.3.2) are objects of $\mathcal{G}_{U_{ij}}$, it simply asserts that one can choose an open cover $\mathcal{U}_{ij} = (U_{ij}^{\alpha})_{\alpha\in J_{ij}}$ of each of the open sets U_{ij}, and a collection of arrows

$$\varphi_{ij}^{\alpha} : x_j|_{U_{ij}^{\alpha}} \longrightarrow x_i|_{U_{ij}^{\alpha}} \tag{2.3.4}$$

in $\mathcal{G}_{U_{ij}^{\alpha}}$ for varying upper indices α. When $i=j$, we may cover U_{ii} by the single open set U_i so that we may in that case drop the upper index, and set $\varphi_{ii} = 1_{x_i}$ as in (2.3.3). The notation, at this stage, becomes somewhat cumbersome, and the reader may at first prefer to assume that arrows (2.3.2) actually exists, thus in effect dropping all the upper indices. He will then obtain from the following discussion a geometric interpretation of the Čech H^2 set associated to the nerve of an open cover $\mathcal{U}$ of X, rather than an interpretation of the full H^2, viewed as the cohomology set associated to limits over the hypercovers $\mathcal{U}'$ of X associated to the various families of open sets $(U_i\ ,\ U_{ij}^{\alpha})$ (for the definition of hypercovers, *see* Verdier's appendix to exposé V of [SGA 4], [A-M]).

Definition 2.4: *Let $\mathcal{G}$ be a gerbe over a space X, $\mathcal{U} = (U_i)_{i\in I}$ an open cover of X, and for each $i, j \in I$, let $\mathcal{U}_{ij} = (U_{ij}^{\alpha})_{\alpha\in J_{ij}}$ be an open cover of the open set U_{ij} . A collection $(x_i)_{i\in I}$ of objects in $\mathcal{G}_{U_i}$, together with families of morphisms φ_{ij}^{α} (2.3.4) in $\mathcal{G}_{U_{ij}^{\alpha}}$ satisfying condition (2.3.3) is called a decomposition of the gerbe $\mathcal{G}$ relative to the hypercover $\mathcal{U}'$ of X determined by the open covers $(\mathcal{U}, \mathcal{U}_{ij})$. When a labeling and a decomposition of $\mathcal{G}$ are both given relative to the same family of objects $(x_i)_{i\in I}$ in $\mathcal{G}$, we will speak of a labeled decomposition of $\mathcal{G}$.*

In the simpler situation in which there exist morphisms φ_{ij} (2.3.2), we will say that $\mathcal{G}$ decomposes relative to the open cover $\mathcal{U}$ of X.

We now show how to associate a G_i - valued cocycle to a given labelled decomposition of a $\{G_i\}$ –gerbe $\mathcal{G}$. One first observes that the morphisms φ_{ij}^{α} induce by conjugation isomorphisms

(2.4.1)
$$\lambda_{ij}^{\alpha} = (\varphi_{ij}^{\alpha})_*: \ \underline{Aut}(x_j)_{|U_{ij}^{\alpha}} \longrightarrow \underline{Aut}(x_i)_{|U_{ij}^{\alpha}}$$
$$g_j \longmapsto (\varphi_{ij}^{\alpha})^{-1} \circ g_j \circ (\varphi_{ij}^{\alpha}) .$$

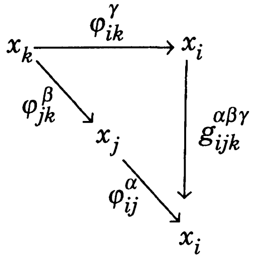
Since the morphisms φ^{α}_{ij} were chosen quite arbitrarily, there isn't any compatibility between them. For any open set

(2.4.2) $$U^{\alpha\beta\gamma}_{ijk} = U^{\alpha}_{ij} \cap U^{\beta}_{jk} \cap U^{\gamma}_{ik} \quad ,$$

the obstruction to such a compatibiliy is measured by the automorphism

(2.4.3) $$g^{\alpha\beta\gamma}_{ijk} = \varphi^{\alpha}_{ij} \circ \varphi^{\beta}_{jk} \circ (\varphi^{\gamma}_{ik})^{-1}$$

of the restriction of x_i to $U^{\alpha\beta\gamma}_{ijk}$, in other words by the following commutative diagram in $\mathcal{G}_{U^{\alpha\beta\gamma}_{ijk}}$



(2.4.4)

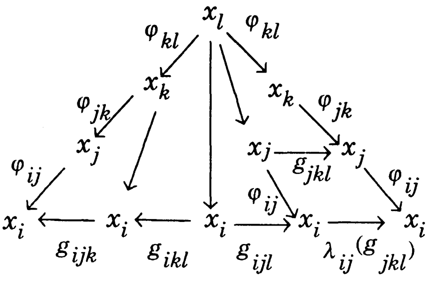
The isomorphisms (2.3.1) allow us to view the arrow $g^{\alpha\beta\gamma}_{ijk}$ as a section of the sheaf G_i on the open set $U^{\alpha\beta\gamma}_{ijk}$, and the arrow λ^{α}_{ij} as sections on U^{α}_{ij} of the sheaf <u>Isom</u> (G_j, G_i) of isomorphisms from G_j to G_i. The family of pairs of sections $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{ijk})$ is called the 2-cochain associated to the labelled decomposition $(x_i, \eta_i, \varphi^{\alpha}_{ij})$ of the $\{G_i\}$–gerbe $\mathcal{G}$. A 2-cochain constructed in this manner automatically satisfies two cocycle conditions. The first one asserts that inner conjugation of sections of $\Gamma(U^{\alpha\beta\gamma}_{ijk}, G_k)$ by either of the two composite arrows in diagram (2.4.4) necessarily yields the same result, *i.e.*, that the identity

(2.4.5) $$\lambda^{\alpha}_{ij} \circ \lambda^{\beta}_{jk} = i(g^{\alpha\beta\gamma}_{ijk}) \circ \lambda^{\gamma}_{ik}$$

between sections on $U^{\alpha\beta\gamma}_{ijk}$ of the sheaf of isomorphisms from G_k to G_i is satisfied, i being the inner conjugation map

(2.4.6)
$$\begin{aligned} i: G &\longrightarrow \underline{Aut}(G) \\ g &\longmapsto (g_1 \longmapsto g g_1 g^{-1}) . \end{aligned}$$

The next condition describes the compatibility between different sections $g^{\alpha\beta\gamma}_{i\,jk}$ defined by (2.4.3). In order to state it, we introduce the following diagram, constructed out of commuting triangles (2.4.4), in which the upper indices have been omitted for greater legibility. It embodies the different paths from x_l to x_i which may be built out of arrows (2.3.4):



(2.4.7)

Note that both diagonal edges of the outer triangle are the same composite map $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} : x_l \longrightarrow x_i$. Since $\mathcal{G}$ is a groupoid, it follows that the two composite arrows comprising the base of the triangle also coincide. Reintroducing the upper indices, this determines the second identity for the pair $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{i\,jk})$:

(2.4.8)
$$\lambda^{\alpha}_{ij}(g^{\beta\,\eta\,\varepsilon}_{j\,k\,l})\, g^{\alpha\varepsilon\delta}_{i\,jl} \;=\; g^{\alpha\beta\gamma}_{ijk}\, g^{\gamma\eta\,\delta}_{i\,k\,l} \,.$$

This formula is to be understood as follows. Let $U^{\delta}_{i\,l} \subset U_i \cap U_l$, $U^{\varepsilon}_{j\,l} \subset U_j \cap U_l$, $U^{\eta}_{k\,l} \subset U_k \cap U_l$ be the three further open sets, and let φ^{δ}_{il}, $\varphi^{\varepsilon}_{jl}$ and φ^{η}_{kl} be corresponding arrows (2.3.4). An arrow such as $g^{\beta\eta\varepsilon}_{j\,kl}$ is a section of the sheaf G_j above the open set $U^{\beta\,\eta\varepsilon}_{j\,kl} = U^{\beta}_{j\,k} \cap U^{\eta}_{k\,l} \cap U^{\varepsilon}_{j\,l}$ on which the three corresponding arrows $\varphi^{\beta}_{j\,k}$, φ^{η}_{kl} and $\varphi^{\varepsilon}_{jl}$ are defined. Each of the four terms appearing in (2.4.8) is the restriction to the open set

(2.4.9)
$$U^{\alpha\beta\gamma\delta\varepsilon\eta}_{ij\,kl} \;=\; U^{\alpha}_{ij} \cap U^{\beta}_{jk} \cap U^{\gamma}_{ik} \cap U^{\delta}_{il} \cap U^{\varepsilon}_{jl} \cap U^{\eta}_{kl}$$

of the corresponding element defined on the appropriate triple intersection, and (2.4.8) is therefore an identity between sections of the group G_i on the open set (2.4.9). In view of condition (2.3.3), the cocycles $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{ijk})$ are normalized as follows. Whenever $i=j$, there are no upper indices for λ_{ij}, and the condition

$$\lambda_{ii} = 1_{G_i} \tag{2.4.10}$$

is automatically satisfied. Furthermore, it follows from the definition (2.4.3) of the section $g^{\alpha\beta\gamma}_{ijk}$ that the upper index α vanishes whenever $i=j$, and that the condition

$$g^{\beta\beta}_{iik} = 1_{x_i} \tag{2.4.11}$$

is satisfied on U^{β}_{ik}. Similarly, the upper index β vanishes when one sets $j=k$, and one obtains, for $\alpha=\gamma$, the normalization condition

$$g^{\alpha\alpha}_{ikk} = 1_{x_i} \tag{2.4.12}$$

on the sets U^{α}_{ik}.

Suppose that we are given a second labelled decomposition of $\mathcal{G}$, this time as a $\{G'_i\}$–gerbe. Passing to a common refinement, we may assume without loss of generality that it is defined with respect to the same hypercover $\mathcal{U}' = (U^{\alpha}_{ij})$ as the original decomposition (x_i, φ_{ij}) of $\mathcal{G}$, so that it too consists of objects x'_i in $\mathcal{G}_{U_i}$, arrows η'_i (2.3.1), and arrows $\varphi'^{\alpha}_{ij}: x'_j \longrightarrow x'_i$ in $\mathcal{G}_{U^{\alpha}_{ij}}$ for varying indices $(i, j; \alpha)$. Refining the open cover once more if necessary, the gerbe axioms allow us to choose an isomorphism

$$\chi_i : x_i \longrightarrow x'_i \tag{2.4.13}$$

in $\mathcal{G}_{U_i}$. This defines an isomorphism $\mu_i : G_i \longrightarrow G'_i$ for which the following diagram commutes

$$\begin{array}{ccc} G_i & \xrightarrow{\mu_i} & G'_i \\ \lambda_i \downarrow & & \downarrow \lambda'_i \\ \underline{Aut}(x_i) & \xrightarrow[(\chi_i)_*]{} & \underline{Aut}(x'_i) \end{array} \tag{2.4.14}$$

$(\chi_i)_* : g \longmapsto \chi_i \circ g \circ (\chi_i)^{-1}$ being the conjugation map induced by the arrow χ_i.

Let $(\lambda'^{\alpha}_{ij}, g'^{\alpha\beta\gamma}_{ijk})$ be the 2-cochain defined by the (φ'^{α}_{ij}). It satisfies the 2-cocycle conditions (2.4.5) , (2.4.8), as well as the normalization conditions (2.4.10) -(2.4.12). We record from [Br 4] (5.3.5) - (5.3.6) the coboundary relation relating the pair of 2-cocycles $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{ijk})$ and $(\lambda'^{\alpha}_{ij}, g'^{\alpha\beta\gamma}_{ijk})$. Let $\delta^{\alpha}_{ij} \in \Gamma(U^{\alpha}_{ij}, G'_i)$ be the section of G'_i which measures the difference between φ'^{α}_{ij} and φ^{α}_{ij}. It is defined by

$$\varphi'^{\alpha}_{ij} = \delta^{\alpha}_{ij} \circ \chi_i \circ \varphi^{\alpha}_{ij} \circ (\chi_j)^{-1}, \tag{2.4.15}$$

so that $\delta_{ii} = 1$. Translated into the present notation, the coboundary relations between the pair of 2-cochains $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{ijk})$ and $(\lambda'^{\alpha}_{ij}, g'^{\alpha\beta\gamma}_{ijk})$ mentioned in *op.cit.*, become

$$\lambda'^{\alpha}_{ij} = i_{\delta^{\alpha}_{ij}} \circ \mu_i \circ \lambda^{\alpha}_{ij} \circ (\mu_j)^{-1} \tag{2.4.16}$$

$$g'^{\alpha\beta\gamma}_{ijk} \circ \delta^{\gamma}_{ik} = \delta^{\alpha}_{ij} \circ \Lambda^{\alpha}_{ij}(\delta^{\beta}_{jk}) \circ \mu_i(g^{\alpha\beta\gamma}_{ijk}), \tag{2.4.17}$$

where the morphism $\Lambda^{\alpha}_{ij} : G'_j \longrightarrow G'_i$ is defined by $\Lambda^{\alpha}_{ij} = \mu_i \circ \lambda^{\alpha}_{ij} \circ (\mu_j)^{-1}$. It is worth remarking that the first of these identities takes place in the group $\Gamma(U^{\alpha}_{ij}, Isom(G'_j, G'_i))$ and the second one in $\Gamma(U^{\alpha\beta\gamma}_{ijk}, G'_i)$.

If we restrict ourselves to decompositions of $\mathcal{G}$ as a $\{G_i\}$–gerbe, for a fixed family of U_i – groups (G_i), the maps μ_i (2.4.14) become automorphisms of the groups G_i, and the identities (2.4.16) and (2.4.17) respectively live in $\Gamma(U_{ij}^{\alpha}, Isom\,(G_j, G_i))$ and $\Gamma(U_{i\,jk}^{\alpha\beta\gamma}, G_i)$. The set of normalized $\{G_i\}$–valued cocycles $(\lambda_{ij}^{\alpha}, g_{i\,jk}^{\alpha\beta\gamma})$, modulo these coboundary relations, is the $\{G_i\}$–valued non-abelian H^2. We prefer to denote it by $H(X, \{G_i\})$ without any upper index at all, since it should in some sense, as we shall see, rather be viewed as an H^1 then as an H^2.

Remark 2.5: Here is a slightly different manner of viewing these coboundary relations. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are respectively a $\{G_i\}$– and an $\{H_i\}$–gerbe on X with given labeled decompositions $(x_i, \varphi_{ij}^{\alpha}, \eta_i)$ and $(y_i, \psi_{ij}^{\alpha}, \eta_i')$ relative to a common hypercover $\mathcal{U}'$ of X, and associated 2-cocycles $(\lambda_{ij}^{\alpha}, g_{i\,jk}^{\alpha\beta\gamma})$ and $(\lambda'^{\alpha}_{ij}, g'^{\alpha\beta\gamma}_{i\,jk})$. Let $\Phi: \mathcal{G} \longrightarrow \mathcal{H}$ be an equivalence between these gerbes. The objects $\Phi(x_i)$ and the arrows $\Phi(\varphi_{ij}^{\alpha}): \Phi(x_j) \longrightarrow \Phi(x_i)$ define a second decomposition of $\mathcal{H}$, labeled by the composed isomorphisms

$$\Phi \circ \eta_i : G_i \longrightarrow \underline{Aut}(x_i) \longrightarrow \underline{Aut}(\Phi(x_i)),$$

and whose associated 2-cocycle is yet again $(\lambda_{ij}^{\alpha}, g_{i\,jk}^{\alpha\beta\gamma})$. It follows that the pair of 2-cocycles $(\lambda_{ij}^{\alpha}, g_{i\,jk}^{\alpha\beta\gamma})$ and $(\lambda'^{\alpha}_{ij}, g'^{\alpha\beta\gamma}_{i\,jk})$ are related by a coboundary relation (2.4.16-2.4.17), which may therefore be viewed as describing the equivalence Φ. An equivalence class of gerbes on a space X thus determines a well-defined degree 2 cohomology class.

2.6 We now show how the previous construction may be reversed, in order to construct a $\{G_i\}$–gerbe $\mathcal{G}$ with labeled decomposition from a given $\{G_i\}$–valued 2-cocycle $(\lambda_{ij}^{\alpha}, g_{i\,jk}^{\alpha\beta\gamma})$. This is most easily done when one starts from a Čech 2-cocycle (λ_{ij}, g_{ijk}) relative to an open cover $\mathcal{U} = (U_i)$ of X, rather than from a more general upper-indexed cocycle associated to a hypercover $\mathcal{U}'$ of X. We will begin by discussing this case. Each of the given sheaves of groups G_i then determines a stack in groupoids $Tors\,(G_i)$ on the open set U_i, and the given isomorphisms

(2.6.1) $$\lambda_{ij}\colon G_j|_{U_{ij}} \longrightarrow G_i|_{U_{ij}}$$

induce "change of structure groups" functors

(2.6.2) $$\tilde{\lambda}_{ij} : Tors\,(G_j)|_{U_{ij}} \longrightarrow Tors\,(G_i)|_{U_{ij}}$$

which are equivalences of stacks on the open sets U_{ij}. The section g_{ijk} satisfies (2.4.5), or rather its simplified Čech version

(2.6.3) $$\lambda_{ij} \circ \lambda_{jk} = i(g_{ijk})\,\lambda_{ik}$$

in which the upper indices have been omitted. By lemma 1.5 *i*), the inverse of this section determines a natural transformation

(2.6.4) $$\tilde{\lambda}_{ij} \circ \tilde{\lambda}_{jk} \Rightarrow \tilde{\lambda}_{ik}$$

between the corresponding functors from $Tors(G_k)|_{U_{ijk}}$ to $Tors(G_i)|_{U_{ijk}}$. The simplified Čech version

(2.6.5) $$\lambda_{ij}(g_{jkl})\,g_{ijl} = g_{ijk}\,g_{ikl}$$

of (2.4.8) asserts that the two natural transformations between the functors $\tilde{\lambda}_{ij} \circ \tilde{\lambda}_{jk} \circ \tilde{\lambda}_{kl}$ and $\tilde{\lambda}_{ik}$ on U_{ijkl} which may be built out of transformations (2.6.2) must coincide. This is simply the tetrahedral condition (1.3.3), so that (2.6.2) and (2.6.4) determine a set of 2-descent data for the family of stacks $Tors(G_i)$. As we have seen in 1.11, the fibered 2-category $Stack_X$ is a 2-stack, so that this data is effective, and therefore determines a stack $\mathcal{G}$ on the space X. Since each of the stacks $Tors\,(G_i)$ is a gerbe, the descended stack $\mathcal{G}$ is also locally non-empty and locally connected, so that it too is a gerbe. The canonical identification of G_i with the sheaf of automorphisms of the trivial G_i torsor $(G_i)_d$ determines a labeling of the gerbe $\mathcal{G}$ by the groups G_i.

2.7 In the general case, the previous argument breaks down for several reasons. First of all, a 2-cocycle defined with respect to the most general sort of hypercover of X is a pair $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{ijk})$ satisfying conditions (2.4.5) and (2.4.8). While it is true that λ^{α}_{ij} is once more a section of the sheaf $Isom(G_j, G_i)$ on the corresponding open set U^{α}_{ij}, there no longer is *a priori* given a section $g^{\alpha\beta\gamma}_{ijk}$ of G_i on the entire set $U^{\alpha\beta\gamma}_{ijk}$ (2.4.2). Instead, one is merely given sections $(g^{\alpha\beta\gamma}_{ijk})_{\lambda}$ of G_i defined on open sets $(V^{\alpha\beta\gamma}_{ijk})_{\lambda}$ which form, as λ varies, an open cover of $U^{\alpha\beta\gamma}_{ijk}$. The identity (2.4.5) now takes place on each of these open sets $(V^{\alpha\beta\gamma}_{ijk})_{\lambda}$ and the identity (2.4.8) on the corresponding sets

(2.7.1) $$V^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl} = (V^{\alpha\beta\gamma}_{ijk})_{\lambda} \cap (V^{\alpha\varepsilon\delta}_{ijl})_{\mu} \cap (V^{\gamma\eta\delta}_{ikl})_{\nu} \cap (V^{\beta\eta\varepsilon}_{jkl})_{\rho}$$

on which the terms which comprise the latter identity are all defined. It should be noted that these open sets depend on four additional indices $(\lambda, \mu, \nu, \rho)$ so that they should really be denoted by the somewhat fearsome expression

(2.7.2) $$(V^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda\mu\nu\rho} .$$

In the sequel, we will whenever possible supress these additional indices from the notation for the open sets, and we will refer to them, when they are really needed, as the hidden indices.

Let us set $i=j$ in the identity (2.4.8), and therefore omit the corresponding upper index α. When we also set $\beta=\gamma$ and $\delta=\varepsilon$, the identity (2.4.8) reduces in view of the normalization condition (2.4.11), once the hidden indices have been taken into account, to the identity

(2.7.3) $$(g^{\beta\eta\varepsilon}_{jkl})_{\rho} = (g^{\beta\eta\varepsilon}_{jkl})_{\nu}$$

on the open sets $(V^{\beta\eta\varepsilon}_{jkl})_{\rho} \cap (V^{\beta\eta\varepsilon}_{jkl})_{\nu}$, for varying indices ρ,ν. It follows that these locally defined sections glue together so that they define a section $g^{\beta\eta\varepsilon}_{jkl}$ of G_j on the entire open set $U^{\beta\eta\varepsilon}_{jkl}$. On the other hand, the homomorphisms

(2.7.4) $$\lambda^{\alpha}_{ij} : G_{j|U^{\alpha}_{ij}} \longrightarrow G_{i|U^{\alpha}_{ij}}$$

do not descend in a similar manner, when α varies, to a homomorphism (2.6.1) on U_{ij} , so that the rest of the argument cannot be carried out exactly as in the Čech case. Let us instead consider the induced maps

(2.7.5) $$\tilde{\lambda}^{\alpha}_{ij} : Tors(G_j) \longrightarrow Tors(G_i)$$

between the corresponding stacks of torsors on U^{α}_{ij} . We now show that these maps glue, as α varies, and so determine a well-defined morphism (2.6.2) on U_{ij}. To begin with, let us set $j=k$ in the identity (2.4.5), and drop the corresponding upper index β. In view of the normalization condition (2.4.10), this identity becomes

$$\lambda^{\alpha}_{ij} = i(g^{\alpha\gamma}_{ijj})\,\lambda^{\gamma}_{ij},$$

so that the inverse of the element $g^{\alpha\gamma}_{ijj}$ determines, by lemma 1.5, a natural transformation

(2.7.6) $$\tilde{\lambda}^{\alpha}_{ij} \Rightarrow \tilde{\lambda}^{\gamma}_{ij}$$

between the corresponding morphisms (2.7.4) on U^{α}_{ij} . Setting $j=k=l$ in (2.4.8), and dropping the corresponding upper indices β, ε and η, we obtain in view of (2.4.11, 2.4.12) the relation

(2.7.7) $$g^{\alpha\delta}_{ijj} = g^{\alpha\gamma}_{ijj}\; g^{\gamma\delta}_{ijj}\,.$$

The natural transformations (2.7.6) are therefore compatible for varying α, and do indeed define a natural transformation (2.6.2) on the entire open set U_{ij}.

Finally, let us consider the condition (2.4.8) when $k=l$ (and the corresponding upper index η has been omitted). This now becomes

$$\lambda^{\alpha}_{ij}(g^{\beta\varepsilon}_{jkk})\; g^{\alpha\varepsilon\delta}_{ijk} = g^{\alpha\beta\gamma}_{ijk}\; g^{\gamma\delta}_{ikk}$$

or rather, if we replace ε by β' and δ by γ', (2.7.8) $$\lambda^{\alpha}_{ij}(g^{\beta\beta'}_{jkk})\ g^{\alpha\beta'\gamma'}_{ijk} = g^{\alpha\beta\gamma}_{ijk}\ g^{\gamma\gamma'}_{ikk}\,.$$

If we set instead $i=j$ (and hence $\alpha = 1$), identity (2.4.8) becomes

$$g^{\beta\eta\varepsilon}_{ikl}\ g^{\varepsilon\delta}_{iil} = g^{\beta\gamma}_{iik}\ g^{\gamma\eta\delta}_{ikl}\,.$$

When we set $\varepsilon=\delta$ in this expression, so that one of the terms vanishes by (2.4.11), this identity becomes, after some relabeling of the indices

(2.7.9) $$g^{\alpha\beta'\gamma'}_{ijk} = g^{\alpha\alpha'}_{iij}\ g^{\alpha'\beta'\gamma'}_{ijk}\,.$$

The substitution of (2.7.9) into (2.7.8) yields a relation between the sections $g^{\alpha\beta\gamma}_{ijk}$ and $g^{\alpha'\beta'\gamma'}_{ijk}$ of G_i. This expresses the compatibility, on the common set of definition, of the two natural transformations

(2.7.10) $$(\lambda^{\alpha}_{ij})_* \circ (\lambda^{\beta}_{jk})_* \Rightarrow (\lambda^{\gamma}_{ik})_* : Tors(G_k) \longrightarrow Tors(G_i)$$
$$(\lambda^{\alpha'}_{ij})_* \circ (\lambda^{\beta'}_{jk})_* \Rightarrow (\lambda^{\gamma'}_{ik})_* : Tors(G_k) \longrightarrow Tors(G_i)$$

determined by these sections with the gluing data (2.7.6) between the pairs of corresponding terms such as $(\lambda^{\alpha}_{ij})_*$ and $(\lambda^{\alpha'}_{ij})_*$ (we refer to diagram (6.5.12) of [Br 2] for an illustration of this). It follows that the transformations (2.7.10) glue to a natural transformation (2.6.4) defined on the entire set U_{ijk}. This natural transformation satisfies the tetrahedral condition (1.3.3), since by (2.4.8) it satisfies it locally. The proof can now be completed just as in the simpler Čech case previously considered, by invoking the gluing properties of the fibered category $Stack_X$.

2.8 The 2-cocycles encountered above are very general. Let us examine several situations in which simplifications occur.

i) Suppose that $\mathcal{G}$ is a G–gerbe, as defined in 2.3. In that case the elements λ^{α}_{ij} and μ_i (*resp.* the elements $g^{\alpha\beta\gamma}_{ijk}$ and δ^{α}_{ij}) may respectively be interpreted as sections of the sheaf $Aut\ (G)$ of automorphisms of G and of the sheaf G itself. The 2-cocycle $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{ijk})$ now takes its values in the crossed module $G \longrightarrow Aut(G)$ of lemma 1.3, and we recover the description given in [Br 2] of classes of G–gerbes as elements of the cohomology set $H^1(\mathcal{U}', G \longrightarrow Aut(G))$.

ii) We return to the case of an arbitrary $\{G_i\}$–gerbe $\mathcal{G}$, but suppose now that the given U_i– groups G_i are abelian. In that case, the formulas (2.4.5) and (2.4.16) simplify, since inner conjugation is trivial in the group G_i, so that the cocycles $g_{ijk}^{\alpha\beta\gamma}$ are uncoupled from the elements λ_{ij}^{α}. The latter glue by (2.7.6) to a section λ_{ij} of $Isom(G_j, G_i)$ on the set U_{ij}, which by (2.6.3) now satisfies the Čech 1-cocycle condition $\lambda_{ij} \circ \lambda_{jk} = \lambda_{ik}$. Thus, in the Čech cohomology situation, the g_{ijk} constitute, in the terminology of [Sc]), a 2-cocycle with values in the twisted complex of sheaves $F_i^{\cdot} = G_i[0]$ associated to the sheaves G_i concentrated in degree 0 (the twisting operation being defined here by the maps λ_{ij}). However, it should be noted that coboundary relations given in [Sc] are somewhat more restrictive than ours, since it is assumed there that the automorphisms μ_i in formulas (2.4.16)-(2.4.17) are the identity, *i.e.*, that the arrows (2.4.13) are compatible with the labelings. This can always be achieved if we are not given the labelings *a priori*.

iii) Combining the hypotheses in *i)* and *ii)*, let us give ourselves an abelian group G defined over X, whose restrictions over the open sets U_i are the abelian groups G_i. The elements λ_{ij} now define an $Aut(G)$–valued 1-cocycle on X, and therefore an $Aut(G)$–torsor P on X. The abelian group

$$^{P}G = P \wedge^{Aut(G)} G$$

defined by twisting G by the cocycle λ_{ij} is locally isomorphic to G, and formula (2.4.8) now simply says that the $g_{ijk}^{\alpha\beta\gamma}$ defines a ^{P}G - valued 2-cocycle on X in the traditional sense. In particular, when all the arrows λ_{ij} are trivial, the $g_{ijk}^{\alpha\beta\gamma}$ determine a traditional 2-cocycle on X with values in the abelian group G.

Those gerbes which are described by these traditional abelian 2-cocycles have a particularly pleasant description. Consider the following variation on definition 2.3

Definition 2.9 ([Gi] IV proposition 2.2.3.4): *Let $\mathcal{G}$ be a gerbe on X and G a sheaf of groups on X. Suppose that there exists, for every object x in a fiber category $\mathcal{G}_U$, an isomorphism of sheaves of groups $\eta_x : G_{|U} \longrightarrow \underline{Aut}(x)$, and that, for any morphism $f: x \longrightarrow y$ in $\mathcal{G}_U$, the corresponding diagram of sheaves on U*

$$\begin{array}{ccc} & G_{|U} & \\ {\scriptstyle \eta_x} \swarrow & & \searrow {\scriptstyle \eta_y} \\ \underline{Aut}(x) & \xrightarrow[\lambda]{} & \underline{Aut}(y) \end{array} \tag{2.9.1}$$

determined by the morphism λ (2.1.2) associated to f commutes. The gerbe $\mathcal{G}$ is then called an abelian G–gerbe on X.

Such an abelian G–gerbe is evidently a G–gerbe, and it follows from the definition (2.4.1) and the commutativity of diagram (2.9.1) that the component λ_{ij}^{α} of any associated 2-cocycle is trivial. We now observe that in this situation the group G is automatically abelian. Indeed, the commutativity of the group law in G may be verified locally, for sections of a sheaf $G_{|U}$. Let g be a section this sheaf and consider the diagram (2.9.1) associated to the corresponding arrow $u = \eta_x(g): x \longrightarrow x$. This is the diagram

$$
\begin{array}{ccc}
 & G_{|U} & \\
 {}^{\eta_x}\swarrow & & \searrow^{\eta_x} \\
 \underline{Aut}(x) & \xrightarrow[i_u]{} & \underline{Aut}(x)
\end{array}
\tag{2.9.2}
$$

since the map λ now is inner conjugation by u in the group $\underline{Aut}(x)$. Commutativity of this diagram implies that the map i_u is the identity map, so that the sheaf $\underline{Aut}(x)$ (and hence $G_{|U}$) is abelian. The abelian gerbe $\mathcal{G}$ is then simply described by a traditional 2-cocycles $g_{ijk}^{\alpha\beta\gamma}$, with values in the abelian group G. We refer to [Gi] IV §3.5, [De 5] 5.3 and [Bry] 5.2 for additional discussion of the relationship between gerbes and cocycles in the abelian Čech context.

2.10 We now discuss the notion of a lien on a space X ([Gi], [D-M]). Such an object is locally defined by a sheaf of groups, but in a category in which the morphisms between groups defined by inner conjugation are ignored. A cocyclic description of such a lien, in the spirit of the present text, therefore goes as follows. Consider an open cover $\mathcal{U}=(U_i)_{i\in I}$ of X, and a family of sheaves of groups G_i, defined on the open sets U_i, but now denoted $lien(G_i)$. The sheaves $lien(G_i)$ and $lien(G_j)$ are glued on the open set U_{ij} by a section ψ_{ij} of the quotient sheaf

$$Out(G_j,G_i)=G_i\backslash Isom(G_j,G_i) \tag{2.10.1}$$

on U_{ij}, the left action of a section g of G_i on a section $u\colon G_j \longrightarrow G_i$ of $Isom(G_j,G_i)$ being defined by left composition of u with the inner automorphism i_g of G_i determined by g. The lien L is thus determined by a family of sections ψ_{ij} of $Out(G_j,G_i)$ on U_{ij} satisfying the 1-cocycle condition

$$\psi_{ij}\circ\psi_{jk}=\psi_{ik} \tag{2.10.2}$$

in $Out(G_k, G_i)$, and the normalization condition

(2.10.3) $$\psi_{ii} = 1$$

for all i. A pair of such liens L and L' on X, locally defined by families of groups (G_α) and (G'_β), are isomorphic whenever there exists a common refinement $\mathcal{V} = (V_i)$ of the defining covers $\mathcal{U}$ and $\mathcal{U}'$, and a family of isomorphism of liens $\chi_i : lien(G_i) \longrightarrow lien(G'_i)$ on the open sets V_i, which are compatible with the gluing data. The cocycles (ψ_{ij}) and (ψ'_{ij}) associated to L and L' are therefore related by the coboundary conditions

(2.10.4) $$\psi'_{ij} = \chi_i \, \psi_{ij} \, \chi_j^{-1} ,$$

the isomorphisms χ_i being viewed as sections on the open sets V_i of the sheaf $Out(G_i) = G_i \backslash Aut(G_i)$ of outer automorphisms of G_i. The set of isomorphism classes of such liens L is thus described by the set[4] $H^0(X, \Gamma)$ of cohomology classes of X with values in the sheaf of groupoids Γ, whose set of objects is the indexing set I for the open cover $\mathcal{U}$ of X, the set of morphisms between a pair of objects j and i in I being the set of sections of the sheaf $Out(G_j, G_i)$. Alternately, one can say that a lien L on X corresponds to a torsor under the groupoid Γ in the sense of Haefliger [H] , or that L is an object in the stack associated to the sheaf of groupoids Γ (we refer to [Br 4] 2.11, 2.13 for a comparison between these points of view). When the lien under consideration are locally of the form $lien(G_i)$, for a family of groups G_i which are the restrictions to the open sets U_i of a given sheaf of groups G on X, the ψ_{ij} are simply section on the open set U_{ij} of the sheaf $Out(G)$ of outer automorphism of G. Such liens, which are locally isomorphic to the lien $lien(G)$, will be called here G–liens. It follows from the present discussion that they are classified by the non-abelian

---

[4] It is traditional to denote to a set of classes of groupoid-valued cocycles ψ_{ij} (2.10.2) by $H^1(X,\Gamma)$, but the notation $H^0(X, \Gamma)$ is more consistent with our conventions.

cohomology set $H^1(X, Out(G))$.This may be restated geometrically as the assertion that the functor $L \longmapsto Isom(lien(G), L)$ defines an equivalence between the category of G–liens and the category of $Out(G)$-principal bundles on X.

To every gerbe $\mathcal{G}$ on X is associated a lien on X, which is denoted by $lien(\mathcal{G})$ (*see* [Gi] IV 2.2), and this association is functorial in $\mathcal{G}$. This map a gerbe to the corresponding lien sends the neutral gerbe $Tors(G)$ to the lien $lien(G)$ represented by the group G. It is described as follows at the cocycle level. Let $(\lambda^{\alpha}_{ij}, g^{\alpha\beta\gamma}_{i\,jk})$ be the 2-cocycle associated to a labeled decomposition of a $\{G_i\}$–gerbe $\mathcal{G}$. To the section λ^{α}_{ij} of the sheaf $Isom(G_j, G_i)$ on the open sets U^{α}_{ij} corresponds the section $\psi^{\alpha}_{ij} = [\lambda^{\alpha}_{ij}]$ of the sheaf $Out(G_j, G_i)$. The situation is analogous to that discussed in 2.8 *ii*), except that we are no longer working in the sheaf $Isom(G_j, G_i)$ but rather in the quotient sheaf $Out(G_j, G_i)$ (and these are not isomorphic when G_i is non-abelian). The sections ψ^{α}_{ij} glue by (2.7.6), for varying upper indices, to a section ψ_{ij} of $Out(G_j, G_i)$ above the entire set U_{ij} and the equation (2.4.5) ensures that this section satisfies the cocycle condition (2.10.2). In the case of G–gerbes, the corresponding map from G–gerbes to G–liens is particularly easy to define in cohomological terms. By 2.8*i*), it simply corresponds to the map of pointed sets

(2.10.5) $$H^1(X, G \longrightarrow Aut(G)) \longrightarrow H^1(X, Out(G))$$

induced by the canonical map

(2.10.6) $$(G \longrightarrow Aut(G)) \longrightarrow (1 \longrightarrow Out(G))$$

between the coefficient crossed modules.

The notion of a lien provides us with the following characterization of a G-gerbe.

Proposition 2.11: *Let G be a sheaf of groups on a space X. A gerbe $\mathcal{G}$ on X is a G–gerbe if and only if its lien is locally isomorphic to lien(G).*

Proof: Since a G-gerbe is locally isomorphic to *Tors*(G), it is immediate that its lien is locally isomorphic to *lien*(*Tors*(G))=*lien*(G). Conversely, let $\mathcal{G}$ be a gerbe whose lien is isomorphic, when restricted to some open cover $\mathcal{U}$ of X, to *lien*(G) for some given sheaf of groups G. One can choose a second open cover $\mathcal{U}'=(U_i)$ of X, for which there exists a family of objects $x_i \in \mathcal{G}_{U_i}$. The gerbe $\mathcal{G}$ is then locally of the form *Tors*(G_i) for the tautological labeling of $\mathcal{G}$ defined by

$$G_i = \underline{Aut}\,(x_i). \tag{2.11.1}$$

It follows that $lien(\mathcal{G})_{|U_i}$ is isomorphic to *lien*(G_i), so that, on the elements V_α of a common refinement $\mathcal{U}''$ of $\mathcal{U}$ and $\mathcal{U}'$, we have isomorphisms of liens

$$lien(G)_{|V_\alpha} \longrightarrow lien(G_i)_{|V_\alpha}\,.$$

Such an isomorphism is defined by sections $[\eta]_\alpha$ on the open sets V_α of the sheaf *Out*(G, G_i). It follow from the definition given by (2.10.1) of this sheaf that these sections lift to sections

$$\eta_\beta : G_{|W_\beta} \longrightarrow G_{i\,|W_\beta}$$

of *Isom*(G, G_i) on the open sets W_β of an appropriate refinement $\mathcal{U}'''$ of $\mathcal{U}''$. In view of (2.11.1), the collection of maps (η_β) determines the sought-after G–gerbe structure on $\mathcal{G}$.

The following proposition supplies a characterization, along similar lines, of abelian gerbes

Proposition 2.12: *Let G be a sheaf of abelian groups on a space X. A gerbe $\mathcal{G}$*

on X is an abelian G–gerbe if and only if lien $(\mathcal{G}) \simeq$ *lien* (G).

Proof: It follows from diagram (2.9.1) that the components λ^{α}_{ij} of the 2-cocycle associated to a decomposed abelian G–gerbe $\mathcal{G}$ are all trivial, so that its lien is globally isomorphic to *lien* (G). Conversely, suppose that $\mathcal{G}$ is a gerbe with lien isomorphic to *lien* (G) for some abelian group G. For any object $x \in \mathcal{G}_U$, we can construct, as in the proof of proposition 2.11, an isomorphism of liens

$$\text{(2.12.1)} \qquad lien\,(\underline{Aut}(x)) \longrightarrow lien\,(\mathcal{G}_{|U}) \longrightarrow lien\,(G_{|U}),$$

which is described by an outer isomorphism $\underline{Aut}\,(x) \longrightarrow G_{|U}$. Since G is now abelian, there is no distinction between such an outer isomorphism and an ordinary isomorphism of sheaves of groups $\underline{Aut}(x) \longrightarrow G_{|U}$. The inverse isomorphism $\eta_x : G_{|U} \longrightarrow \underline{Aut}\,(x)$ defines an abelian G–gerbe structure on $\mathcal{G}$. The required commutativity of diagram (2.10.1) is equivalent here, since it involves abelian groups, to the commutativity of the corresponding diagram of liens. This in turn follows, by applying the lien functor to diagram (2.1.3).

2.13 The mechanism of cocycles and coboundaries described for G–gerbes in 2.8 *i*) remains valid when the coefficient crossed module $G \longrightarrow Aut\,(G)$ is replaced by an arbitrary crossed module $\delta: G \longrightarrow \Pi$. In that case, it was shown in [Br 2] that the corresponding set $H^1(X, G \longrightarrow \Pi)$ classifies the set of equivalence classes of torsors on X under the gr–stack $\mathcal{G}$ associated to the crossed module in question. The distinguished element in this set, defined by the trivial cocycle, is of course the class of the trivial torsor. When the crossed module is endowed with a braiding (1.8.4), the commutativity of the corresponding group law on $\mathcal{G}$ implies that the multiplication law m (1.2.1) on $\mathcal{G}$ is a morphism of gr–stacks (for the obvious product gr–structure on $\mathcal{G} \times \mathcal{G}$). It follows, by a line of reasoning familiar in topology, that we may define the "contracted product" morphism of $\mathcal{G}$-torsors as the composed morphism of stacks

$$(2.13.1)\qquad \begin{array}{ccccc} Tors(\mathcal{G})\times Tors(\mathcal{G}) & \longrightarrow & Tors(\mathcal{G}\times\mathcal{G}) & \longrightarrow & Tors(\mathcal{G}) \\ (\mathcal{C},\mathcal{D}) & \longmapsto & \mathcal{C}\times\mathcal{D} & \longmapsto & m_*(\mathcal{C}\times\mathcal{D}) , \end{array}$$

where $\mathcal{G}\times\mathcal{G}$ acts componentwise on the product stack $\mathcal{C}\times\mathcal{D}$. This defines a *gr*–structure on the 2-stack $\mathbb{T} =: Tors(\mathcal{G})$, in the sense to be discussed in §8 below. In particular, the group law (2.13.1) induces a group law on the set $H^1(X, G\longrightarrow \Pi)$ of equivalence classes of $\mathcal{G}$-torsors.

Let us now consider the natural transformation

$$\eta : m \Rightarrow m\circ\pi$$

between the multiplication law m and the opposite law

$$(2.13.2)\qquad m\circ\pi : \mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}$$

obtained by composing with m the permutation π of the factors of $\mathcal{G}\times\mathcal{G}$. When the crossed module δ is stable, so that, as we have seen in 1.8, $\mathcal{G}$ is a Picard stack, the natural transformation η is compatible with the *gr*-structures, in the sense made explicit for example in [Br 3] (1.1.2.5), so that it induces a commutativity condition for the group law (2.13.1) on the 2-stack *Tors* ($\mathcal{G}$). In particular, the induced group law on the set $H^1(X, G\longrightarrow \Pi)$ of equivalence classes is then abelian. Note that this fact is immediate when the crossed module $G\longrightarrow\Pi$ satisfies the stronger condition (1.8.6), since $G\longrightarrow\Pi$ is then quasi-isomorphic to a length one complex of abelian groups $A_1 \longrightarrow A_0$. In that case, $H^1(X, G\longrightarrow\Pi) = H^1(X, A_1\longrightarrow A_0)$ is a traditional hypercohomology group with values in a complex of abelian groups, so that it is automatically endowed with a commutative group structure. For the crossed module $\delta : G^{sc}\longrightarrow G$ (1.9.1), the quasi-isomorphism (1.9.4) induces an isomorphism between the corresponding cohomology group $H^1(X,\, G^{sc}\longrightarrow G)$ and Borovoi's [Bo] abelianized degree one cohomology group defined by

$$(2.13.3)\qquad H^1_{ab}(X, G) =: H^1(X, Z^{sc}\longrightarrow Z).$$

A similar geometric interpretation for the second abelianized cohomology group H^2_{ab}(X, G) will be given in 4.14.

The previous discussion specializes, in the case of the strict crossed module $A \longrightarrow 1$ defined by an abelian group A, and of its associated strict Picard stack $\mathcal{A} = Tors(A)$, to the assertion that the set $H^1(X, A[1])$ classifies the set of $\mathcal{A}$-torsors on X. Since there exists a canonical isomorphism

(2.13.4) $$H^2(X,A) = H^1(X, A[1]),$$

this provides one more interpretation for the set of equivalence classes of abelian A-gerbes, which in geometrical terms goes as follows.

Proposition 2.14: *Let A be a sheaf of abelian groups on X, and $\mathcal{A} = Tors(A)$ the associated stack of A–torsors. There exists a canonical equivalence between the 2–stack of abelian A–gerbes and the 2–stack of $\mathcal{A}$–torsors on X.*

Here is a direct proof of this proposition, along the same lines as the characterization of G–gerbes as torsors under the stack $Bitors(G)$ given in [Br 2] proposition 7.3. Let L be the lien on X defined by the abelian group A. Taking into account proposition 2.12, we may associate to any A-gerbe $\mathcal{G}$ the stack

$$\Psi(\mathcal{G}) =: \mathcal{E}q_L(Tors(A), \mathcal{G})$$

of equivalences of gerbes which induce the identity on the corresponding lien. It follows from lemma 1.7 that $\Psi(\mathcal{G})$ is a right torsor under the gr-stack $\mathcal{A}$. The opposite arrow associates to an arbitrary $\mathcal{A}$-torsor $\mathcal{P}$ on X the contracted product of stacks

$$\Phi(\mathcal{P}) =: \mathcal{P} \wedge^{\mathcal{A}} \mathcal{A}$$

for the action of $\mathcal{A}$ on itself by translation defined as in *op.cit.*, 6.7. As observed in (1.7.2), this action induces the identity at the lien level, so that $lien(\Phi(\mathcal{P})) = lien(\mathcal{A}) = L$ and $\Phi(\mathcal{P})$ is indeed, by proposition 2.12, an abelian

A-gerbe.The 2-functors Ψ and Φ are quasi-inverse, and determine the sought-after equivalence.

Remark 2.15: The previous proposition points the way, for any sheaf of abelian groups A, to an iterative definition for all $n \geq 1$ of the Picard $(n+1)$–stack of abelian n–A-gerbes $\mathcal{A}_n$ as the $(n+1)$–stack $Tors(\mathcal{A}_{n-1})$ of torsors under the Picard n–stack $\mathcal{A}_{n-1}$. This is consistent with the isomorphisms

$$H^n(X,A) = \ldots = H^1(X,A[n-1]) = H^0(X,A[n]).$$

However, a more geometric definition of such an abelian n–A-gerbe, along the lines of definitions 2.9 and of 4.13 below, is much to be preferred. Another approach to this problem, which relies on simplicial techniques, is given in [Du 2]. We refer to §8 below for the definition of a Picard 2-stack.

3. The definition of a 2-gerbe

We will now introduce the objects which embody degree three cohomology. Here and in the sequel, we have at times denoted by $f \circ g$ the composition of two 1-arrows f and g in a 2-category, in order to distinguish this operation from the horizontal or vertical composition of 2-arrows (or its degenerate versions, such as the composition

$$vg : f \circ g \Rightarrow h \circ g$$

of a 2-arrow $v: f \Rightarrow h$ with a 1-arrow g).

Definition 3.1: *A 2-gerbe $\mathbb{C}$ over a space X is a 2-stack on X which satisfies the following conditions:*

(G1) $\mathbb{C}$ is locally non-empty: *there exists an open cover $\mathcal{U} = (U_i)$ of X for which the set of objects of the 2-category $\mathbb{C}_{U_i}$ is non-empty.*

(G2) $\mathbb{C}$ is locally connected: *for each pair of objects x and y in some fiber 2-category $\mathbb{C}_U$, there exists an open cover $\mathcal{V} = (V_i)_{i \in I}$ of the open set U such that, for all $i \varepsilon I$, the set of 1-arrows from $x_{|V_i}$ to $y_{|V_i}$ in $\mathbb{C}_{V_i}$ is non-empty.*

(G3) 1-arrows are weakly invertible: *for any 1-arrow $f: x \longrightarrow y$ in a fiber 2–category $\mathbb{C}_U$, there exists an arrow $g: y \longrightarrow x$ in $\mathbb{C}_U$ which is both a left and a right inverse of f (up to a pair of 2–arrows λ: $g \circ f \Rightarrow 1_x$ and*

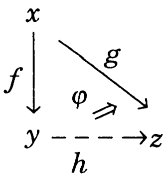
$\rho: f \circ g \Rightarrow 1_y$).

(G4) <u>2-arrows are invertible</u>: *for any 2-arrow* $u: f \Rightarrow g$ *in some fibre 2-category* $\mathbb{C}_U$ *, there exists a 2-arrow* $v: g \Rightarrow f$ *in* $\mathbb{C}_U$ *which is both a left and a right inverse for* u*. (By a cancelling argument familiar from group theory, such an inverse* v *of* u *is then in fact unique).*

A morphism (*resp.* 2- morphism, *resp.* 3-morphism) between gerbes is by definition a Cartesian morphism (*resp.* 2-morphism, *resp.* 3-morphism) between the underlying 2-stacks.

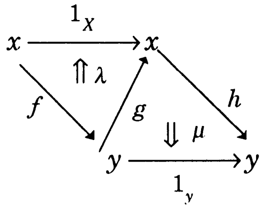
Here are various other ways of stating the axiom (G3). Consider first of all the following condition:

(G3') Given a pair of 1-arrows $f: x \longrightarrow y$ *and* $g: x \longrightarrow z$ *in a fiber 2-category* $\mathbb{C}_U$ *with a common object* x *as source, there exists a 1-arrow* $h: y \longrightarrow z$ *and a 2-arrow* $\varphi: h \circ f \Rightarrow g$ *in* $\mathbb{C}_U$*:*



This is equivalent to the requirement that any 1-arrow $f: x \longrightarrow y$ has a weak left inverse $g: y \longrightarrow x$ in $\mathbb{C}_U$, endowed with an associated 2-arrow $\lambda: g \circ f \Rightarrow 1_x$. Since the 1-arrow g itself then has a weak left inverse h and an associated 2-arrow $\mu: h \circ g \Rightarrow 1_y$, we may introduce the diagram of 2-arrows

(3.1.1)



When axiom ($G4$) is also satisfied, this defines a composite 2-arrow

$$\nu = \lambda\,\mu^{-1} : f \Rightarrow 1_y \circ f \Rightarrow h \circ g \circ f \Rightarrow h \circ 1_x \Rightarrow h, \tag{3.1.2}$$

from f to its "double left inverse" h, which may be used to define a composite 2-arrow $\rho = \mu\,(\nu\,g)$:

$$\rho:\ f \circ g \Rightarrow h \circ g \Rightarrow 1_y. \tag{3.1.3}$$

This shows that the left inverse g of f is also a right inverse of f. It follows that conditions ($G3$) and ($G3'$) are equivalent whenever ($G4$) is satisfied.

In fact, under hypothesis ($G4$), the given 2-arrow $\lambda : g \circ f \Rightarrow 1_x$ is compatible with the 2-arrow $\rho : f \circ g \Rightarrow 1_y$ defined by (3.1.3). By this we mean that the pair of 2-arrows

(3.1.4.i) $$f\lambda : f \circ (g \circ f) \Rightarrow f$$

and

(3.1.4.ii) $$\rho f : (f \circ g) \circ f \Rightarrow f$$

in $\mathcal{A}r_{\mathbb{C}_U}(x,y)$ coincide, as do the pair of 2-arrows

(3.1.5.i) $$g\rho : g \circ (f \circ g) \Rightarrow g$$

and

(3.1.5.ii) $$\lambda g :\ (g \circ f) \circ g \Rightarrow g$$

in $\mathcal{A}r_{\mathbb{C}_U}(y,x)$. Indeed, the identification of the pair of 2-arrows (3.1.4) follows easily from the definition of ρ and the axioms for 2-categories. To prove that the two 2-arrows (3.1.5) also coincide is somewhat more delicate. Observing that the double left inverse h of f itself has a left inverse $k : y \longrightarrow x$, endowed with a 2-arrow $\nu :\ kh \Rightarrow 1_X$, we may apply construction (3.1.3) to g. This yields a 2-arrow

$$\sigma : g \circ h \Rightarrow k \circ h \Rightarrow 1_x$$

in $\mathcal{A}r(x,x)$ and the identification of the two 2-arrows (3.1.4) associated to the morphism g shows that the pair of 2-arrows $g\mu$ and σg: $g\circ h\circ g\Rightarrow g$ in $\mathcal{A}r(y,x)$ coincide. This is therefore also the case of the composites of each of these 2-arrows with the 2-arrow $g\rho g: g\circ f\circ g\Rightarrow g\circ h\circ g$. But it follows from the definition (3.1.3) of ρ that the first of these composites is just $g\rho$, while the axioms for 2-categories imply that the second is precisely λg. This finishes the proof that the two 2-arrows (3.1.5) coincide.

We summarize the previous discussion by stating that the following axiom ($G3''$), while apparently more restrictive then ($G3$), is in fact equivalent to it whenever ($G4$) is satisfied:

($G3''$) For any 1-arrow $f: x \longrightarrow y$ in a fiber 2–category $\mathbb{C}_U$, there exists an arrow g: $y \longrightarrow x$ in $\mathbb{C}_U$ which is both left and right inverse of f, up to a pair of 2–arrows λ: $g\circ f\Rightarrow 1_x$ and ρ: $f\circ g \Rightarrow 1_y$ satisfying the compatibility conditions (3.1.4) *and* (3.1.5).

We say that g is a coherent inverse of f whenever the conditions ($G3''$) are satisfied. A 2-stack $\mathscr{C}$ which satisfies conditions ($G3$) and ($G4$) deserves to be called a 2-stack in 2-groupoids (or, more simply, a 2-groupoid stack), since each fiber 2-category has coherently invertible 1-arrows, and invertible 2-arrows. Note, however, that a 2-category satisfying the more restrictive condition that both 1- and 2-arrows be strictly invertible is sometimes referred to as a 2-groupoid (*see* for example [Mo-Sv]).

An inverse g of a 1-arrow f is unique up to a unique 2-isomorphism: if we are given a second arrow g': $y\longrightarrow x$, and a pair of corresponding compatible 2-arrows λ' and ρ', then the composite 2-arrow

$$(\lambda')^{-1}\,\lambda : g\circ f\Rightarrow 1\Rightarrow g'\circ f$$

yields, by right cancellation of f, a 2-arrow η: $g\Rightarrow g'$, and the compatibility condition (3.1.4) then implies that the diagram

(3.1.6)

$$\begin{array}{ccc} gf & \overset{\eta f}{\Rightarrow} & g'f \\ & {}_{\lambda}\!\searrow\!\!\!\searrow \quad \swarrow\!\!\!\swarrow_{\lambda'} & \\ & 1_x & \end{array}$$

commutes. In fact the 2-arrow η is also compatible with the right cancellation 2-arrows ρ and ρ'. This is verified by observing that the two maps $(f\eta)\circ\rho'$ and $\rho\colon fg \Rightarrow 1$ coincide, since right composition of each of these two maps with f yields a diagram

(3.1.7)

$$\begin{array}{ccc} (fg)f & \overset{(f\eta)f}{\Rightarrow} & (fg')f \\ & {}_{\rho f}\!\searrow\!\!\!\searrow \quad \swarrow\!\!\!\swarrow_{\rho' f} & \\ & f & \end{array}$$

whose commutativity follows from the compatibility condition (3.1.4) and from the commutativity of diagram (3.1.6).

There also exist local versions of the previous axioms. Here, for example, is the local version of (G3'):

$(G3'_{loc})$ Given a pair of arrows $f\colon x \longrightarrow y$ and $g\colon x \longrightarrow z$ in a fiber 2-category $\mathbb{C}_U$ with a common object x as source, there exists an open cover $\mathcal{U} = (U_i)_{i\in I}$ of U, and, for each $i \in I$, an arrow $h_i\colon y_{|U_i} \longrightarrow z_{|U_i}$ and a 2-arrow $\varphi_i\colon h_i \circ f_{|U_i} \Rightarrow g_{|U_i}$ in $\mathbb{C}_{U_i}$:

$$\begin{array}{ccc} x_{|U_i} & & \\ {\scriptstyle f_{|U_i}}\downarrow & \overset{\varphi_i}{\nearrow\!\!\!\nearrow} \searrow^{g_{|U_i}} & \\ y_{|U_i} & \xrightarrow[h_i]{} & z_{|U_i} \end{array}$$

(3.1.8)

Similary, here is the local version of the requirement that any 1-arrow have a weak left inverse:

For every 1-arrow $f\colon x \longrightarrow y \in \mathbb{C}_U$, there exists an open cover $\mathcal{U} = (U_i)$

of U, and a family of local weak left inverses $g_i : y_{|U_i} \longrightarrow x_{|U_i}$ of $f_{|U_i}$ in $\mathbb{C}_{U_i}$ (with associated 2-arrows $u_i : g_i \circ f_{|U_i} \Rightarrow 1_{x|U_i}$).

The line reasoning used above in the global situation now shows that, under condition ($G4$), the local left weak inverses g_i are also local weak right inverses (after passing to a possibly finer cover of the open set U). In fact, these local versions of ($G3$) and its variants are equivalent, whenever ($G4$) is satisfied, to the a priori more restrictive global versions. Suppose for example that one is given as above an arrow $f: x \longrightarrow y \in \mathbb{C}_U$ and a family of local weak inverses g_i in $\mathbb{C}_{U_i}$ for some open cover $\mathcal{U} = (U_i)$ of U. The restrictions of g_i and of g_j above U_{ij} are both left inverses of $f_{|U_{ij}}$, so that the uniqueness argument for inverses for 1-arrows shows that there exists a 2-arrow $\eta_{ij} : g_{j\,|U_{ij}} \longrightarrow g_{i\,|U_{ij}}$ such that the corresponding (3.1.6) diagram above U_{ijk}

$$\begin{array}{ccc} g_j \circ f & \overset{\eta_{ij} f}{\Rightarrow} & g_i \circ f \\ & {}_{\lambda}\!\searrow \quad \swarrow_{\lambda'} & \\ & 1_{x\,|U_{ij}} & \end{array}$$

commutes. It follows that the relation

$$(\eta_{ij} f)\,(\eta_{jk} f) = \eta_{ik} f$$

is satisfied by the 2-arrows in question from $g_k \circ f$ to $g_i \circ f$, so that right cancellation of the 1-arrow f by one of its inverses above U_{ijk} implies that the 2-arrows η_{ij} satisfy the cocycle condition $\eta_{ij} \circ \eta_{jk} = \eta_{ik}$ on this open set. Since $\mathbb{C}$ is a 2-stack, the local inverses g_i therefore descend to a global 1-arrow $g \in \mathbb{C}_U$, which is the sought-after global left inverse of f.

Remark 3.2: A simpler version of the previous discussion shows that condition ($G4$) can also be replaced by one of the following a priori weaker condition:

(G4′): every 2–arrow $\varphi: f \Rightarrow g$ in a fibre category $\mathbb{C}_U$ is left invertible

($G4'_{loc}$): every 2-arrow $u: f \Rightarrow g$ in a fibre category $\mathbb{C}_U$ is locally left invertible

($G4_{loc}$): *every 2-arrow u: $f \Rightarrow g$ in a fibre category $\mathbb{C}_U$ is locally invertible.*

3.3 Let $\mathbb{C}$ be a 2-groupoid stack over a space X. For any object $x \in \mathbb{C}_U$, the prestack in groupoids $\underline{Aut}(x)$ of self-arrows of x is endowed with a (stricly associative) monoidal structure determined by composition of 1-arrows, and it follows from ($G3$), that this is group-like. In fact, we have just seen that local inverses for 1-arrows always descend to global ones, so that $\underline{Aut}(x)$ is actually a gr–stack on U, as defined in 1.2, once specific inverses for 1-arrows have been chosen.

Let $\mathcal{U}=(U_i)_{i\in I}$ be an open cover of X, and let us give ourselves, for each $i \in I$, a gr–stack $\mathcal{G}_i$ on the corresponding open set U_i. The following definition is modeled on definition 2.3.

Definition 3.4: *A 2–gerbe $\mathbb{C}$ is said to be relevant to the family of gr–stacks $\{\mathcal{G}_i\}_{i\in I}$, or is simply be called a $\{\mathcal{G}_i\}_{i\in I}$ -2-gerbe, if there exists a family of objects $x_i \in \mathbb{C}_{U_i}$ and a family of equivalences of gr–stacks*

$$\eta_i : \mathcal{G}_i \longrightarrow \underline{Aut}(x_i) \tag{3.4.1}$$

above U_i. When the gr–stacks $\mathcal{G}_i$ are all restrictions to the open sets U_i of a given gr–stack $\mathcal{G}$ on X, $\mathbb{C}$ will be called a $\mathcal{G}$–2–gerbe. The choice of a collection of objects x_i and of the corresponding equivalences (3.4.1) is called a labeling of the $\{\mathcal{G}_i\}$–2–gerbe $\mathbb{C}$.

In particular, one may consider the 2-gerbes on a space X which are relevant to the family of *gr*–stacks $\mathcal{G}_i$ associated to a collection of crossed modules $(G_i \longrightarrow \Pi_i)$ defined on an open cover $(U_i)_{i \in I}$ of X. Of special interest is the case in which one starts from a fixed crossed module $(G \longrightarrow \Pi)$ on X and considers the collection of all $\mathcal{G}$–2–gerbes on X, for $\mathcal{G}$ the *gr*–stack associated of the presheaf of categories defined by $(G \longrightarrow \Pi)$.

4. From 2-gerbes to 3-cocycles

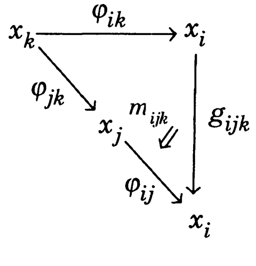
4.1 We will now explain how to attach a 3-cocycle to a 2-gerbe $\mathbb{C}$. The method is the one by which we associated a 2-cocycle to an ordinary gerbe in § 2, but pushed one step further. We begin by observing that, by axiom $(G1)$, we may choose an open cover $\mathcal{U} = (U_i)_{i \in I}$ of X and a family of objects x_i in the fiber 2-categories $\mathbb{C}_{U_i}$. Axiom $(G2)$ then allows us to choose an open cover $\mathcal{U}_{ij} = (U_{ij}^{\alpha})$ of each open set U_{ij} and a family of 1-arrows φ_{ij}^{α} in $\mathbb{C}_{U_{ij}^{\alpha}}$, as in (2.3.4). In order to simplify the present discussion, by keeping the notation under control, let us temporarily replace axiom $(G2)$ by the following overly restrictive hypothesis

(G2') $\underline{\mathbb{C}\text{ is connected}}$: *for each pair of objects x and y in some fiber category $\mathbb{C}_U$, the set of arrows in $\mathbb{C}_U$ from x to y is non–empty.*

When this condition is satisfied, we may choose, for each pair (i, j), a 1-arrow

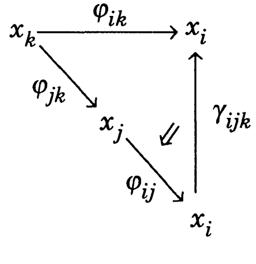
(4.1.1) $$\varphi_{ij} : x_j \longrightarrow x_i$$

in $\mathbb{C}_{U_{ij}}$ as in (2.3.2), rather than simply a family of arrows φ_{ij}^{α} for varying α's as in (2.3.4). We may even suppose by $(G3'')$ that the arrow (4.1.1) is invertible, with given compatible left and right inverses, up to specified 2-arrows. When $i=j$, we make the obvious choice $\varphi_{ij}=1$. One can no longer, as in (2.4.3), measure the compatibility of the arrows $\varphi_{ij} \circ \varphi_{jk}$ and φ_{ik} by introducing an arrow g_{ijk} for which diagram (2.4.4) commutes. However, axiom $(G3')$ allows us to choose a (weak) automorphism g_{ijk} of $x_{i\,|\,U_{ijk}}$ and a 2-arrow $m_{ijk} : g_{ijk} \circ \varphi_{ik} \Rightarrow \varphi_{ij} \circ \varphi_{jk}$ in $\mathbb{C}_{U_{ijk}}$, and to build a diagram



(4.1.2)

in the fiber 2-category $\mathbb{C}_{U_{ijk}}$. *We will denote this diagram by* T_{ijk} . We make the obvious choice $(g_{ijk}, m_{ijk}) = (1,1)$ whenever $i=j$ or $j=k$, so that the diagram T_{ijk} is trivial whenever two successive indices are equal. We will not in the sequel make an explicit distinction between diagram (4.1.2) and those which may be derived from it by elementary operations, particularly the corresponding diagram in which the 2-arrow m_{ijk} is replaced by its inverse, or the one in which the arrow g_{ijk} is reversed. The latter is essentially the diagram $\gamma_{ijk} T_{ijk}$ obtained by composing the 2-arrow m_{ijk} on the left with an inverse γ_{ijk} of the 1-arrow g_{ijk} :



(4.1.3)

Definition 4.2: *Let* $\mathbb{C}$ *be a 2–gerbe over a space* X *and* $\mathcal{U} = (U_i)_{i \in I}$ *an open cover of* X *. A collection of objects* $(x_i)_{i\in I}$ *in* $\mathbb{C}_{U_\alpha}$*, of families of* 1*–arrows* φ_{ij} (4.1.1) *in* $\mathbb{C}_{U_{ij}}$ *, of objects* g_{ijk} *in the category* $\mathcal{A}ut(x_{i\,|\,U_{ijk}})$ *and of arrows* m_{ijk} (4.1.2) *in the category* $\mathcal{A}r(x_{k\,|\,U_{ijk}}, x_{i\,|\,U_{ijk}})$ *is called a decomposition of the*

2–gerbe $\mathbb{C}$ relative to the cover $\mathcal{U}$. When a labeling (3.4.1) *and a decomposition of $\mathbb{C}$ are both given relative to the same family of objects $(x_i)_{i \in I}$ of $\mathbb{C}$, we will speak of a labeled decomposition of the $\{\mathcal{G}_i\}$–2–gerbe $\mathbb{C}$.*

Let us suppose that we have been able to chose such a labeled decomposition of $\mathbb{C}$ relative to $\mathcal{U}$. We saw in 3.1 that the inverse ψ_{ij} (in the weak sense made precise in axiom ($G3''$) of the invertible arrow φ_{ij} is uniquely defined, up to a canonical 2-arrow. The choice of a specific such inverse ψ_{ij} determines an equivalence of gr–stacks

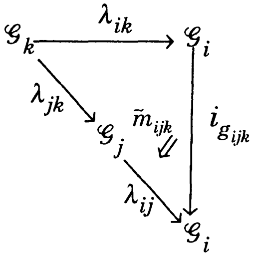
(4.2.1)
$$\begin{aligned} \lambda_{ij} : \mathcal{G}_j|_{U_{ij}} &\longrightarrow \mathcal{G}_i|_{U_{ij}} \\ g &\longmapsto \varphi_{ij} \circ g \circ \psi_{ij} \end{aligned}$$

on U_{ij} which is the analog, in the present context, of (2.4.1). This may be viewed as a section over U_{ij} of the stack $\mathcal{E}q\,(\mathcal{G}_j, \mathcal{G}_i)$, whose fiber $\mathcal{E}q\,(\mathcal{G}_j, \mathcal{G}_i)_V$ on each open set $V \in U_{ij}$ is the category $\mathcal{E}q\,(\mathcal{G}_j|_V, \mathcal{G}_i|_V)$. It follows from the definition of λ_{ij} that, for every object $g \in \mathcal{G}_j|_{U_{ij}}$, there exists a canonical 2-arrow

$$\begin{array}{ccc} x_j & \xrightarrow{\quad g \quad} & x_j \\ {\scriptstyle \varphi_{ij}}\big\downarrow & \overset{\eta}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle \varphi_{ij}} \\ x_i & \xrightarrow[\lambda_{ij}(g)]{\qquad} & x_i \end{array}$$

(4.2.2)

There are two further sets of data which may be extracted from the given decomposition of the 2-gerbe $\mathbb{C}$. First of all, we may use the various components of diagram (4.1.2) in order to conjugate elements of $\mathcal{G}_k$. If we respectively denote by λ_{jk} and λ_{ik} the equivalences associated as in (4.2.1) to φ_{jk} and φ_{ik}, and to their chosen inverses ψ_{jk} ψ_{ik}, and by $\lambda_{(ik)}$ the corresponding equivalence from $\mathcal{G}_k$ to $\mathcal{G}_i$ determined by the composite arrow $\varphi_{ij} \circ \varphi_{jk}$ and its inverse $\psi_{jk} \circ \psi_{ij}$, then the essential uniqueness of inverses of 1-arrows in $\mathbb{C}$ yields a natural transformation $\lambda_{ij} \circ \lambda_{jk} \Rightarrow \lambda_{(ik)}$. It then follows that the 2-arrow m_{ijk} (4.1.2) induces a natural transformation $\tilde{m}_{ijk}$



(4.2.3)

between functors from $\mathcal{G}_k$ to $\mathcal{G}_i$, where for an object g in a group-stack $\mathcal{G}$ with chosen inverse object, we have denoted by

$$i_g : \mathcal{G} \longrightarrow \mathcal{G}$$

the inner conjugation functor defined by the object g in the gr–stack $\mathcal{G}$. This natural transformation $\tilde{m}_{ijk}$ yields, for any object $\gamma \in \mathcal{G}_k$, an arrow

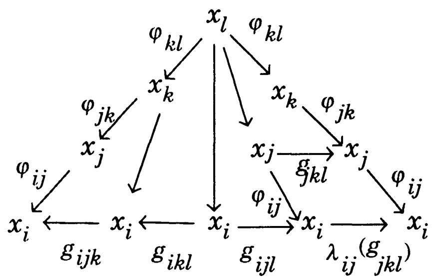
$$\tilde{m}_{ijk}(\gamma) : \; i_{g_{ijk}} \circ \lambda_{ik}(\gamma) \longrightarrow \lambda_{ij} \circ \lambda_{jk}(\gamma) \tag{4.2.4}$$

in $\mathcal{G}_i$. By right multiplication by the inverse object of its source, this arrow corresponds to an arrow

$$\{\tilde{m}_{ijk}, \gamma\} : 1_{x_i} \longrightarrow \lambda_{ij} \circ \lambda_{jk}(\gamma) \circ (i_{g_{ijk}} \circ \lambda_{ik}(\gamma))^{-1} \tag{4.2.5}$$

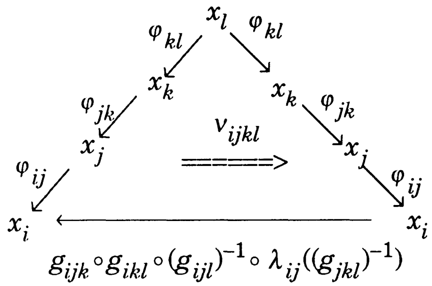
in $\mathcal{G}_i$, sourced at the identity, The natural transformation $\tilde{m}_{ijk}$, together with the associated family of arrows (4.2.5), is the first of the two additional sets of data which we associate to the chosen decomposition of $\mathbb{C}$.

In order to obtain the second additional set of data determined by the decomposition of $\mathbb{C}$, let us consider, following the scheme set up in diagram (2.4.7), the following diagram of 2-arrows built up from the four triangles $T_{ijk}, T_{ikl}, T_{ijl}, T_{jkl}$ (4.1.2) and the square (4.2.2).



(4.2.6)

Inverting the requisite 2-arrows in the two left-hand cells of this diagram, and the requisite 1-arrows in the two right hand triangles, as indicated in (4.1.3), we see that diagram (4.2.6) yields a composite 2-arrow

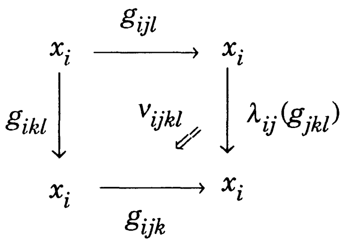


(4.2.7)

This in turn determines, when multiplied on the right by the inverse of the oblique arrow, an arrow

(4.2.8)
$$\nu_{ijkl}\colon 1_{x_i} \Rightarrow g_{ijk} \circ g_{ikl} \circ (g_{ijl})^{-1} \circ \lambda_{ij} (g_{jkl})^{-1}$$

in $\mathcal{G}_{i\,|\,U_{ijkl}}$ sourced at the identity. The latter arrow is the final set of data which can be extracted from the chosen decomposition of $\mathbb{C}$. It can also be viewed as a 2-arrow



(4.2.9)

This 2-arrow ν_{ijkl} may be characterized more explicitly by the following identity between arrows (4.1.2) in the category $\mathcal{A}r(x_l, x_i)$, which is simply an algebraic formulation for the manner in which it has been constructed in diagram (4.2.6) from the corresponding triangles T_{ijk} (here the effect of the tautological 2-arrow (4.2.2) is disregarded):

$$m_{ijk}({}^{g_{ijk}}m_{ikl})\,\nu_{ijkl} = (\varphi_{ij} \circ m_{jkl})({}^{\lambda_{ij}(g_{jkl})}m_{ijl})\,. \qquad (4.2.10)$$

In this formula, the following conventions have been taken into account. For any 2-arrow $m\colon \lambda \Rightarrow \lambda'$ in $\mathcal{A}r(y, x)$ and any arrow $g\colon x \longrightarrow x$, the induced 2-arrow $g\lambda \Rightarrow g\lambda'$, defined by left composition with g, has been denoted by ${}^{g}m$. On the other hand, the map $\lambda h \Rightarrow \lambda' h$ induced by *right* composition of a 2-arrow m with an arbitrary arrow $h\colon z \longrightarrow y$ is simply denoted by m. While these notations are not the categorical notations of §3, we find them to be more convenient for cocycle calculations, particularly when we interpret these cocycles, as we shall in 4.7, as taking their values in a crossed module of gr–stacks.

With these same conventions, it follows, by applying the inner conjugation functor i to formula (4.2.10), that the 1-arrow i_ν determined in $\mathcal{E}q(\mathcal{G}_i)$ by the 1-arrow $\nu = \nu_{ijkl}$ in $\mathcal{G}_i$ satisfies the corresponding identity

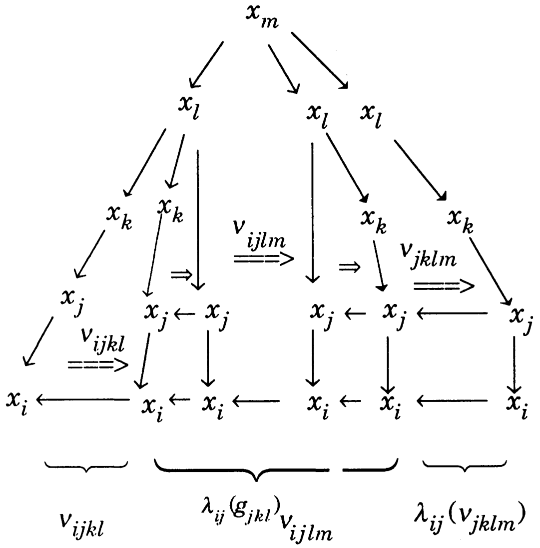
$$\tilde{m}_{ijk}\,({}^{g_{ijk}}\tilde{m}_{ikl})\,i_\nu = (\lambda_{ij}(\tilde{m}_{jkl}))\,({}^{\lambda_{ij}(g_{jkl})}\tilde{m}_{ijl}) \qquad (4.2.11)$$

in $\mathcal{E}q(\mathcal{G}_l, \mathcal{G}_i)$. Here an expression such as ${}^{g}\tilde{m}$ describes the arrow in $\mathcal{E}q(\mathcal{H}, \mathcal{G})$ associated to an arrow $\tilde{m} \in \mathcal{E}q(\mathcal{H}, \mathcal{G})$ and an object $g \in \mathcal{G}$, according to the pattern

$$\mathcal{H} \underset{\longrightarrow}{\overset{\longrightarrow}{\tilde{m}\Downarrow}} \mathcal{G} \xrightarrow{\;\; i_g \;\;} \mathcal{G}$$

where i_g is the inner conjugation by g, viewed as an object in $\mathcal{E}q(\mathcal{G})$.

The arrow (4.2.8) in $\mathcal{G}_i$, satisfying condition (4.2.11), is the sought after non-abelian 3-cochain associated to the decomposed 2-gerbe $\mathbb{C}$. In order to state the 3-cocycle condition which it satisfies, let us begin by considering the following diagram, which is defined on the open set U_{ijklm} :



(4.2.12)

The labeled triangles in this diagram refer to the restrictions to this open set of the corresponding 2-arrows (4.2.7). We will adopt the convention that any unlabeled triangle in diagram (4.2.12) is of the form (4.1.2) and any square is of the form (4.2.2). Observe that diagram (4.2.12) is built out of three large triangles with a horizontal basis, set side by side, whose vertices are x_m, x_i and x_i and whose non-horizontal edges are all defined by the composite arrow

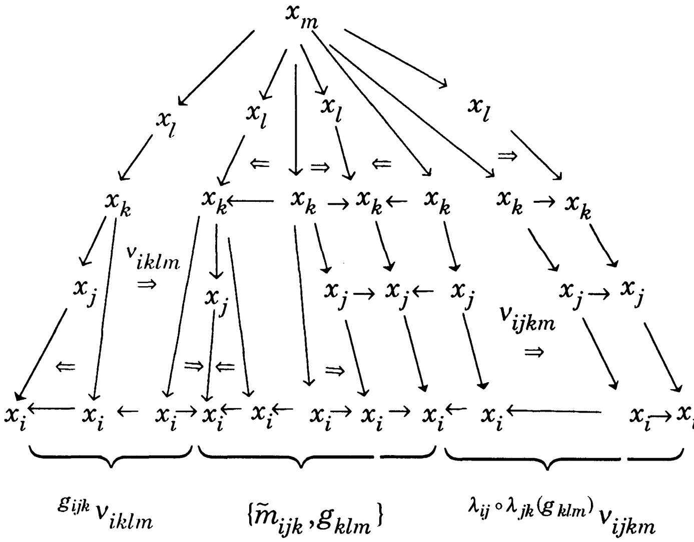
(4.2.13) $$\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} \circ \varphi_{lm} \colon\ x_m \to x_l \to x_k \to x_j \to x_i \ .$$

These triangles may respectively be denoted, once the auxiliary small triangles (4.1.2) and squares (4.2.2) are taken into account, by v_{ijkl} , ${}^{\lambda_{ij}(g_{jkl})}v_{ijlm}$, and $\lambda_{ij}(v_{jklm})$. Indeed, after right composition with an inverse of arrow (4.2.13), each of these triangles may be identified with an arrow in $\mathcal{G}_i$ sourced at the identity, and whose target is the base of the corresponding triangle. The notation just chosen for these triangles is then consistent with the standard notation ${}^{g}v$ for the conjugation of an arrow v sourced at the identity in a gr–stack $\mathcal{G}$ by an object g in $\mathcal{G}$, and with the notation $\lambda(v)$ for the image by a functor $\lambda: \mathcal{G}_j \longrightarrow \mathcal{G}_i$ of an arrow v in $\mathcal{G}_j$. Since juxtaposition of the triangles corresponds to multiplication of arrows in the gr–stack $\mathcal{G}_i$, under the group law of $\mathcal{G}_i$, the full diagram (4.2.12) is described by the arrow

(4.2.14)
$$\lambda_{ij}(v_{jklm})\,({}^{\lambda_{ij}(g_{jkl})}v_{ijlm})\;\; v_{ijkl}$$

in $\mathcal{G}_i$ sourced at 1_{x_i} .

Diagram (4.2.12) can now be compared with the analogous diagram



(4.2.15)

which is defined on the same open set U_{ijklm} .We adopt the same conventions here as in (4.2.12), except that we have chosen, in order to emphasize certain cancellations, not to make explicit the inversion of certain 1- and 2-arrows. Again, this diagram is built up from three adjacent large triangles with the same vertices x_m , x_i , x_i and same oblique edges (4.2.13) as in those composing (4.2.12). The first and the third of these triangles are respectively described, with the same notation as above, by the arrows ${}^{g_{ijk}}v_{iklm}$ and ${}^{\lambda_{ij}\circ\lambda_{jk}(g_{klm})}v_{ijkm}$ in $\mathcal{G}_i$, sourced at 1_{x_i} . In view of the cancellation between the two 2-arrows μ_{klm} at the top of the second triangle, the middle one simply corresponds to the arrow $\{\tilde{m}_{ijk}, g_{klm}\}$ defined in (4.2.5). It follows from this discussion that diagram (4.2.15) is described by the composite arrow

(4.2.16) $$({}^{\lambda_{ij}\circ\lambda_{jk}(g_{klm})}v_{ijkm})\{\tilde{m}_{ijk}, g_{klm}\}\ ({}^{g_{ijk}}v_{iklm})$$

in $\mathcal{G}_i$.

By construction, the two arrows (4.2.14) and (4.2.16) in $\mathcal{G}_i$ have 1_{x_i} as a common source, and it may be verified, by making explicit by (4.2.5) and (4.2.8) the target of each of their constituents, that they also have the same target. They may therefore be compared, and the cocycle condition which we have been seeking asserts that they coincide, in other words that the following relation holds between 1-arrows sourced at the identity in $\mathcal{G}_{i|U_{ijklm}}$:

(4.2.17)
$$\lambda_{ij}(v_{jklm})\ ({}^{\lambda_{ij}(g_{jkl})}v_{ijlm})\ v_{ijkl} = ({}^{\lambda_{ij}\circ\lambda_{jk}(g_{klm})}v_{ijkm})\ \{\tilde{m}_{ijk}, g_{klm}\}\ ({}^{g_{ijk}}v_{iklm}).$$

We will henceforth denote this identity by the symbol I_{ijklm}. It may be verified, by decomposing each of the five 2-arrows v (4.2.7) appearing in this formula into its constituent 2-arrows, that this cocycle condition is indeed always satisfied by the 2-cochain v defined in (4.2.7). This verification is most efficiently carried diagrammatically, by inserting as in (4.2.6) into each of the five heptagons v ocurring in diagrams (4.2.12) and (4.2.15) the appropriate four triangles (4.1.2) and the corresponding square (4.2.2), and by working out the requisite cancellations.

The previous discussion may now be summarized as follows. Let $\mathbb{G}$ be a 2-gerbe on a space X. Let us choose as in definition 4.2, for some open cover $\mathcal{U}$ of X, a labeled decomposition (x_i , φ_{ij} , g_{ijk} , m_{ijk} ; η_i) of $\mathbb{G}$ relative to

$\mathcal{U}$. To this data, and to the chosen local gr–stacks $\mathcal{G}_i$ defined above U_i we have associated the following elements:

4.2.i) an object λ_{ij} (4.2.1) in the category $\mathcal{E}q\,(\mathcal{G}_j\,|\,_{U_{ij}}, \mathcal{G}_i\,|\,_{U_{ij}})$ of equivalences between the group-stacks $\mathcal{G}_j\,|\,_{U_{ij}}$ and $\mathcal{G}_i|_{U_{ij}}$.

4.2.ii) an arrow g_{ijk} (4.1.2), viewed as an object in the fiber of the stack $\mathcal{G}_i$ over the open set U_{ijk}.

4.2.iii) an arrow $\tilde{m}_{ijk}$ (4.2.3) in the category $\mathcal{E}q\,(\mathcal{G}_j, \mathcal{G}_i)_{U_{ijk}}$.

4.2.iv) an arrow ν_{ijkl} (4.2.8) in the category $(\mathcal{G}_i)_{U_{ijkl}}$, for which the identity (4.2.11) is valid, and which satisfies the cocycle condition (4.2.17) in $(\mathcal{G}_i)_{U_{ijklm}}$.

Definition 4.3: *A quadruple of elements* $(\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij})$ *defined as in* 4.2.i)–4.2.iv) *is called a* $\{\mathcal{G}_i\}_{i\in I}$ *–valued non–abelian 3–cocycle on the space X.*

Neglecting certain harmless identifications, the following normalization conditions follow from the choices which we have made in the degenerate cases of (4.1.1) and (4.1.2):

4.3.i) $\lambda_{ij}=1$ whenever $i=j$.

4.3.ii) $(g_{ijk}, \tilde{m}_{ijk})=(1, 1)$ whenever $i=j$ or $j=k$.

4.3.iii) $\nu_{ijkl} = 1$ whenever $i=j$, $j=k$, or $k=l$.

Condition 4.3.ii) illustrates the fact that the elements g_{ijk} and $\tilde{m}_{ijk}$ are to a certain extent linked.

The terminology introduced in definition 4.3 takes into account the fact that a 3-cocycle, as defined here, takes its values in the family of gr–stacks $\mathcal{G}_i$. When we wish to emphasize this in the notation, we will denote the cocycle by $(\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij}\,;\, \mathcal{G}_i)$. One might instead put the emphasis on the pair (v_{ijkl}, g_{ijk}), and consider that it defines in itself a twisted cocycle taking its values in the gr–category $\mathcal{G}_i$, the twisting being determined by the additional data $(\tilde{m}_{ijk}, \lambda_{ij})$. As we have observed in 2.10 in the simpler case of 1-gerbes, the twisting data $(\tilde{m}_{ijk}, \lambda_{ij})$ cannot in general be uncoupled from the cocycle pair (v_{ijkl}, g_{ijk}), unless some commutativity hypothesis on the group law of $\mathcal{G}_i$ is postulated. It is however worth pointing out that one could mimic Giraud's approach to gerbes, by introducing the appropriately defined 2-lien $\mathcal{L}$ on X, determined, along the lines reviewed for ordinary liens in 2.8 (and also in 6.1 below), by the appropriately defined outer isomorphism class of the pair $(\tilde{m}_{ijk}, \lambda_{ij})$. One could then view the cocycle pair (v_{ijkl}, g_{ijk}) as defining a class in a corresponding $\mathcal{L}$-valued cohomology set $H^2(X, \mathcal{L})$. A third option is to focus on v_{ijkl} , and to consider that it embodies by itself the sought-after non-abelian 3-cocycle, with the triplet $(\tilde{m}_{ijk}, g_{ijk}, \lambda_{ij})$ providing some auxiliary twisting data.

4.4 In order for the description by cocycles of a 2-gerbe $\mathbb{C}$ to be of intrinsic significance, it is essential that we understand how the cocycle in question varies when we pass to a second decomposition of $\mathbb{C}$. Let us give ourselves a new family of gr–stacks $\mathcal{H}_i$ defined on the same open cover $\mathcal{U}$ of X, and a labeling

$$\eta'_i : \mathcal{H}_i \longrightarrow \mathcal{A}ut(y_i)$$

of $\mathbb{C}$ as an $\{\mathcal{H}_i\}$–2–gerbe. A decomposition $(y_i, \psi_{ij}, \gamma_{ijk}, \mu_{ijk})$ of $\mathbb{C}$ relative to the labeling η'_i yields a cocycle $(v'_{ijkl}, \tilde{\mu}_{ijk}, \gamma_{ijk}, \lambda'_{ij} ; \mathcal{H}_i)$. Let us compare it with the cocycle $(v_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij} ; \mathcal{G}_i)$ previously associated to the 2-gerbe $\mathbb{C}$. Since $\mathbb{C}$ satisfies the overly restrictive connectedness axiom (G2'), one may choose a family of 1-arrows $\rho_i : y_i \longrightarrow x_i$ in $\mathbb{C}_{U_i}$. By axiom $(G3')$, it is then possible to make the further choice of a family of objects $h_{ij} \in \mathcal{G}_{i\,|\,U_{ij}}$ and of 2-arrows χ_{ij} in $\mathbb{C}_{U_{ij}}$

$$\begin{array}{ccc} y_j & \xrightarrow{\psi_{ij}} & y_i \\ & & \downarrow \rho_i \\ \rho_j \downarrow & \chi_{ij} \Swarrow & x_i \\ & & \uparrow h_{ij} \\ x_j & \xrightarrow[\varphi_{ij}]{} & x_i \end{array} \tag{4.4.1}$$

in $\mathbb{C}_{U_{ij}}$. We set

$$h_{ii} = 1 \tag{4.4.2}$$

and $\chi_{ij} = 1$ whenever $i=j$. Conjugation by the arrow ρ_i induces a morphism of gr-stacks

$$(\rho_i)_* : \mathcal{H}_i \longrightarrow \mathcal{G}_i \tag{4.4.3}$$

and the 2-arrow χ_{ij} in $\mathbb{C}$ determines an 2-arrow

$$\tilde{\chi}_{ij} \colon (\rho_i)_* \circ \lambda'_{ij} \Rightarrow i_{h_{ij}} \circ \lambda_{ij} \circ (\rho_j)_* . \tag{4.4.4}$$

satisfying the identity

$$\tilde{\chi}_{ii} = 1_{(\rho_i)_*} . \tag{4.4.5}$$

Since the 2-arrow $\tilde{\chi}_{ij}$ lives in the fiber above U_{ij} of the stack $\mathcal{E}q(\mathcal{H}_j, \mathcal{G}_i)$, it determines, for every object h in $\mathcal{H}_i$, a 1-arrow

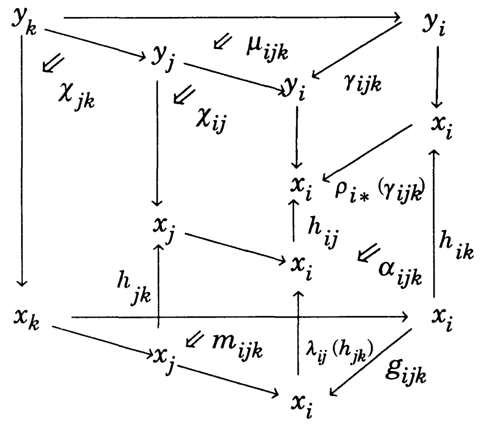
$$\tilde{\chi}_{ij}(h) \colon (\rho_i)_* \circ \lambda'_{ij}(h) \longrightarrow i_{h_{ij}} \circ \lambda_{ij} \circ (\rho_j)_* (h) \tag{4.4.6}$$

in $\mathcal{G}_i$. The corresponding arrow sourced at the identity in $\mathcal{G}_i$ is denoted by

$$\{\{\tilde{\chi}_{ij}, h\}\} \colon 1_{x_i} \longrightarrow i_{h_{ij}} \circ \lambda_{ij} \circ (\rho_j)_* (h)((\rho_i)_* \circ \lambda'_{ij}(h))^{-1}. \tag{4.4.7}$$

It is somewhat analogous in its construction to the arrow defined in (4.2.5).

Let us now consider the following prism, whose faces are 2-arrows in $\mathbb{C}_{U_{ijk}}$:



(4.4.8)

The rear vertical face of this diagram is defined by the 2-arrow χ_{ik}. The remaining unlabelled square in the left-hand vertical face is defined by (4.2.2) and the square at the top of the right-hand vertical face is the analogous 2-arrow describing the conjugation by ρ_i (4.4.3). Only the square located at the bottom of the right-hand vertical face of diagram (4.4.8) has so far been left unaccounted for. The choices $(\rho_i, h_{ij}, \chi_{ij})$ which have been made therefore determine it uniquely, as the 2-arrow α_{ijk} in $\mathbb{C}$ for which the diagram (4.4.8) commutes. The labeling η_i(3.4.1) transforms this into an arrow

(4.4.9) $$a_{ijk} : (\rho_i)_*(\gamma_{ijk}) \circ h_{ik} \Rightarrow h_{ij} \circ \lambda_{ij}(h_{jk}) \circ g_{ijk}$$

in $(\mathcal{G}_i)_{U_{ijk}}$. It satisfies the normalization conditions

(4.4.10) $$a_{iik} = a_{ikk} = 1.$$

Commutativity of the diagram (4.4.8) ensures that 1-arrow

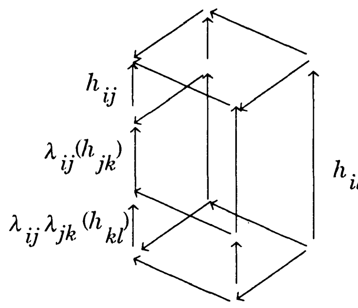
(4.4.11) $$i_{a_{ijk}} : i_{(\rho_i)_*(\gamma_{ijk})\, h_{ik}} \Rightarrow i_{h_{ij}\, \lambda_{ij}(h_{jk}) g_{ijk}} ,$$

which (4.4.9) induces in the category $\mathcal{E}q\,(\mathcal{G}_i)_{U_{ijk}}$, satisfies the relation

(4.4.12) $$\lambda_{ij}(\tilde{\chi}_{jk})\ \tilde{\chi}_{ij}\ \ \rho_{i_*}(\tilde{\mu}_{ijk}) = ({}^{h_{ij}\ \lambda_{ij}(h_{jk})}\ \tilde{m}_{ijk})\ i_{a_{ijk}}\ {}^{\rho_{i*}(\gamma_{ijk})}\tilde{\chi}_{ik}$$

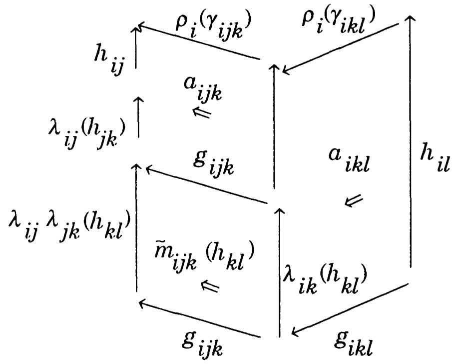
the notation being the same as in formula (4.2.11).

Finally, we obtain a relation between the two 3-cocycle terms ν_{ijkl} and ν'_{ijkl} determined by the two given labeled decompositions of the 2-gerbe $\mathbb{G}$. Consider the following cube, all of whose vertices are equal to x_i, and whose top and bottom faces are respectively defined by the 2-arrows $(\rho_i)_* \nu'_{ijkl}$ and ν_{ijkl} (4.2.9):

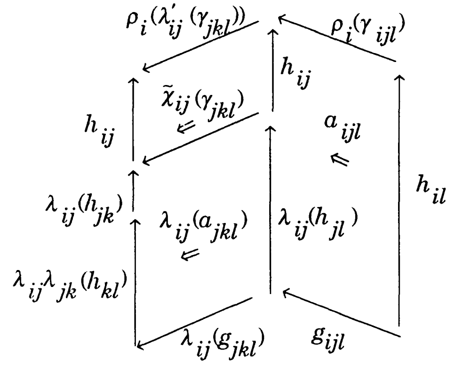


(4.4.13)

The vertical faces of this cube are most easily visualized by cutting it along the two marked vertical edges. The vertical faces then decompose into the following pair of rectangular diagrams of 2-arrows



and



(4.4.14)

We now display the algebraic formula asserting the commutativity of diagram (4.4.13), just as we worked out formula (4.2.17), which described the equality between diagrams (4.2.12) and (4.2.15). It is a generalization of formula (2.3.4.11) of [Br 3]. Each term in the following identity is an arrow in $\mathcal{G}_i$ sourced at the identity. The notation follows the same conventions as in (4.2.10) and (4.2.14). The two sorts of curly brackets appearing here were respectively defined in (4.2.5) and in (4.4.7).

(4.4.15)

$$ {}^{h_{ij}\,\lambda_{ij}(h_{jk})}\{\tilde{m}_{ijk}, h_{kl}\} \;\; a_{ijk} \;\; {}^{(\rho_i)_*(\gamma_{ijk})}a_{ikl} \;\; (\rho_i)_*(\nu'_{ijkl}) = $$

$$ = {}^{h_{ij}\,\lambda_{ij}(h_{jk})\,(\lambda_{ij}\lambda_{jk}(h_{kl}))}(\nu_{ijkl}) \;\; {}^{h_{ij}}\lambda_{ij}(a_{jkl}) \;\; \{\{\tilde{\chi}_{ij}, \gamma_{jkl}\}\} \;\; {}^{(\rho_i)_*(\lambda'_{ij}(\gamma_{jkl}))}a_{ijl} \,. $$

The complexity of such a formula provides a good illustration for R. Brown's point of view [R. Br], according to which the "2-dimensional algebra" describing the commutativity of diagrams of 2-arrows such as (4.2.12) - (4.2.15) or (4.4.13) is preferable, despite its apparent complexity, to the corresponding traditional 1-dimensional algebraic formulation, as given by formulas (4.2.17) and (4.4.15), or by the comparable formulas in [Ded 2].

The following definition takes the previous discussion into account.

Definition 4.5 *Two non–abelian* 3*–cocycles* $(\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij}\,; \mathcal{G}_i)$ *and* $(\nu'_{ijkl}, \tilde{\mu}_{ijk}, \gamma_{ijk}, \lambda'_{ij}\,; \mathcal{H}_i)$ *are cohomologous if there exists a family of objects* $(\rho_i)_* \in \mathcal{E}q(\mathcal{H}_i, \mathcal{G}_i)$ *and a family of objects* $h_{ij} \in (\mathcal{G}_i)_{U_{ij}}$, *together with a family of morphisms* (4.4.4) $\tilde{\chi}_{ij} \in \mathcal{E}q(\mathcal{H}_j, \mathcal{G}_i)_{|U_{ij}}$ *and of morphisms* (4.4.9) $a_{ijk} \in (\mathcal{G}_i)_{U_{ijk}}$), *for which the identities* (4.4.12) *and* (4.4.15), *and the normalization conditions* (4.4.2),(4.4.5) *and* (4.4.10) *are satisfied.*

One can also say that the quadruple $((\rho_i)_*, \tilde{\chi}_{ij}, h_{ij}, a_{ijk})$ determines a coboundary relation

(4.5.1) $$(\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij}\,; \mathcal{G}_i) \sim (\nu'_{ijkl}, \tilde{\mu}_{ijk}, \gamma_{ijk}, \lambda'_{ij}\,; \mathcal{H}_i)$$

between the corresponding 3-cocycle quintuples. An equivalence class of quintuples for this relation will be called a Čech (non-abelian) degree 3 cohomology class on X, relative to the open cover $\mathcal{U} = (U_i)_{i \in I}$.

Let us now choose a fixed family of gr–stacks $(\mathcal{G}_i)_{i \in I}$ defined on the open sets U_i. The set of classes of $\mathcal{G}_i$–valued non-abelian 3-cocycles, as defined in 4.3, for the equivalence relation (4.5.1) (with $\mathcal{G}_i = \mathcal{H}_i$ for all i) will be denoted by $\check{H}(\mathcal{U}, \{\mathcal{G}_i\})$. The following proposition has now been proved.

Proposition 4.6: *Let* $\mathcal{U} = (U_i)_{i \in I}$ *be an open cover of* X *and* $(\mathcal{G}_i)_{i \in I}$ *a family of gr–stacks on* U_i. *The previous construction associates to any* $\{\mathcal{G}_i\}$*–2-gerbe* $\mathbb{G}$ *on* X *satisfying the connectedness axiom* (G2′) *an element in the Čech cohomology set* $\check{H}(\mathcal{U}, \{\mathcal{G}_i\})$ *which is independent of the choice of a labeled decomposition of* $\mathbb{G}$.

Passing to the limit over the various open covers $\mathcal{U}$ of X, one obtains in this manner a Čech non-abelian cohomology set $\check{H}(X)$.

4.7 Let $\mathcal{G}$ be a given *gr*–stack defined on the space X. Just as in the case of G-gerbes discussed in 2.8 *i*), the set of classes of $\mathcal{G}$-2-gerbe on X is somewhat simpler to classify than the full set of 2-gerbes on X. The data 4.2.*i*)–*iv*) associated to such a $\mathcal{G}$-2-gerbe takes on the following form:

4.7.*i*) an object λ_{ij} in the fiber category $\mathcal{E}q(\mathcal{G})_{U_{ij}}$ above U_{ij} of the *gr*–stack $\mathcal{E}q(\mathcal{G})$ of self-equivalences of $\mathcal{G}$, and an object g_{ijk} (4.1.2) in $\mathcal{G}_{U_{ijk}}$.

4.7.*ii*) an arrow $\tilde{m}_{ijk}$ (4.2.3) in $\mathcal{E}q(\mathcal{G})_{U_{ijk}}$ and an arrow ν_{ijkl} (4.2.8) in $\mathcal{G}_{U_{ijkl}}$ satisfying the cocycle condition (4.2.17) in $\mathcal{G}_{U_{ijklm}}$.

Such a quadruple may be viewed as a Čech cocycle relative to the open cover $\mathcal{U}$ with value in the "crossed module" of *gr*–stacks $\mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G})$ defined by the inner conjugation functor in $\mathcal{G}$. Two such cocycles $(\nu_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij})$ and $(\nu'_{ijkl}, \tilde{\mu}_{ijk}, \gamma_{ijk}, \lambda'_{ij})$ are cohomologous whenever there exist

4.7.*iii*) a family of objects $(\rho_i)_*$ in $\mathcal{E}q(\mathcal{G})_{U_i}$ and a family of objects h_{ij} in $\mathcal{G}_{U_{ij}}$.

4.7.*iv*) a family of arrows $\tilde{\chi}_{ij}$ (4.4.4) in the category $\mathcal{E}q(\mathcal{G})_{U_{ij}}$ and a family of morphisms a_{ijk} (4.4.9) in $\mathcal{G}_{U_{ijk}}$ for which the identities (4.4.12) and (4.4.15) are satisfied.

The set of corresponding cohomology classes, will be denoted by $H^1(\mathcal{U}, \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G}))$. Once more, we may now define the Čech cohomology set $\check{H}^1(X, \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G}))$ by passing to the limit over open covers of X. We state as a corollary the following special case of proposition 4.6.

Corollary 4.8: *Let $\mathcal{G}$ be a gr–stack on a space X. The previous construction associates to any connected $\mathcal{G}$-2–gerbe* $\mathbb{C}$ on X *a well defined element of the Čech cohomology set $\check{H}^1(X, \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G}))$, which is independent of the choice of* *a labeled decomposition of* $\mathbb{C}$.

Note that the 2-stack $\mathbb{C} = Tors(\mathcal{G})$ is a $\mathcal{G}$-2-gerbe, and that $\mathcal{G}$ itself, viewed as the trivial $\mathcal{G}$-torsor, is an object x in the fiber 2-category $Tors(\mathcal{G})_X$, for which there is exists a canonical labeling $\mathcal{G} \longrightarrow \mathcal{E}q(x)$ of $\mathbb{C}$. The class of $Tors(\mathcal{G})$ therefore determines a distinguished element in the cohomology set $\check{H}^1(X, \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G}))$, which is simply the class of the trivial quadruple (1, 1, 1, 1) consisting of neutral objects and identity functors . Conversely, let $\mathbb{C}$ be a 2-gerbe on X posessing a global non-trivial object $x \in \mathbb{C}_X$, whose gr–stack of self-arrows is $\mathcal{G}$. The map of 2-prestacks $\mathcal{G}[1] \longrightarrow \mathbb{C}$ defined in 1.11.*ii*), which sends to x the unique object in the 2-prestack $\mathcal{G}[1]$, induces a canonical equivalence of $\mathcal{G}$–2-gerbes a: $Tors(\mathcal{G}) \longrightarrow \mathbb{C}$. Any cocycle quadruplet for $\mathbb{C}$ determined by a labeling of $\mathbb{C}$ compatible with that determined by x belongs to the class of the trivial quadruple.

4.9 Let us now drop the connectedness assumption ($G2'$), which ensured that the 1-arrows (4.1.1) existed. We will show that, as in the discussion carried out in 2.4 for ordinary gerbes, the Čech cocycles just obtained are then replaced by cocycles defined on an appropriate hypercover refinement of the Čech cover.

Let us choose open covers $\mathcal{U}_{ij} = (U_{ij}^{\alpha})$ of each open set $U_{ij} \in \mathcal{U}$, and 1-arrows (2.3.4) φ_{ij}^{α} in $\mathbb{C}_{U_{ij}^{\alpha}}$. Formula (2.4.1) now yields sections λ_{ij}^{α} of the stack $\mathcal{E}q(\mathcal{G}_j, \mathcal{G}_i)$ on each of the open sets U_{ij}^{α} . The second and third components of the cocycle quadruple defined in 4.3 must now be respectively replaced by an object $g_{i\,j\,k}^{\alpha\beta\gamma}$ in $\mathcal{G}_i$ and by an arrow

$$\tilde{m}_{ijk}^{\alpha\beta\gamma} : i_{g_{ijk}^{\alpha\beta\gamma}} \circ \lambda_{ik}^{\gamma} \Rightarrow \lambda_{ij}^{\alpha} \circ \lambda_{jk}^{\beta} \tag{4.9.1}$$

in the category $\mathcal{E}q(\mathcal{G}_j, \mathcal{G}_i)$, which both live above the open set $U_{i\,jk}^{\alpha\,\beta\gamma}$ (2.4.2). Finally, construction (4.2.7) yields an arrow

(4.9.2)
$$\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl} : 1_{x_i} \Rightarrow g^{\alpha\beta\gamma}_{ijk} \circ g^{\gamma\eta\delta}_{ikl} \circ (g^{\alpha\varepsilon\delta}_{ijl})^{-1} \circ \lambda^{\alpha}_{ij} (g^{\beta\eta\varepsilon}_{jkl})^{-1}$$

in the fiber category of $\mathcal{G}_i$ above the open set (2.4.9), and this arrow satisfies the upper-indexed analog of conditions (4.2.11). We do not not write down this upper-indexed formula here, since the upper indices which are to be inserted in (4.2.11) are precisely those which appear in the corresponding terms of (4.9.2). Nor do we write down the upper indices in the 3-cocycle formula corresponding to (4.2.17), which the arrow (4.9.2) satisfies, since they may be obtained from the corresponding terms in the diagrams (5.8.2) below. We simply note here that this cocycle formula is defined in the fiber of $\mathcal{G}_i$ above that multiple intersection of the U^{α}_{ij} 's on which the six terms which comprise it are defined. The elements of this cocycle quadruple may be chosen to satisfy the following generalizations of the normalization conditions 4.3.*i*)–*iii*):

(4.9.3)
$$\lambda_{ii} = 1_{G_i} .$$

Whenever $i=j$, the upper index α may be omitted and the conditions

(4.9.4)
$$g^{\beta\beta}_{iik} = 1 \quad \text{and} \quad \tilde{m}^{\beta\beta}_{iik} = 1$$

are satisfied on U^{β}_{ik}. Similarly, when $j=k$, the upper index β vanishes and the conditions

(4.9.5)
$$g^{\alpha\alpha}_{ikk} = 1 \quad \text{and} \quad \tilde{m}^{\alpha\alpha}_{ikk} = 1$$

are satisfied on U^{α}_{ik}. Finally, the three following normalization conditions for ν replace the corresponding condition 4.3*iii*) (the symbol $*$ in each formula denotes the position of the missing upper index):

(4.9.6)
$$\nu^{*\beta\beta\delta\delta\eta}_{iikl} = 1, \quad \nu^{\alpha*\alpha\delta\varepsilon\varepsilon}_{ijjk} = 1 \quad \text{and} \quad \nu^{\alpha\beta\gamma\gamma\beta*}_{ijkk} = 1 .$$

If now give ourselves a second decomposition $(y_i, \psi^{\alpha}_{ij}, \gamma^{\alpha\beta\gamma}_{ijk}, \mu^{\alpha\beta\gamma}_{ijk})$ of the 2--gerbe $\mathbb{C}$, relative to the same hypercover determined by the open covers $\mathcal{U}_{ij}$, of the open sets U_{ij} we may no longer, as in 4.4, choose a morphism $\rho_i \colon y_i \longrightarrow x_i$ between the corresponding elements, but only a family of arrows $\rho_i^{\alpha}: y_i \longrightarrow x_i$ defined on each term of an appropriate refinement (U_i^{α}) of the open sets U_i on which both x_i and y_i were defined. It is more expedient to relabel this new open cover of X as $\mathcal{V} = (V_i)$, and the restrictions to it of the objects x_i, y_i and the arrows ρ_i correspondingly, so that the x_i, y_i and ρ_i are now all defined above the open sets $V_i \in \mathcal{V}$. We are now essentially in the situation examined in 4.4, and we may introduce the open cover $\mathcal{V}'_{ij}$ of the open sets V_{ij} defined by the open sets

(4.9.7)
$$V_{ij}^{\alpha} = V_i \cap V_j \cap U_{ij}^{\alpha} .$$

The 1- and 2-arrows h_{ij} and χ_{ij} appearing in (4.4.1) must then be replaced by families h_{ij}^{α} and χ_{ij}^{α} of 1- and 2-arrows defined, for varying α, on the open sets V_{ij}^{α} on which the analog

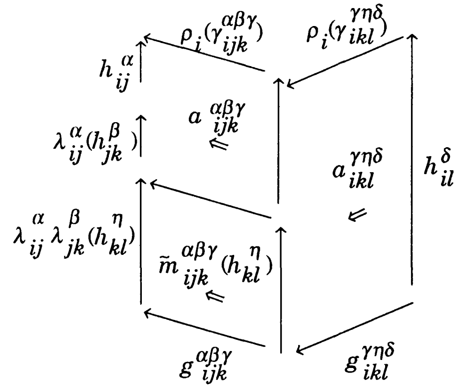
(4.9.8)
$$\begin{array}{ccc} y_j & \xrightarrow{\psi_{ij}^{\alpha}} & y_i \\ & \chi_{ij}^{\alpha} \swarrow\!\!\swarrow & \downarrow \rho_i \\ \rho_j \downarrow & & x_i \\ & & \uparrow h_{ij}^{\alpha} \\ x_j & \xrightarrow[\varphi_{ij}^{\alpha}]{} & x_i \end{array}$$

of diagram (4.4.1) can be defined. These 1- and 2-arrows in turn determine as in 4.4 a family of quadruples $((\rho_i)_*, \tilde{\chi}_{ij}^{\alpha}, h_{ij}^{\alpha}, a_{ijk}^{\alpha\beta\gamma})$ with

(4.9.9)
$$\tilde{\chi}_{ij}^{\alpha} : (\rho_i)_* \circ \lambda'^{\alpha}_{ij} \Rightarrow i_{h_{ij}^{\alpha}} \circ \lambda_{ij}^{\alpha} \circ (\rho_j)_* ,$$

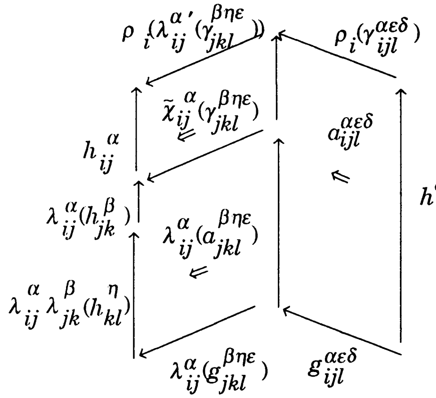
(4.9.10)
$$a_{ijk}^{\alpha\beta\gamma} : (\rho_i)_*(\gamma_{ijk}^{\alpha\beta\gamma}) \circ h_{ik}^{\gamma} \Rightarrow h_{ij}^{\alpha} \circ \lambda_{ij}^{\alpha}(h_{jk}^{\beta}) \circ g_{ijk}^{\alpha\beta\gamma} .$$

The 2-arrows (4.9.10) live in the following upper-indexed versions of diagrams (4.4.14):



(4.9.11)

and



(4.9.12)

The cube analogous to (4.4.13), but with lateral faces described by (4.9.11) and (4.9.12), and upper and lower horizontal faces respectively by $(\rho_i)_*(\nu'{}^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})$ and by $\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl}$, now defines an upper-indexed coboundary relations analogous to (4.4.15) between the corresponding non-abelian 3-cocycle quintuplets. We do not spell this out, since its upper indices can simply be read off from the corresponding terms in diagrams (4.9.11) and (4.9.12). This 3-coboundary relation, which is relative to the new hypercover $\mathcal{V}'$ determined by the open covers $(\mathcal{V},\mathcal{V}'_{ij})$ may be normalized by the conditions:

(4.9.13) $$h_{ii}^{\alpha\alpha}=1 \text{ and } \tilde{\chi}_{ii}^{\alpha\alpha}=1$$
$$a_{ijj}^{\alpha\beta\beta}=a_{iik}^{\alpha\alpha\gamma}=1$$

The set of such $\{\mathcal{G}_i\}$–valued cocycles relative to a hypercover $\mathcal{U}'$ of X, with coboundary with respect to an appropriate refinement $\mathcal{V}'$ of $\mathcal{U}'$ will be denoted $H(\mathcal{U}', \{\mathcal{G}_i\})$ without introducing $\mathcal{V}'$ into the notation.

We have now proved the following generalization of proposition 4.6

Proposition 4.10: *Let $\mathcal{U}=(U_i)_{i\in I}$ be an open cover of X and $(\mathcal{G}_i)_{i\in I}$ a family of gr–stacks on U_i. The previous construction associates to any $\{\mathcal{G}_i\}$–2-gerbe $\mathbb{G}$ on X an element in the cohomology set $H(\mathcal{U}', \{\mathcal{G}_i\})$ associated to an appropriate hypercover refinement $\mathcal{U}'$ of $\mathcal{U}$. This element is independent of the choice of a labeled decomposition of $\mathbb{G}$.*

Remark 4.11: The reader may find it suprising that the hypercover $\mathcal{U}'$ appearing in proposition 4.10 is not the most general possible, since it did not prove necessary to introduce, as in 2.7, a refinement $(V_{ijk}^{\alpha\beta\gamma})_\lambda$ of the cover $U_{ijk}^{\alpha\beta\gamma}$. The reason for this is that we allowed ourseves to choose, as coefficients in which the cocycles were to take their values, a *gr*–stack $\mathcal{G}$ on X (or a family of *gr*–stacks $\mathcal{G}_i$ defined on an open cover $\mathcal{U}=(U_i)_{i\in I}$ of X). We might instead have begun with a fixed crossed module $G \longrightarrow \Pi$ on X (*resp.*, a family of crossed modules $G_i \longrightarrow \Pi_i$ defined on the open sets U_i), and considered the associated *gr*–stack $\mathcal{G}$ (*resp.*, $\mathcal{G}_i$). While the procedure carried out above would then have yielded, as in 4.2, 4.9, a cocycle quadruple taking its values in objects and arrows of these *gr*–stacks $\mathcal{G}$ and $\mathcal{E}q(\mathcal{G})$ (*resp.*, $\mathcal{G}_i$ and $\mathcal{E}q(\mathcal{G}_j, \mathcal{G}_i)$), we might then have sought refined cocycles, taking their values in the appropriate crossed modules themselves, rather than in the associated *gr*–stacks. The objects $g_{ijk}^{\alpha\beta\gamma}$ in the fiber of $\mathcal{G}_i$ on the open set $U_{ijk}^{\alpha\beta\gamma}$ would then have been described by a family of sections $(g^{\alpha\beta\gamma}_{ijk})_\lambda$ of the sheaf Π_i on the various open sets $(V^{\alpha\beta\gamma}_{ijk})_\lambda$, and the corresponding arrows v_{ijkl} by sections of the sheaves G_i on the multi-indexed open sets (2.7.1). Finally, the 3-cocycle condition (4.2.17) would then have taken place on the intersection of the five multi-indexed open sets on which the each of the G_i -valued 3-cochains v, and the corresponding term (4.2.5), would have been defined.

It does not, however, in general seem possible to describe the equivalence (4.2 *i*)) $\lambda^\alpha_{ij} \in \mathcal{E}q(\mathcal{G}_j, \mathcal{G}_i)_{U^\alpha_{ij}}$ in an analogous manner, by a morphism between the restrictions to some localization of U^α_{ij} of the crossed modules $(G_j \longrightarrow \Pi_j)$ and $(G_i \longrightarrow \Pi_i)$ which respectively define $\mathcal{G}_j$ and $\mathcal{G}_i$. There are nevertheless some special situations in which this will be possible. Suppose for example that the sheaf Π_j is representable by an object of the site under consideration, for example if each Π_i is the trivial sheaf on U_i, or if $\mathcal{G}_j$ is a (strictly associative) *gr*–stack which is algebraic in the sense of [De-Mu], [L-M] def. 3.1. In such cases, there will indeed exist a local representation of λ^α_{ij} in terms of a morphism of crossed modules

$$(4.11.1) \qquad (G_j \longrightarrow \Pi_j) \longrightarrow (G_i \longrightarrow \Pi_i).$$

As it was pointed out in [Br 3] in a related context, such map will in general not be a homomorphism of crossed modules in the traditional sense, but merely homomorphisms up to coherent homotopy.

4.12 As in the case of 1-gerbes, the cocycles associated to a 2-gerbe simplify when commutativity conditions on the coefficients are introduced. The following definition is the analog of 2.9.

Definition 4.13: *Let* $\mathbb{C}$ *be a 2–gerbe on X, and let* $\mathcal{G}$ *be a gr–stack on X. Suppose that there exists, for every object x in a fiber 2–category* $\mathbb{C}_U$, *an isomorphism of sheaves of gr–stacks* $\eta_x : \mathcal{G}_{|U} \longrightarrow Aut(x)$, *and for any* *morphism* $f{:}x \longrightarrow y$ *in* $\mathbb{C}_U$, *a* 2*–arrow* $\eta_f : \lambda_f \circ \eta_x \longrightarrow \eta_y$

$$\begin{array}{ccc} & \mathcal{G}_{|U} & \\ {}^{\eta_x}\swarrow & & \searrow^{\eta_y} \\ \underline{Aut}(x) & \xrightarrow[\lambda_f]{\ \eta_f \nearrow\ } & \underline{Aut}(y) \end{array} \tag{4.13.1}$$

where λ_f *is the morphism of gr–stacks* (2.1.2) *defined by* f. *The natural transformations* η_f *are required to respect the group structures, and satisfy the following transitivity and normalization conditions*:

4.13.*i*) *For any pair of composable morphisms* $f{:}\ x \longrightarrow y$ *and* $g{:}y \longrightarrow z$ *in* $\mathbb{C}_U$, *the composite* 2*– arrow obtained by pasting* η_f *and* η_g *is equal to* η_{gf}.

4.13.*ii*) *For any* 2*–arrow* $\varphi{:}f \Rightarrow g$ *between a pair of morphisms* f *and* $g{:}x \longrightarrow y$ *in* $\mathbb{C}_U$, $\eta_f = \eta_g \circ \lambda_\varphi$, *where* $\lambda_\varphi : \lambda_f \Rightarrow \lambda_g$ *is the conjugation by* φ.

4.13.*iii*) *For every* x *in* $\mathbb{C}_U$, $\eta_{1_x} = 1$

A 2*–gerbe* $\mathbb{C}$ *satisfying these conditions will be called an abelian* $\mathcal{G}$*–*2*–gerbe on the space* X.

Specializing diagram (4.13.1) to the case in which $x = y$, it follows that for any $f \in \underline{Aut}(x)$, the 2-arrow η_f defines an equivalence between the "inner conjugation by f" functor i_f for the gr–stack $\underline{Aut}(x)$ and the identity functor $1_{\underline{Aut}(x)}$. Axioms 4.13 *i*) and *ii*) then imply that this commutativity condition on $\underline{Aut}(x)$ induces a braiding on the gr–stack $\mathcal{G}$. Furthermore, the axioms in question imply the triviality of the terms $(\lambda_{ij}, \tilde{m}_{ijk})$ associated to a decomposition of $\mathbb{C}$ compatible with the labeling by the η_x. The remaining data (g_{ijk}, ν_{ijkl}) (4.1.2, 4.2.8) determines in the limit a class in what should properly be called the Čech cohomology set $\check{H}^2(X, \mathcal{G})$ with values in the braided stack $\mathcal{G}$. We refer to [B-C], [Ul] and [Br 3] (2.4.6) for related definitions, in various contexts, of cohomology groups with values in braided (or Picard) categories.

Examples 4.14: *i*) Suppose that $\mathbb{C}$ is an abelian $\mathcal{G}$-2-gerbe on X, and that, for some sheaf of abelian groups A on X, the additional condition

(4.14.1) $$\mathcal{G} = Tors(A)$$

is satisfied. Since $\mathcal{G}$ is the stack associated to the crossed module $A \longrightarrow 1$, we may pass to a further refinement of the open cover of X, as discussed in 4.11. The term g_{ijk} in the cocycle then also disappears, and only a traditional A–valued 3-cocycle v_{ijkl} remains. A convenient way of stating condition (4.14.1) is to require that any pair of self-arrows $f, g: x \longrightarrow x$ in a fibre 2-category $\mathbb{C}_U$ be locally connected by some 2-arrow $u: f \Rightarrow g$. In that case, the stack $\underline{Aut}(x)$ is a gerbe on U, and it is automatically a neutral one since the identity map 1_x provides it with a global section. A labelling by A of this gerbe, in other words an isomorphism of sheaves $A \longrightarrow Aut(1_x)$ then determines an equivalence (4.14.1). This geometric description of traditional 3-cocycles as abelian 2-gerbes satisfying these additional connectedness and labeling conditions is the one introduced by Brylinski and McLaughlin [Br-M]. Passing to the limit over the open covers $\mathcal{U}$, the corresponding set $\check{H}^2(X,\mathcal{G}) = lim\, H^2(\mathcal{U},\mathcal{G})$ coincides with the ordinary third[5] Čech cohomology group $\check{H}^3(X,A)$.

ii) Let G be a reductive group, and let $G^{sc} \longrightarrow G$ be the crossed module of example 1.9, with associated Picard stack $\mathcal{A}$. The quasi-isomorphism (1.9.4) identifies the group $H^2(X, \mathcal{A})$ of abelian $\mathcal{A}$-2-gerbes on X with Borovoi's abelianized degree 2 cohomology group defined by the formula $H^2_{ab}(X,G) = H^2(X, Z^{sc} \longrightarrow Z)$.

---

[5] The grading is consistent with the formulas $H^2(X, G[1]) = H^3(X,G)$ and $H^1(X,\mathcal{G}[1]) = H^2(X,\mathcal{G})$.

***Remark* 4.15:** It is possible, when $\mathcal{G}$ is Picard, to define a group law on the set $H^2(X, \mathcal{G})$ of classes of abelian $\mathcal{G}$-gerbes at the cocycle level and it would be interesting to describe this group law in geometric terms. Let us outline a method for doing this. One should define, as already advocated in 4.3, the 2-lien associated to a given 2-gerbe $\mathbb{G}$. One could then prove, along the same lines as in the proof of proposition 2.14, that the 3-stack of abelian $\mathcal{G}$-2-gerbe is equivalent to that of torsors under the 2–stack $\mathbb{T} = \mathit{Tors}(\mathcal{G})$, endowed with the group law defined by (2.13.1). We have seen that this group law on $\mathbb{T}$ is abelian so that a (commutative) law is defined in this manner on the 3-stack of $\mathbb{T}$-torsors.

5.The 2-gerbe associated to a 3-cocycle

5.1 We now show how the construction carried out in the last section can be reversed, thereby associating a 2-gerbe $\mathbb{C}$ on X to a given 3-cocycle. Once more, we begin by dealing with the Čech cohomology case. Let $(v_{ijkl}, \tilde{m}_{ijk}, g_{ijk}, \lambda_{ij} ; \mathcal{G}_i)$ be a Čech 3-cocycle quintuplet relative to some open cover $\mathcal{U} = (U_i)_{i \in I}$ of X, satisfying conditions 4.2.*i*) – 4.2.*iv*). The term λ_{ij} (4.2.1) determines an equivalence of 2-prestacks $\lambda_{ij}[1]: \mathcal{G}_j[1] \longrightarrow \mathcal{G}_i[1]$ on the open set U_{ij}, inducing an equivalence

$$(5.1.1) \qquad \tilde{\lambda}_{ij}: Tors(\mathcal{G}_j) \longrightarrow Tors(\mathcal{G}_i)$$

between the associated 2-stacks, which is simply the "change of structural *gr*–stack" 2-functor defined by the equivalence λ_{ij} (*see* [Br 2] 6.7.4).

Let $\mathcal{G}$ and $\mathcal{H}$ be a pair of *gr*–stacks on X. We will now describe in an explicit manner certain elements in the 2-category $\mathcal{E}q(Tors(\mathcal{H}), Tors(\mathcal{G}))$ of equivalences between the 2-stacks $Tors(\mathcal{H})$ and $Tors(\mathcal{G})$. The first part of the following lemma is a higher order analog of lemma 1.5. It is a weak version of the higher Morita theorem stated in [Br 3] §4.1.

Lemma 5.2 : *i) Let λ , λ' : $\mathcal{H} \longrightarrow \mathcal{G}$ be a pair of morphism of gr–stacks on X, and let*

(5.2.1) $$\tilde{\lambda}, \tilde{\lambda}' : Tors(\mathcal{H}) \longrightarrow Tors(\mathcal{G})$$

be the associated 2–functors. A natural transformation between the pair of 2–functors $\tilde{\lambda}$ and $\tilde{\lambda}'$ is determined by an object $g \in \mathcal{G}_X$ and a 1–arrow

(5.2.2) $$m: i_g \circ \lambda \Rightarrow \lambda'$$

in $\mathcal{E}q(\mathcal{H}, \mathcal{G})$. Such a transformation will be denoted by (g,m). Neglecting a canonical isomorphism, the ("vertical") composition

(5.2.3) $$Tors(\mathcal{H}) \begin{array}{c} \xrightarrow{\tilde{\lambda}} \\ (g,m)\Downarrow \\ \xrightarrow{\tilde{\lambda}'} \\ (g_1,m_1)\Downarrow \\ \xrightarrow[\tilde{\lambda}'']{} \end{array} Tors(\mathcal{G})$$

of two such composable natural transformations is described by the rule

(5.2.4) $$(g_1, m_1) \circ (g, m) = (g_1 g, m_1 {}^{g_1}m),$$

where the notation is the same as in (4.2.11). *A 2–arrow*

(5.2.5) $$Tors(\mathcal{H}) \quad \begin{array}{c} \xrightarrow{\tilde{\lambda}} \\ (g,m)\Downarrow \ \overset{v}{\Rrightarrow} \ \Downarrow (g',m') \\ \xrightarrow[\tilde{\lambda}']{} \end{array} \quad Tors(\mathcal{G})$$

between two such natural transformations is determined by an arrow $v{:}g \longrightarrow g'$ in $\mathcal{G}_X$ such that the diagram of 2–arrows

(5.2.6) $$\begin{array}{ccc} i_g \circ \lambda & \xRightarrow{i_v \circ \lambda} & i_{g'} \circ \lambda \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu'} & \\ & \lambda' & \end{array}$$

in $\mathcal{E}q(\mathcal{H}, \mathcal{G})$ commutes. Left vertical composition of such a 2-arrow

$v\colon (g,m) \Rrightarrow (g',m')$ with a natural transformation $(g_1 m_1)$, according to the scheme

$$
\mathit{Tors}(\mathcal{H}) \quad
\begin{array}{c}
\tilde{\lambda} \\
\xrightarrow{\hspace{8em}} \\
(g,m)\Downarrow \ \overset{v}{\Rrightarrow} \ \Downarrow (g',m') \\
\xrightarrow{\hspace{8em}} \\
(g_1,m_1)\Downarrow \quad \tilde{\lambda}' \\
\xrightarrow{\hspace{8em}} \\
\tilde{\lambda}''
\end{array}
\quad \mathit{Tors}(\mathcal{G})
$$

yields a 2– arrow

(5.2.7) $$(g_1, m_1)\,v\colon\ (g_1 g, m_1\, {}^{g_1}m) \Rrightarrow (g_1 g', m_1\, {}^{g_1}m')$$

in $\mathcal{E}q(\mathit{Tors}(\mathcal{H}), \mathit{Tors}(\mathcal{G}))$, which is described by the 1–arrow

$$ {}^{g_1}v : g_1 g \longrightarrow g_1 g' $$

in $\mathcal{G}_X$ deduced from v by left multiplication by the object g_1. Finally, let $(g',m') \Rrightarrow (g'',m'')$ be another 2-arrow in $\mathcal{E}q(\mathit{Tors}(\mathcal{H}), \mathit{Tors}(\mathcal{G}))$, which is described as above by an arrow $v'\colon g' \longrightarrow g''$ in $\mathcal{G}_X$. The composite of these two 2-arrows is the 2-arrow $(g,m) \Rrightarrow (g'',m'')$ in $\mathcal{E}q(\mathit{Tors}(\mathcal{H}), \mathit{Tors}(\mathcal{G}))$ described by the composite 1–arrow $v'v\colon g \longrightarrow g''$ in $\mathcal{G}_X$.

Proof: Since the "associated 2-stack" 2-functor is fully faithful, it suffices to prove the corresponding statement for natural transformations and 2-arrows between the morphisms of 2-prestacks

(5.2.8) $$\lambda[1],\ \lambda'[1]\colon \mathcal{H}[1] \longrightarrow \mathcal{G}[1]$$

defined by λ and λ'. The statement then follows from the definition of a natural transformation and of a 2-arrow in $\mathcal{E}q(\mathcal{H}[1], \mathcal{G}[1])$. A natural transformation between the two 2-functors is given, for the unique object e of $\mathcal{H}[1]$, by a 1-arrow $g\colon \lambda[1](e) \longrightarrow \lambda'[1](e)$, which is "functorial in e up to a natural transformation m", in other words, by a line of reasoning analogous to that of diagram (1.2.7), up to a 2-arrow

$$\begin{array}{ccc} e & \xrightarrow{\;g\;} & e \\ {\scriptstyle\lambda(h)}\Big\downarrow & \overset{m}{\nearrow\!\!\!\!\nearrow} & \Big\downarrow{\scriptstyle\lambda'(h)} \\ e & \xrightarrow[\;g\;]{} & e \end{array}$$

This is nothing else than (5.2.2). The commutativity of (5.2.4) is verified in the same way, by making explicit at the 2-prestack level the description of a 2-arrow between two such natural transformations. The rest of the proof of the lemma follows directly from these observations, and the explicit definitions of the various compositions of 1- and 2-arrows.

It is also of interest to describe explicitly certain elements associated to the natural pairing of 2-categories

$$\begin{array}{rcl} \mathcal{E}q(\mathcal{G},\mathcal{F})\times\mathcal{E}q(\mathcal{H},\mathcal{G}) & \longrightarrow & \mathcal{E}q(\mathcal{H},\mathcal{G})\ , \\ (\rho,\lambda) & \longmapsto & \rho\circ\lambda \end{array}$$

which we will refer to as "horizontal" composition. The following lemma discusses this for the compositions of 1- and 2-arrows with a 1-arrow, according to the schemes

$$Tors(\mathcal{H})\ \ \begin{array}{c} \xrightarrow{\quad\tilde{\lambda}\quad} \\ (g,m)\Downarrow\ \overset{\nu}{\Longrightarrow}\ \Downarrow(g',m') \\ \xrightarrow[\quad\tilde{\lambda}'\quad]{} \end{array}\ \ Tors(\mathcal{G})\ \xrightarrow{\quad\tilde{\rho}\quad}\ Tors(\mathcal{F})$$

(5.2.9)

and

(5.2.10)

$$Tors(\mathcal{K})\ \xrightarrow{\quad\tilde{\sigma}\quad}\ Tors(\mathcal{H})\ \ \begin{array}{c} \xrightarrow{\quad\tilde{\lambda}\quad} \\ (g,m)\Downarrow\ \overset{\nu}{\Longrightarrow}\ \Downarrow(g',m') \\ \xrightarrow[\quad\tilde{\lambda}'\quad]{} \end{array}\ \ Tors(\mathcal{G})$$

Lemma 5.3: *Let $\rho\colon \mathcal{G} \longrightarrow \mathcal{F}$ and $\sigma\colon \mathcal{K} \longrightarrow \mathcal{H}$ be two additional morphisms of gr–stacks on X.*

i) The natural transformation $\tilde{\rho}(g,m) : \tilde{\rho} \circ \tilde{\lambda} \Rightarrow \rho \circ \tilde{\lambda}'$, induced in $\mathcal{E}q(Tors(\mathcal{H}), Tors(\mathcal{F}))$ from (g,m) by left composition with $\tilde{\rho}$, is determined by the pair $(\rho(g), \rho.m))$, where $\rho.m$ is the composite arrow

$$\rho.m : i_{\rho(g)} \circ \rho \circ \lambda \Rightarrow \rho \circ (i_g \circ \lambda) \Rightarrow \rho \circ \lambda'$$

in $\mathcal{E}q(\mathcal{H}, \mathcal{F})$ defined by composing the canonical arrow with the arrow $\rho \circ m$ induced by m. Similarly, consider the 2–arrow (5.2.5) determined in $\mathcal{E}q(Tors(\mathcal{H}), Tors(\mathcal{G}))$ by an arrow v in $\mathcal{G}_X$. The corresponding 2-arrow $\rho \circ (g,m) \Rrightarrow \rho \circ (g',m'))$, defined as in (5.2.9), is determined by the 1-arrow $\rho(v)$ in $\mathcal{F}_X$.

ii) The natural transformation $(g,m) \circ \sigma : \tilde{\lambda} \circ \tilde{\sigma} \Rightarrow \tilde{\lambda}' \circ \tilde{\sigma}$ induced as in (5.2.10) in $\mathcal{E}q(Tors(\mathcal{K}), Tors(\mathcal{G}))$ by $\tilde{\sigma}$ is determined by the pair $(g, m.\sigma)$, where $m.\sigma$ is the 1–arrow

$$m.\sigma : i_g \circ \lambda \circ \sigma \Rightarrow \lambda' \circ \sigma$$

in $\mathcal{E}q(\mathcal{K}, \mathcal{G})$ induced by m. Similarly, the 2-arrow $(g,m).\sigma \Rrightarrow (g',m).\sigma$, obtained in $\mathcal{E}q(Tors(\mathcal{K}), Tors(\mathcal{G}))$ by composing the 2–arrow in $\mathcal{E}q(Tors(\mathcal{H}), Tors(\mathcal{G}))$ determined by the arrow v in $\mathcal{G}_X$ with the 1–arrow σ, is described by the arrow v.

5.4 Finally, observe that there are two possible ways of defining the "horizontal" composition of a pair of 2-arrows (g,m) and (f,n), according to the following scheme

$$\mathcal{H}[1] \underset{\lambda_2}{\overset{\lambda_1}{\underset{(g,m)\Downarrow}{\longrightarrow}}} \mathcal{G}[1] \underset{\rho_2}{\overset{\rho_1}{\underset{(f,n)\Downarrow}{\longrightarrow}}} \mathcal{F}[1]$$

These are respectively defined as follows by the "vertical" composition (5.2.3) of the appropriate 2-arrows:

$$(f,n)\lambda_2 \,.\, \rho_1(g,m) : \rho_1\lambda_1 \Rightarrow \rho_1\lambda_2 \Rightarrow \rho_2\lambda_2$$

and

$$\rho_2(g,m).\,(f,n)\lambda_1 \;:\; \rho_1\lambda_1 \Rightarrow \rho_2\lambda_1 \Rightarrow \rho_2\lambda_2.$$

By the formula (5.2.4) for the vertical composition of 2-arrows, and lemma 5.3, these two 2-arrows are respectively described, in the terminology introduced in lemma 5.2, by the pairs $(f\rho_1(g),\ (n.\lambda_2)^{\,f}(\rho_1.m))$ and $(\rho_2(g)f,\ (\rho_2.m)^{\,\rho_2(g)}(n.\lambda_1))$. While it is part of the axioms in any 2-category that the two such possible ways of defining "horizontal" composition of 2-arrows coincide, this need no longer be the case in a 3-category, such as the 3-category $2\text{-}Stack_X$ presently under discussion. Indeed, for arrows of the form (5.2.1), it can be verified explicitly, by working as above at the 2-prestack level, that the difference between these two sorts of horizontal compositions, which is portrayed by the diagram

$$\begin{array}{ccc} & \rho_1(g,m) & \\ \rho_1\lambda_1 & \Longrightarrow & \rho_1\lambda_2 \\ (f,n)\lambda_1 \Downarrow & \overset{n(g)}{\swarrow} & \Downarrow (f,n)\lambda_2 \\ \rho_2\lambda_1 & \Longrightarrow & \rho_2\lambda_2 \\ & \rho_2(g,m) & \end{array} \tag{5.4.1}$$

in $\mathcal{E}q(\mathcal{H},\mathcal{F})$, is described by the 1-arrow

$$n(g)\colon f\rho_1(g) \longrightarrow \rho_2(g)f \tag{5.4.2}$$

in $\mathcal{F}$ which is obtained by evaluating the natural transformation $n\colon i_f \circ \rho_1 \Rightarrow \rho_2$ in $\mathcal{E}q(\mathcal{G},\mathcal{F})$ on the object $g \in \mathcal{G}$. The 1-arrow in $\mathcal{F}$ sourced at the identity

$$\{n,g\}\colon 1_{\mathcal{F}} \longrightarrow \rho_2(g) f \rho_1(g)^{-1} f^{-1} \tag{5.4.3}$$

which corresponds to $n(g)$ will be denoted, as in (4.2.5), by $\{n,g\}$.

5.5 Returning to the construction of the 2-gerbe $\mathbb{C}$ undertaken in 5.1, it now follows from lemma 5.2 that the terms g_{ijk} , $\tilde{m}_{ijk}$ (4.2 *ii*)–*iii*)) in the given Čech cocycle quadruplet determine a natural transformation of 2-functors $\psi_{ijk} = (g_{ijk}, \tilde{m}_{ijk})$:

(5.5.1) $$\tilde{\lambda}_{ik} \Rightarrow \tilde{\lambda}_{ij} \circ \tilde{\lambda}_{jk} : Tors(\mathcal{G}_k) \longrightarrow Tors(\mathcal{G}_i)$$

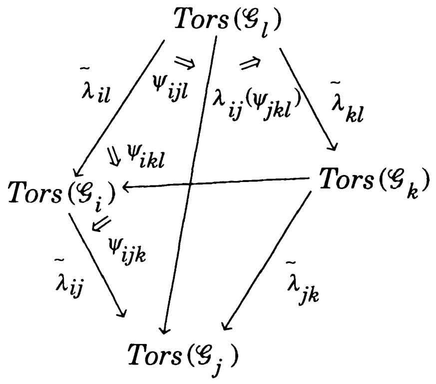
on U_{ijk} , and diagram (4.2.9) and formula (4.2.11) imply by lemma 5.2 that the 3-cochain ν_{ijkl} given in 4.2 *iv*) defines a 2-arrow

(5.5.2) $$\tilde{\nu}_{ijkl} : \tilde{\lambda}_{ij}(\psi_{jkl})\, \psi_{ijl} \Longrightarrow \psi_{ijk}\, \psi_{ikl}$$

in the fiber of the 2-category $\mathcal{E}q\,(Tors(\mathcal{G}_l), Tors(\mathcal{G}_i))$ on the open set U_{ijkl}. It may therefore represented there by the diagram

(5.5.3) $$\begin{array}{ccc} \tilde{\lambda}_{il} & \xrightarrow{\ \psi_{ijl}\ } & \tilde{\lambda}_{ij} \circ \tilde{\lambda}_{jl} \\ {\scriptstyle \psi_{ikl}} \downarrow & \begin{array}{c}\tilde{\nu}_{ijkl} \\ \Downarrow\end{array} & \downarrow {\scriptstyle \tilde{\lambda}_{ij}(\psi_{jkl})} \\ \tilde{\lambda}_{ik} \circ \tilde{\lambda}_{kl} & \xrightarrow[\ \psi_{ijk} \circ \tilde{\lambda}_{kl}\ ]{} & \tilde{\lambda}_{ij} \circ \tilde{\lambda}_{jk} \circ \tilde{\lambda}_{kl} \end{array}$$

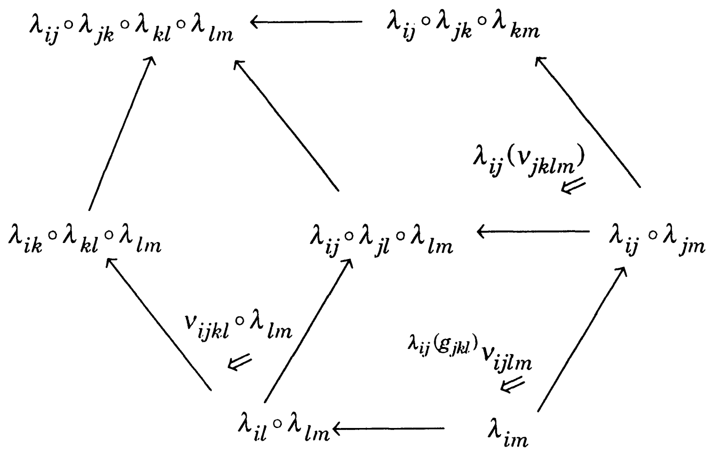
Alternately, we may work directly in the fibered 3-category $2\text{–}Stack_X$. The 2-arrow (5.5.2) is then viewed as a 3-arrow filling the following non-commuting tetrahedron of 2-arrows, in the fiber of $2\text{-}Stack_X$ on the open set U_{ijkl}



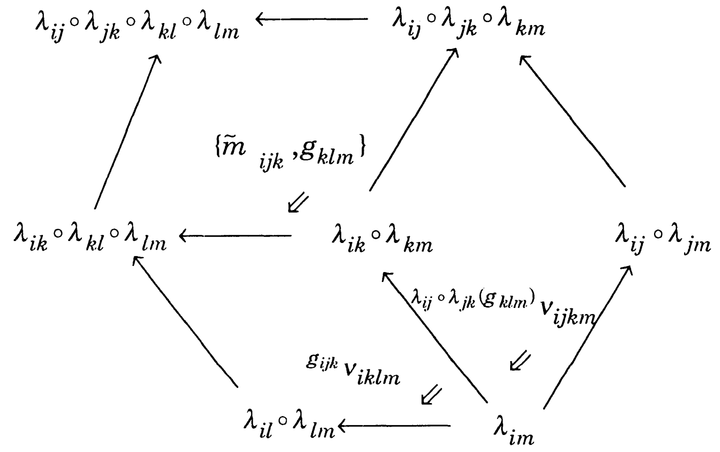
(5.5.4)

(each of the four triangular faces of the tetrahedron thus corresponds to one of the edges of the square (5.5.3)).

This non-commuting tetrahedron is a diagram of the form considered in (1.10.3). As was pointed out in remark 1.12, the fibered 3-category 2-$Stack_X$ is actually a 3-stack. The 2-arrow $\tilde{v}_{ijkl}$ will therefore define a 2-stack $\mathbb{C}$ on X, locally equivalent over each of the open sets U_i to the corresponding 2-stack $Tors(\mathcal{G}_i)$, if and only if the 3-arrow (5.5.4) which it describes satisfies the 3-descent condition. As we have said in remark 1.12, this condition is difficult to visualize directly in our three-dimensional world, since it is an identity between a pair of 3-arrows in the 3-category 2-$Stack_X$. It is therefore more convenient to represent each of the constituent 3-arrows in this 3-descent condition by the corresponding 2-arrow of the form (5.5.3). Each of the two following diagrams, which are built from the 2-arrows (5.5.3) describes a composite 2-arrow, which is an element in the fibre of the 2-category $\mathcal{E}q\,(Tors(\mathcal{G}_m), Tors(\mathcal{G}_i))$ over the open set U_{ijklm} (we will henceforth replace an expression such as $\tilde{\lambda}$ in a diagram by the corresponding symbol λ):



(5.5.5)



(5.5.6)

Since these composite 2-arrows have the same source and target, they may be compared, and the 3-descent condition may now most simply be stated as the assertion that they both coincide, in other words that the cube of 2-arrows which they constitute is commutative. It now follows from lemmas 5.2, 5.3 and from 5.4 that these composite 2-arrows respectively correspond to the expressions (4.2.14) and (4.2.16), so that the cocycle condition (4.2.17) asserts that the 2-arrows respectively defined by diagrams (5.5.5) and (5.5.6) are equal. The 2-gerbes $Tors(\mathcal{G}_i)$ thus descend to a 2-stack $\mathbb{C}$ defined on the entire space X. This 2-stack is automatically a

2-gerbe, since it is by construction locally equivalent above U_i to the 2-gerbe $Tors\,(\mathcal{G}_i)$, and since the conditions defining a 2-gerbe are all of a local nature. This argument provides the main part of the proof of the following theorem.

Theorem 5.6: *The previous construction associates a* $\{\mathcal{G}_i\}_{i\in I}$ *-2-gerbe* $\mathbb{C}$ *on* X *to the family of equivalences* $\tilde{\lambda}_{ij}$ (5.1.1), *the family of natural transformations* ψ_{ijk} (5.5.1) *and the family of 2–arrows* $\tilde{v}_{ijkl}$ (5.5.2) *determined by a Čech cohomology quintuplet* $(v_{ijkl}\,, \tilde{m}_{ijk}\,, g_{ijk}\,, \lambda_{ij}\,; \mathcal{G}_i)$. *It is the reverse of the construction given in proposition* 4.6, *and it associates a pair of equivalent 2-gerbes to a pair of cohomologous cocycle quintuplets.*

The verification of the last assertion is carried out in an analogous manner, by appealing once more to lemmas 5.2, 5.3 in order to show that a coboundary relation yields a local description of the sought-after equivalence of gerbes.

Remark 5.7: Since a 3-descent condition is modeled on the 4-simplex $\Delta(4)$, it should consist of five 3-arrows, associated to the five tetrahedral faces of $\Delta(4)$. It may therefore seem somewhat surprising that there are three squares in each of the two diagrams (5.5.5), (5.5.6), corresponding to the three factors on either side of equality (4.2.17). The reason for the appearance of the extra term $\{\tilde{m}_{ijk}\,, g_{jkl}\}$ in diagram (5.5.5) is the ambiguity mentioned in 5.4 in the definition of the horizontal composition of two 2-arrows in a 3-category. This ambiguity is already apparent in the definition of the highest term in the aforementioned diagram O_4 of [St] p.290. The two possible ways in which this term might be interpreted differ from each other by a 2-arrow which yields, in the present context, the 2-arrow $\{\tilde{m}_{ijk}\,, g_{jkl}\}$. We also observe that diagrams similar to (5.5.5)-(5.5.6) appear elsewhere in the litterature in related contexts, for example in [Du 4] and in [Le].

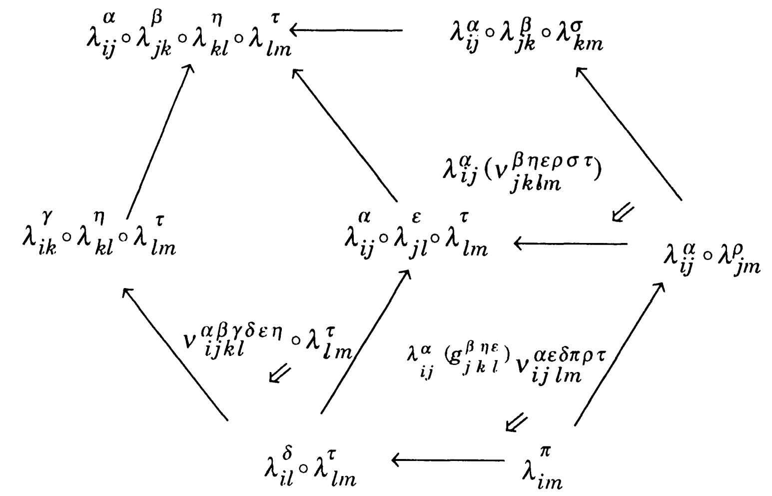
5.8 There remains the question of extending the discussion of (5.1)-(5.7) from the Čech to the hypercover case. We start out with a quintuple cocycle

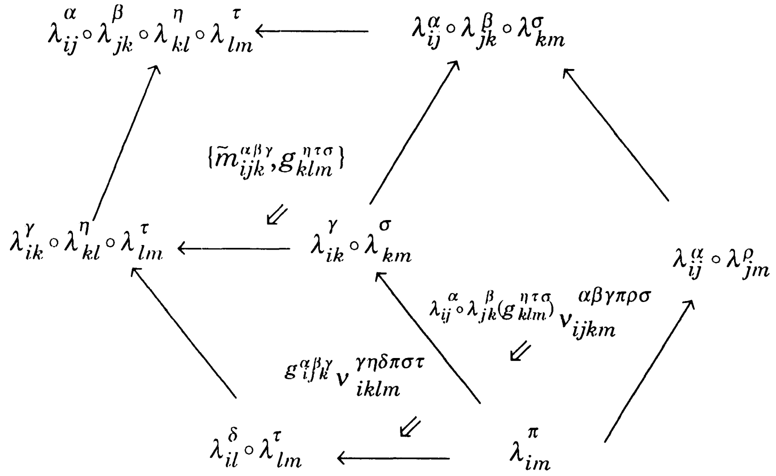
$$(5.8.1) \qquad (v_{ijkl}^{\alpha\beta\gamma\delta\varepsilon\eta}, \tilde{m}_{ijk}^{\alpha\beta\gamma}, g_{ijk}^{\alpha\beta\gamma}, \lambda_{ij}^{\alpha}; \mathcal{G}_i)$$

defined as in 4.9, and which satisfies the normalization conditions (4.9.3)-(4.9.6). We work here in full generality, so that the terms $\tilde{m}_{ijk}^{\alpha\beta\gamma}$ and $g_{ijk}^{\alpha\beta\gamma}$ in this cocycle quintuple are defined on the open sets $(V_{ijk}^{\alpha\beta\gamma})_\lambda$ introduced in 2.7, and the 3-cochains $v_{ijkl}^{\alpha\beta\gamma\delta\varepsilon\eta}$ live on open sets of an open cover of the set (2.7.2). We will denote an element of this open cover by $W_{ijkl}^{\alpha\beta\gamma\delta\varepsilon\eta}$, thereby suppressing from the notation both the indices which were hidden in the notation $V_{ijkl}^{\alpha\beta\gamma\delta\varepsilon\eta}$ (2.7.1) for the set (2.7.2) and a new (and final!) index ω which labels each constituent of this open cover. Let us now introduce four new open sets U_{im}^{π}, U_{jm}^{ρ}, U_{km}^{σ}, U_{lm}^{τ} defined as in definition 2.4. The upper-indexed version of the cocycle condition (4.2.17) which this 3-cochain satisfies is an identity which now lives above the intersection $Y_{ijklm}^{\alpha\beta\gamma\delta\varepsilon\eta\pi\rho\sigma\tau}$ of the five following open sets

$$W_{ijkl}^{\alpha\beta\gamma\delta\varepsilon\eta}, W_{iklm}^{\gamma\eta\delta\pi\sigma\tau}, W_{ijlm}^{\alpha\varepsilon\delta\pi\rho\tau}, W_{ijkm}^{\alpha\beta\gamma\pi\rho\sigma}, W_{jklm}^{\beta\eta\varepsilon\rho\sigma\tau}.$$

We do not write down this upper-indexed cocycle formula here, since this is more an exercise in typography than in mathematics, but remark, as we already did in 4.9, that it may be obtained from (4.2.17) by replacing each term by the corresponding one in one of the two diagrams (5.8.2) below. We have seen that the cocycle condition (4.2.17) could be interpreted as an identification of the pair of diagrams (5.5.5) and (5.5.6) in the fibre 2-category $\mathcal{E}q\,(Tors\,(\mathcal{G}_m), Tors\,(\mathcal{G}_i)$ above the open set U_{ijklm}. The corresponding interpretation of the upper-indexed cocycle identity is simply the identification of the following upper-indexed versions of these two diagrams, which both live in the fibre of this 2-category above the set $Y_{ijklm}^{\alpha\beta\gamma\delta\varepsilon\eta\pi\rho\sigma\tau}$:





(5.8.2)

Let us review here, for future reference, the interpretation of these two diagrams. For a fixed pair of indices (i,j), each arrow $\lambda^{\alpha}_{ij} : \mathcal{G}_j \longrightarrow \mathcal{G}_i$ induces on the open set U^{α}_{ij} an equivalence

$$\ddot{\lambda}^{\alpha}_{ij} : Tors(\mathcal{G}_j) \longrightarrow Tors(\mathcal{G}_i)$$

which we have simply denoted by λ^{α}_{ij}. By lemma 5.2, the upper-indexed version

(5.8.3) $$\tilde{m}^{\alpha\beta\gamma}_{ijk} : i_{g^{\alpha\beta\gamma}_{ijk}} \circ \lambda^{\gamma}_{ik} \longrightarrow \lambda^{\alpha}_{ij} \circ \lambda^{\beta}_{jk}$$

of (4.2.4) determines a natural transformation

$$\tilde{\psi}^{\alpha\beta\gamma}_{ijk} : \tilde{\lambda}^{\gamma}_{ik} \Rightarrow \tilde{\lambda}^{\alpha}_{ij} \circ \tilde{\lambda}^{\beta}_{jk}$$

on the open sets $(V^{\alpha\beta\gamma}_{ijk})_{\lambda}$ and the latter defines the corresponding 1-arrow in the diagrams (5.8.2). Finally, the 2-arrows

(5.8.4) $$\begin{array}{ccc} \lambda^{\delta}_{il} & \xrightarrow{\psi^{\alpha\varepsilon\delta}_{ijl}} & \lambda^{\alpha}_{ij} \circ \lambda^{\varepsilon}_{jl} \\ {\scriptstyle \psi^{\gamma\eta\delta}_{ikl}} \Big\downarrow & \swarrow\!\!\!\!\swarrow \; v^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl} & \Big\downarrow {\scriptstyle \lambda^{\alpha}_{ij}(\psi^{\beta\eta\varepsilon}_{jkl})} \\ \lambda^{\gamma}_{ik} \circ \lambda^{\eta}_{kl} & \xrightarrow[\psi^{\alpha\beta\gamma}_{ijk} \circ \lambda^{\eta}_{kl}]{} & \lambda^{\alpha}_{ij} \circ \lambda^{\beta}_{jk} \circ \lambda^{\eta}_{kl} \end{array}$$

which the 3-cochains $v^{\alpha\beta\gamma\partial\varepsilon\eta}_{ijkl}$ determine live, as we have said, in the fibres of $\mathcal{E}q\,(Tors\,(\mathcal{G}_l),\; Tors\,(\mathcal{G}_i))$ above the open sets $W^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl}$ .We will henceforth use the symbol $I^{\alpha\beta\gamma\delta\varepsilon\eta\pi\rho\sigma}_{ijklm}$ as short-hand for the assertion that the two composite 2-arrows determined by the two diagrams (5.8.2) are equal. From now on, we refer to this assertion as the 3-cocycle identity, without belabouring the fact that it is now the upper-indexed version of it which we have in mind.

It follows immediately from the specialization $I^{\alpha\beta\gamma\delta\varepsilon\eta\delta\varepsilon\eta}_{ijkll}$ of the 3-cocycle identity that the 2-arrows $v^{\alpha\beta\gamma\partial\varepsilon\eta}_{ijkl}$, which were a priori defined on the various open sets $W^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl}$ with fixed hidden indices $(\lambda, \mu, \nu, \rho)$ but varying additional index ω, are in fact compatible on their common domain of definition. Since $\mathcal{E}q\,(Tors\,(\mathcal{G}_l),\; Tors\,(\mathcal{G}_i))$ is a 2-stack, these 2-arrows therefore glue to a 2-arrow which we will also denote by $v^{\alpha\beta\gamma\partial\varepsilon\eta}_{ijkl}$, but which is now defined on the full set $V^{\alpha\beta\gamma\partial\varepsilon\eta}_{ijkl}$ (2.7.1). We may thus henceforth dispense with the additional index ω (note that this argument is analogous to the one used in (2.7.3), but takes place at the higher level of 2-arrows, rather than at that of 1-arrows).

Let us now specialize diagram (5.8.4) by setting $i=j$ so that $\beta=\gamma$, $\delta=\varepsilon$ and α may be omitted. This defines a natural transformation

(5.8.5) $$\nu^{*\beta\beta\delta\delta\eta}_{jjkl} : (g^{\beta\eta\delta}_{jkl})_\rho \Rightarrow (g^{\beta\eta\delta}_{jkl})_\nu$$

which replaces the identity (2.7.3), for any pair of hidden indices ν and ρ. Such a natural transformation may also be obtained as

(5.8.6) $$\nu^{\beta\eta\delta\delta\eta*}_{jkll} : (g^{\beta\eta\delta}_{jkl})_\mu \Rightarrow (g^{\beta\eta\delta}_{jkl})_\lambda$$

for any pair of omitted indices μ and λ. The specialization $I^{\beta\beta\delta\delta\delta\eta\delta\delta\eta}_{jjkll}$ of the 3-cocycle identity then implies that the transformations (5.8.5) satisfy the transitivity condition for varying pairs of these omitted indices, so that (5.8.5) provides glueing data, as ρ varies, for the arrows $(g^{\beta\eta\delta}_{jkl})_\rho$ in the stack $\mathcal{G}_j$. These arrows therefore glue to a global arrow $g^{\beta\eta\delta}_{jkl}$ defined on the entire set $U^{\beta\eta\delta}_{jkl}$ (2.4.2). A similar argument, applied to the arrows $\tilde{m}^{\alpha\beta\gamma}_{ijk}$, and in which the arrow (4.9.2) is replaced by the upper-indexed version of (4.2.11), shows that the arrows $\tilde{m}^{\alpha\beta\gamma}_{ijk}$ also glue to a global arrow, defined in the fiber of the stack $\mathcal{E}q(\mathcal{G}_k, \mathcal{G}_i)$ above the entire open set $U^{\alpha\beta\gamma}_{ijk}$. Lemma 5.2 now implies that the corresponding natural transformations $\tilde{\psi}^{\alpha\beta\gamma}_{ijk}$ are also defined on the open sets $U^{\alpha\beta\gamma}_{ijk}$.

We will now show that diagram (5.8.4) lives entirely in the fiber above the open set (2.4.9). This is now certainly the case for its edges, but the 2-arrow $\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl}$ is, for the time being, only defined on the smaller open set (2.7.1). In order to emphasize this 2-arrows' dependence on the hidden indices, we will temporarily denote it by the unwieldy notation $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda\mu\nu\rho}$. Once more, we appeal to specializations of the 3-cocycle identity. To begin, the specialization $I^{\alpha\beta\gamma\delta\varepsilon\eta\delta\varepsilon\eta}_{ijkll}$ shows that, for an appropriate choice of the hidden indices, the 2-arrow $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda\mu\nu\rho}$ is transformed by the requisite 2-arrows of type (5.8.5) into the corresponding arrow $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda\mu'\nu'\rho'}$. A similar specialization compares 2-arrows of the type $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda\mu\nu\rho}$ and $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda'\mu'\nu'\rho}$ so that any pair of 2-arrows $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda\mu\nu\rho}$ and $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl})_{\lambda'\mu'\nu'\rho'}$ are in fact compatible. This finishes the proof that $\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{ijkl}$ (and therefore the entire diagram (5.8.4)) is well-defined on the entire open set $U^{\alpha\beta\gamma\delta}_{ijkl}$ (2.4.9).

The next stage of the argument concerns the arrows $\lambda^{\alpha}_{ij}: \mathcal{G}_j \longrightarrow \mathcal{G}_i$ .We know that these arrows induce equivalences $\tilde{\lambda}^{\alpha}_{ij}: Tors(\mathcal{G}_j) \longrightarrow Tors(\mathcal{G}_i)$ on each of the corresponding open sets U^{α}_{ij} . We now show the equivalences in question actually determine an equivalence $\tilde{\lambda}_{ij}$ (5.1.1) on the entire set U_{ij}. First of all, observe that the arrow $\tilde{\psi}^{\alpha\beta\gamma}_{ijk}$, when it is specialized to the case $j=k$ (so that the upper index β may be omitted), reduces to a natural transformation

$$\tilde{\psi}^{\alpha\gamma}_{ijj}: \tilde{\lambda}^{\gamma}_{ij} \Rightarrow \tilde{\lambda}^{\alpha}_{ij} \tag{5.8.7}$$

in the fibre of $\mathcal{E}q\,(Tors(\mathcal{G}_j),\ Tors(\mathcal{G}_i))$ above the open set $U^{\alpha}_{ij} \cap U^{\gamma}_{ij}$. Similarly, the 2-arrow $\nu^{\alpha\beta\gamma\partial\varepsilon\eta}_{ijkl}$ specializes in the case $j=k=l$ (with β, η, ε omitted) to a 2-arrow

$$\tilde{\nu}^{\alpha\gamma\delta}_{ijjj} : \tilde{\psi}^{\alpha\delta}_{ijj} \Rrightarrow \tilde{\psi}^{\alpha\gamma}_{ijj}\ \tilde{\psi}^{\gamma\delta}_{ijj} \tag{5.8.8}$$

in the fibre of $\mathcal{E}q\,(Tors(\mathcal{G}_j),\ Tors(\mathcal{G}_i))$ above the open set $U^{\alpha}_{ij} \cap U^{\gamma}_{ij} \cap U^{\delta}_{ij}$, which we denote by $\tilde{\nu}^{\alpha\gamma\delta}_{ij}$. Finally, when (5.8.3) is specialized to the case $j=k=l=m$ (and the corresponding upper indices are omitted), it becomes the identity

$$\nu^{\alpha\delta\pi}_{ij}\ \nu^{\alpha\gamma\delta}_{ij} = \nu^{\alpha\gamma\pi}_{ij}\quad g^{\alpha\gamma}_{ijj}(\nu^{\gamma\delta\pi}_{ij}) \tag{5.8.9}$$

on the open set $U^{\alpha}_{ij} \cap U^{\gamma}_{ij} \cap U^{\delta}_{ij} \cap U^{\pi}_{ij}$. Since (5.8.9) is simply the tetrahedral condition (1.10.3) for the pair $(\tilde{\psi}^{\alpha\gamma}_{ijj},\ \nu^{\alpha\gamma}_{ij\delta})$, this pair defines a set of

2-descent data for the objects $\tilde{\lambda}^{\alpha}_{ij}$ of the 2-stack $\mathcal{E}q\,(Tors\,(\mathcal{G}_j),\ Tors\,(\mathcal{G}_i))$. It follows that the equivalences $\tilde{\lambda}^{\alpha}_{ij}$ do indeed descend to an equivalence $\tilde{\lambda}_{ij}: Tors\,(\mathcal{G}_j) \longrightarrow Tors\,(\mathcal{G}_i)$ defined on the entire open set U_{ij}.

Let us now fix three lower indices (i,j,k). We will now verify in a similar manner that the previously constructed arrows $\psi^{\alpha\beta\gamma}_{i\,j\,k}$ can be descended to an arrow ψ_{ijk} (5.5.1) defined on the entire open set U_{ijk}. This involves the comparison, on their common set of definition, of the two corresponding sets of arrows (5.8.3),which are respectively defined on two open sets $U^{\alpha\,\beta\,\gamma}_{i\,j\,k}$ and $U^{\alpha'\beta'\gamma'}_{i\;j\;k}$. This comparison is achieved by composing in the following manner the 2-arrow $\nu^{\alpha\beta\gamma\gamma'\beta'}_{i\,j\,k\,k}$ with the inverse of $\nu^{\alpha\alpha'\gamma'\gamma'\beta'}_{i\;i\;j\;k}$:

$$\begin{array}{ccccc}
\lambda^{\gamma'}_{ik} & = = = = & \lambda^{\gamma'}_{ik} & \longrightarrow & \lambda^{\gamma}_{ik} \\
\psi^{\alpha'\beta'\gamma'}_{ijk}\Big\downarrow & \nu^{\alpha\alpha'\gamma'\gamma'\beta'}_{iijk} \Swarrow & \Big\downarrow & \Nearrow\, \nu^{\alpha\beta\gamma\gamma'\beta'}_{ijkk} & \Big\downarrow \psi^{\alpha\beta\gamma}_{ijk} \\
\lambda^{\alpha'}_{ij}\circ\lambda^{\beta'}_{jk} & \longrightarrow & \lambda^{\alpha}_{ij}\circ\lambda^{\beta'}_{jk} & \longrightarrow & \lambda^{\alpha}_{ij}\circ\lambda^{\beta}_{jk}
\end{array} \tag{5.8.10}$$

The composite 2-arrow may be viewed as an arrow between the restrictions of the two objects $\psi^{\alpha\beta\gamma}_{i\,j\,k}$ and $\psi^{\alpha'\beta'\gamma'}_{i\;j\;k}$ in the fibered category $\mathcal{A}r\,(\lambda_{ik}, \lambda_{ij}\circ\lambda_{jk})$. Since the pair of objects λ_{ik} and $\lambda_{ij}\circ\lambda_{jk}$ live in the 2-stack $\mathcal{E}q\,(Tors\,(\mathcal{G}_k), Tors\,(\mathcal{G}_i))$, the fibered category of 1-arrows $\mathcal{A}r\,(\lambda_{ik}, \lambda_{ij}\circ\lambda_{jk})$ is in fact a stack on the open set U_{ijk}. In order to show that the locally defined objects $\psi^{\alpha\beta\gamma}_{i\,j\,k}$ of this stack descend to a global object, it is sufficient to verify that the 2-arrows (5.8.10) constitute a set of descent data for the locally defined objects in question. In concrete terms, this means that if a third open set $U^{\alpha''\beta''\gamma''}_{i\;j\;k}$ is introduced, we must verify the transitivity property relating (once the 2-arrows (5.8.3) are taken into account) the three 2-arrows which may be constructed for varying upper indices as in (5.8.10). For this, we consider the specializations $I^{\alpha\beta''\gamma''\gamma'\beta'*\gamma\beta**}_{i\,j\;k\;k\;k}$ and $I^{***\alpha\alpha'\beta''\gamma''\gamma''\gamma''\beta''}_{i\,i\,i\,k}$ of the identity (5.8.2) (in which the symbol $*$ once more means that the corresponding upper index has been omitted). These respectively contain, among other terms, the 2-arrow $v_{ij\,k\,k}^{\alpha\beta''\gamma''\gamma\beta}$ and the 2-arrow $v_{i\;ij\;k}^{\alpha''\alpha\gamma''\gamma''\beta''}$ which are the constituents of the 2-arrow (5.8.5) associated to the pair of upper indices (α,β,γ) and $(\alpha'',\beta'',\gamma'')$. When we replace each of these two 2-arrows by the corresponding terms determined by the identities in question, and when we then permute two of the terms of the resulting diagram by applying the further identity $I_{ii\,j\,k\,k}^{*\alpha'\alpha\gamma''\gamma''\beta''\gamma'\gamma'\beta'*}$, we obtain a diagram consisting of two adjacent copies of the 2-arrows (5.8.10), respectively associated to the pairs of upper indices (α,β,γ), (α',β',γ') and (α',β',γ'), $(\alpha'',\beta'',\gamma'')$. This proves the transitivity of the construction (5.8.5), and therefore shows that the family of arrows (5.8.2) do indeed define a natural transformation ψ_{ijk} (5.5.1) on the entire open set U_{ijk} .

Let us finally consider, for fixed lower indices (i,j,k,l) and varying upper indices $(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$, the collection of 2-arrows $v_{ijk\,l}^{\alpha\beta\gamma\,\partial\,\varepsilon\eta}$ (5.8.4), each of which is defined on an open set (2.4.9). We will, during this discussion, simply denote such a 2-arrow by $v^{\alpha\beta\gamma\delta\varepsilon\eta}$ whenever there is no ambiguity in the determination of the lower indices (i,j,k,l). Since the both pairs of arrows $\lambda_{ij}(\psi_{jkl})\,\psi_{ijl}$ and $\psi_{ijk}\,\psi_{ikl}$ in the 2-stack $\mathcal{E}q\,(Tors\,(\mathcal{G}_l\,),Tors\,(\mathcal{G}_i)$ are now defined over the open set U_{ijkl} , each of the 2-arrows $v^{\alpha\beta\gamma\delta\varepsilon\eta}$ may now be viewed as an arrow in the stack $\mathcal{A}r\,(\lambda_{ij}(\psi_{jkl})\,\psi_{ijl}\,,\,\psi_{ijk}\,\psi_{ikl})$, which is defined above the open set (2.4.9). These arrows glue to an arrow v defined on the entire open set U_{ijkl} whenever they agree on their common set of definition. Instead of comparing the restrictions of an arbitrary pair of 2-arrows $v^{\alpha\beta\gamma\delta\varepsilon\eta}$, $v^{\alpha'\beta'\gamma'\delta'\varepsilon'\eta'}$ to their common set of definition, it is more expedient to modify the upper indices one, or two, at a time. Once more, such a comparison is achieved in each case by considering the appropriate specialization of the 3-cocycle identity. For example, one may pass from the 2-arrow upper-indexed by $(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ to that indexed by $(\alpha,\beta,\gamma,\delta,\varepsilon,\eta')$ by applying the identity $I_{ijkll}^{\alpha\beta\gamma\delta\varepsilon\eta'\delta\varepsilon\eta*}$. We now simply list the identities, which relate the latter sextuple successively to $(\alpha,\beta,\gamma,\delta',\varepsilon',\eta')$, to $(\alpha,\beta',\gamma,\delta',\varepsilon',\eta')$, to $(\alpha,\beta',\gamma',\delta',\varepsilon',\eta')$ and finally to the sought-after $(\alpha',\beta',\gamma',\delta',\varepsilon',\eta')$. These are respectively the four following identities:

$$I_{i\,j\,kl\,l}^{\alpha\beta\gamma\delta'\varepsilon'\eta'\delta\varepsilon\eta'*}\,,\; I_{i\,j\,k\,kl}^{\alpha*\alpha\gamma\beta\beta'\delta'\varepsilon'\varepsilon'\eta'}\,,\; I_{i\,i\,j\,k\,l}^{*\alpha\alpha\gamma'\gamma\beta'\delta'\delta'\varepsilon'\eta'}\,,\; I_{ii\,j\,k\,l}^{*\alpha'\alpha\gamma'\gamma'\beta'\delta'\delta'\varepsilon'\eta'}\,.$$

This finishes the construction of a 2-arrow $v = v_{ijkl}$ (5.5.3) defined on the entire set U_{ijkl}. Such a 2-arrow v_{ijkl} automatically satisfies the Čech 3-cocycle condition (4.2.17), since its restriction $v^{\alpha\beta\gamma\delta\varepsilon\eta}$ satisfies the corresponding upper-indexed 3-cocycle condition. The two diagrams (5.5.5) and (5.5.6) have now been reconstructed from their local versions (5.8.2), and have been shown to agree with one another. We are now reduced to the Čech situation, and the construction of the associated gerbe $\mathbb{G}$ may therefore now be carried out exactly as in the proof of theorem 5.6. This finishes the proof of the following generalization of that theorem:

***Theorem* 5.9**: *The previous construction associates a* $\{\mathcal{G}_i\}_{i\in I}$*–2–gerbe* $\mathbb{G}$ *on* X *to the family of equivalences* $\tilde{\lambda}_{i\,j}^{\alpha}$ *, the family of natural transformations* $\tilde{\psi}_{i\;jk}^{\;\alpha\beta\gamma}$ *and the family of 2–arrows* $\tilde{v}_{ijkl}^{\alpha\beta\gamma\delta\varepsilon\eta}$ (5.8.4) *which are determined by a cohomology quintuple* $(v_{i\,j\,kl}^{\alpha\beta\gamma\delta\varepsilon\eta}, \tilde{m}_{ijk}^{\alpha\beta\gamma}, g_{ijk}^{\alpha\beta\gamma}, \lambda_{ij}^{\alpha}\,; \mathcal{G}_i)$. *It is the reverse of the construction given in proposition* 4.10, *and it associates a pair of equivalent* 2–*gerbes to a pair of cohomologous cocycle quintuples.*

6. The 2-gerbe of realizations of a lien

6.1 As a first illustration of the theory of 2-gerbes, we now examine under what conditions a given lien L on a space X is isomorphic to a lien of the form *lien* $(\mathcal{G})$, for some gerbe $\mathcal{G}$ on X. Such a gerbe $\mathcal{G}$ is then said to be a realization[6] of the lien L. Consider the fibered 2-category $R(L)$ on X, whose fiber $R(L)_U$ over an open set U is the 2-category of pairs $(\mathcal{G}, a)$, consisting of a gerbe $\mathcal{G}$ on U, and an isomorphism a: $lien(\mathcal{G}) \longrightarrow L_{|U}$ in the category of liens on U. A 1-arrow $u: (\mathcal{G}, a) \longrightarrow (\mathcal{G}', a')$ in $R(L)_U$ is a morphism of gerbes $u: \mathcal{G} \longrightarrow \mathcal{G}'$ on X, such that the induced diagram of liens on U

$$\begin{array}{ccc} lien\,(\mathcal{G}) & \xrightarrow{lien(u)} & lien\,(\mathcal{G}') \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle a'} & \\ & L_{|U} & \end{array} \tag{6.1.1}$$

is commutative and a 2-arrow $u_1 \Rightarrow u_2$ is simply a natural transformation between the underlying morphisms u_i in the 2-category of gerbes on U.

Let us prove that $R(L)$ is a 2-gerbe on X. Since the lien L on X is locally isomorphic to a lien of the form *lien* (G), it may be realized by the neutral gerbe *Tors* (G) associated to the sheaf of groups G, so that the fibered

[6] Not to be confused with a representative of the lien L, as defined in [Gi] IV 1.2.1.

2-category $R(L)$ is locally non-empty. In order to show that $R(L)$ is locally connected, one might simply refer to [Gi] IV corollary 2.3.3, but it is more instructive to give a direct proof of this fact. Let $(\mathcal{G}, a)$ and $(\mathcal{G}', a')$ be a pair of objects in $R(L)$, which are both defined on the open set U in X. The gerbes $\mathcal{G}$ and $\mathcal{G}'$ are assumed, after localization, to be respectively of the form $Tors\,(G)$ and $Tors\,(G')$, for some pair of sheaves of groups G and G'. Let $v\colon lien\,(G) \longrightarrow lien\,(G')$ be the unique morphism of liens such that $v \circ a' = a$. It is described by a section ψ of the sheaf $Out\,(G, G')$ (2.10.1), and this may be lifted locally to a genuine group isomorphism $\varphi\colon G \longrightarrow G'$. By construction, the morphism of gerbes $\varphi_*\colon Tors\,(G) \longrightarrow Tors\,(G')$ determined by φ satisfies the equation $lien\,(\varphi_*) = v$, so that the map φ_* defines the sought-after 1-arrow in $R(L)$ between the appropriate restrictions of the given objects $(\mathcal{G}, a)$ and $(\mathcal{G}', a')$. It is easy to show, by similar local arguments, that every 1-arrow in $R(L)$ is an equivalence (*i.e.*, is invertible up to a 2-arrow), and we refer for this to *op.cit.*, IV corollary 2.2.7. Finally, it is automatic that every 2-arrow $\eta\colon u \Rightarrow v$ in $R(L)$ is invertible, since the common target of the pair of arrows u and v is fibered in groupoids (see *op.cit.*, IV 2.3.2.2). We end this discussion by observing that the construction of the 2-gerbe $R(L)$ is functorial in L, so that we have in effect constructed a morphism of 2-categories

(6.1.2)
$$\begin{aligned} R\colon Lien_X &\longrightarrow 2\text{–}Gerbe_X \\ L &\longmapsto R(L) \end{aligned}$$

(the 1-category $Lien_X$ of liens on X being viewed here as a 2-category with no non-trivial 2-arrows).

6.2 Let us now suppose that the given lien L on X is described, as explained in 2.10, by a family of sections ψ_{ij} of the sheaves $Out\,(G_j, G_i)$ on the open sets U_{ij}, satisfying the condition (2.10.2). These sections may be lifted, on a family of refinements U_{ij}^{α} of the open sets U_{ij}, to a family of isomorphisms $\varphi^{\alpha}_{ij}: G_j \longrightarrow G_i$, such that

$$\varphi_{ii} = 1 \tag{6.2.1}$$

and it then follows from the cocycle condition (2.10.2), that one can choose sections $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$ of G_i on each set of a cover $(V^{\alpha\beta\gamma}_{i\,j\,k})_\lambda$ of the open sets $U^{\alpha\beta\gamma}_{i\,j\,k}$ (2.4.2), for which the equation

$$\varphi^{\alpha}_{ij} \circ \varphi^{\beta}_{jk} = i(\gamma^{\alpha\beta\gamma}_{i\,j\,k})\, \varphi^{\gamma}_{ik} \tag{6.2.2}$$

is satisfied. The normalization condition (6.2.1) implies one may, whenever as in (4.9) two consecutive lower indices in $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$ coincide, dispense with the corresponding upper index, and set

$$\begin{aligned} \gamma_{i\,i\,k}^{\;\;\beta\,\beta} &= 1 \\ \gamma^{\alpha\;\;\alpha}_{i\,kk} &= 1. \end{aligned} \tag{6.2.3}$$

We remark in passing, though this is not strictly relevant for our purpose, that the classes $[\gamma^{\alpha\beta\gamma}_{i\,j\,k}]$ which these sections $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$ define in the sheaf $Inn\,(G_i) = G_i/ZG_i$ of inner automorphisms of G_i, together with the isomorphisms

$$\begin{aligned} \lambda^{\alpha}_{ij}: Inn\,(G_j) &\longrightarrow Inn\,(G_i) \\ u &\longmapsto (\varphi^{\alpha}_{ij})^{-1} \circ u \circ (\varphi^{\alpha}_{ij}) \end{aligned}$$

defined by the φ^{α}_{ij}, determine a twisted $\{Inn(G_i)\}$–valued degree 2 cohomology class, which is that of the gerbe of $REP(L)$ of representatives of the lien L (*op.cit*,. IV 3.2.1). Instead of focussing on this class, we now associate to $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$ the element

$$\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{i\,j\,k\,l} = \varphi^{\alpha}_{ij}(\gamma^{\beta\,\eta\,\varepsilon}_{j\,k\,l})\, \gamma^{\alpha\varepsilon\delta}_{i\,j\,l}\, (\gamma^{\gamma\eta\,\delta}_{i\,k\,l})^{-1}\, (\gamma^{\alpha\beta\gamma}_{ijk})^{-1} \tag{6.2.4}$$

which, in view of (6.2.3), satisfies the following normalization conditions whenever two consecutive lower indices, and the constituents of the two corresponding pairs of upper indices, are equal:

$$\nu^{\alpha\beta\gamma\,\gamma\beta\,*}_{i\,j\,k\,k} = 1 \tag{6.2.5}$$

$$\nu^{\alpha*\alpha\,\delta\,\varepsilon\,\varepsilon}_{i\,j\,j\,\;k} = 1$$

$$\nu^{*\gamma\gamma\,\delta\,\delta\,\eta}_{i\,i\,j\;k} = 1$$

(the symbol * once more denotes a missing upper index). The element ν (6.2.4) is a priori a section above the set (2.7.1) of the group G_i , but a comparison between the two possible ways of expressing $\varphi^{\alpha}_{ij} \circ \varphi^{\beta}_{jk} \circ \varphi^{\gamma}_{kl}$ in terms of φ^{δ}_{il} shows that this section actually takes its values in the center ZG_i of the group G_i . It is immediate that this ZG_i–valued 3-cochain ν satisfies a cocycle condition above the open set on which diagrams (5.8.2) are defined. Dropping the upper indices, the cocycle condition in question is the simplified version

$$\varphi_{ij}(\nu_{jklm})\, \nu_{ijlm}\, \nu_{ijkl} \quad = \quad \nu_{ijkm}\, \nu_{iklm} \tag{6.2.6}$$

of the identity (4.2.17). The appropriate upper, and hidden, indices may be retrieved by examining the corresponding terms of the diagrams (5.8.2). When the lien L is a G–lien, the corresponding ψ_{ij} are 1-cocycles which, as we have seen in 2.10, take their values in the sheaf $Out\,(G)$. The identity (6.2.6) then states that in that case the 3-cochain $\nu^{\alpha\beta\,\gamma\,\delta\,\varepsilon\,\eta}_{i\,j\;k\,l}$ is an abelian 3-cocycle, with values in the group $(ZG)^L$ obtained by twisting the center ZG of G by the class of the 1-cocycle ψ_{ij} . The correspondence $(\psi_{ij} \longmapsto \nu_{ijkl})$ just considered is then just the coboundary map

$$H^1\,(X, Out\,(G)) \longrightarrow H^3(X, (ZG)^L) \tag{6.2.7}$$

associated to the non-abelian long exact sequence of sheaves

$$1 \longrightarrow ZG \longrightarrow G \longrightarrow Aut\,(G) \longrightarrow Out\,(G) \longrightarrow 1$$

on X . In particular, when X is the classifying space $B\Pi$ of a group Π and G is the constant sheaf defined by a group G, the map (6.2.7) is simply given by the well-known construction of Eilenberg-MacLane ([MacL 2] IV theorem 8.7), which associates to a G-valued abstract kernel $\psi\colon \Pi \longrightarrow Out\,(G)$ an obstruction $\nu \in H^3(\Pi, ZG)$ to the realization of ψ by an extension of Π by G.

6.3 Returning to the general situation, we now relate the cocycles attached by formulas (6.2.2)-(6.2.4) to a lien L on X to the cocycle quadruple associated by the general construction in §4 to a 2-gerbe such as $R(L)$. This will show that Eilenberg-MacLane's construction, and the generalization which we have just considered, yield an explicit description of the morphism of 2-categories (6.1.2).

We have already seen that whenever a lien L is locally of the form $lien(G_i)$, then there exist non-trivial objects x_i in the fiber 2-categories $R(L)_{U_i}$. These are defined by the equivalences

$$x_i = Tors(G_i), \tag{6.3.1}$$

and by the chosen isomorphisms $a_i: lien(G_i) \longrightarrow L_{|U_i}$. The sections of the gr–stacks $\mathcal{G}_i = \mathcal{E}q(x_i)$ associated to these objects are self-equivalences g of the stacks $Tors(G_i)$, for which the induced map $lien(g)$ satisfies the compatibility condition (6.1.1). It follows from lemma 1.7 that the gr–stack $\mathcal{E}q(x_i)$ of self-equivalences in $R(L)$ of the object x_i is equivalent to the Picard stack $\mathcal{G}_i = Tors(ZG_i)$. Pursuing our description of the cocycle quadruple associated to the 2-gerbe $R(L)$, we may now choose, possibly after base change, an arrow in $R(L)$ from x_j to x_i, in other words an equivalence between the gerbes $Tors(G_j)$ and $Tors(G_i)$. The assertion which follows the statement of lemma 1.4 ensures that such an equivalence may be locally defined as the map

$$\tilde{\varphi}^{\alpha}_{ij} : Tors(G_j) \longrightarrow Tors(G_i)\,, \tag{6.3.2}$$

induced by a section

$$\varphi^{\alpha}_{ij} : G_j \longrightarrow G_i. \tag{6.3.3}$$

of the sheaf $Isom(G_j, G_i)$ over a sufficiently small open set U^{α}_{ij}, (it is understood that (6.3.2) is the identity map whenever $i=j$). Since the arrow (6.3.2) satisfies the compatibility condition (6.1.1), the class $[\varphi_{ij}^{\alpha}]$ in $Out(G_j, G_i)$ of the isomorphism φ_{ij}^{α} is the restriction ψ_{ij}^{α} to the open set U_{ij}^{α} of the outer automorphism ψ_{ij} which describes L. Taking into account the description given above of the *gr*–stack $\mathcal{E}q(x_i)$, we see that the equivalence (6.3.2) induces by conjugation, as in (4.2.1), an equivalence of stacks

$$\lambda_{ij}^{\alpha}: Tors(ZG_j) \longrightarrow Tors(ZG_i). \tag{6.3.4}$$

which sends the trivial ZG_j–torsor to the trivial ZG_i–torsor. The description given in lemma 1.5 of the effect of left or right composition of morphisms such as (6.3.2) with the arrows in the category $Tors(G_j)$, implies that (6.3.4) is simply the equivalence $(\varphi_{ij}^{\alpha}\,|_{ZG_j})^{\sim}$ determined by the restriction

$$\varphi_{ij}^{\alpha}\,|_{ZG_j} : ZG_j \longrightarrow ZG_i \tag{6.3.5}$$

to ZG_j of the isomorphism $\varphi_{ij}^{\alpha} : G_j \longrightarrow G_i$. Since the group G_i acts trivially on ZG_i by inner conjugation, the map (6.3.5) depends only on the class ψ_{ij}^{α} of φ_{ij}^{α} in the group of sections of $Out(G_j, G_i)$ on the open set U_{ij}^{α} .On the other hand, we have seen in (2.7.6)-(2.7.7) in a more general setting that the locally defined maps (6.3.4) glue to a morphism

$$\lambda_{ij}: Tors(ZG_j) \longrightarrow Tors(ZG_i) \tag{6.3.6}$$

defined on the entire open set U_{ij}. This is reflected here by the fact that the arrows (6.3.5) are the restrictions to the open sets U_{ij}^{α} of a well-defined isomorphism

$$\psi_{ij}: ZG_j \longrightarrow ZG_i\,, \tag{6.3.7}$$

which induces λ_{ij} (6.3.6). This arrow is simply defined on the entire open set U_{ij} by restricting to ZG_j the given section ψ_{ij} of $Out(G_j, G_i)$.

In order to associate to the 2-gerbe $R(L)$ a cocycle quadruple, we need to give ourselves a decomposition of the 2-gerbe. We have seen that its first two constituents are the family of objects x_i (6.3.1) and the arrow (2.3.4) defined in (6.3.2). By lemma 1.5 *i*), the section $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$ of G_i chosen in (6.2.2) defines a 2-arrow

$$\begin{array}{ccc} Tors(G_k) & \longrightarrow & Tors(G_j) \\ & \searrow \quad \nearrow\!\!\!\!\nearrow \quad \swarrow & \\ & Tors(G_i) & \end{array} \qquad (6.3.8)$$

Comparing this diagram with the upper indexed version of diagram T_{ijk} (4.1.2), we observe that the arrow $g^{\alpha\beta\gamma}_{i\,j\,k}$ of that diagram is the identity in the present context, and that the 2-arrow $m^{\alpha\beta\gamma}_{i\,j\,k}$ appearing in diagram (4.1.2) is now the 2-arrow determined by the chosen section $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$ .Furthermore, the triviality of the action of a group G_i on its center by inner conjugation implies that the 2-arrow $\tilde{m}^{\alpha\beta\gamma}_{i\,j\,k}$ associated as in (4.2.3)-(4.9.1) to the 2-arrow (6.3.8) is trivial. The arrows (6.3.6) thus satisfy the strict compatibility condition

$$\lambda_{ij} \circ \lambda_{jk} = \lambda_{ik} \ , \qquad (6.3.9)$$

a fact which also follows immediately from the corresponding condition for the arrows ψ_{ij} (6.3.7). Examining the manner in which a 3-cocycle $\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{i\,j\,k\,l}$ (4.9.2) is assembled in (4.2.6) from upper-indexed triangles T_{ijk} , we see that in the present situation the 3-cocycle in question is precisely the one defined by formula (6.2.4). The cocycle associated to the given labeled decomposition of the 2-gerbe $R(L)$ is therefore the quintuple $(\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{i\,j\,k\,l}, 1, 1, \lambda_{ij}; Tors\,(ZG_i))$. One may even go one step further, by applying here the same degenerate versions of the cocycle condition (6.2.6) as in the discussion following diagram (5.8.4). This shows that the sections $\nu^{\alpha\beta\gamma\delta\varepsilon\eta}_{i\,j\,k\,l}$ of the sheaf ZG_i glue, for varying upper indices, to a section ν_{ijkl} of this sheaf on the entire set U_{ijkl} , which then automatically satisfies the cocycle condition (6.2.6). We summarize the present discussion as follows:

Proposition 6.4: *Let L be a lien which admits a local family of representatives lien(G_i) on an open cover $\mathcal{U}=(U_i)_{i\in I}$ of X, and let ψ_{ij} be the corresponding sections of Out(G_j, G_i) on U_{ij}, satisfying the cocycle condition* (2.10.2)–(2.10.3). *The local sections $x_i = Tors(G_i)$ of the* 2*–gerbe R(L) of realizations of L determine a labeling $\eta_i : \mathcal{A}ut(x_i) \to Tors(ZG_i)$ of R(L). A decomposition of R(L) may be defined by the arrows* (6.3.2), *by the identity* 1*–arrows and the* 2*–arrows determined by the chosen sections $\gamma^{\alpha\beta\gamma}_{i\,j\,k}$* (6.2.2) *of G_i. The* 3*–cocycle which by proposition* 4.10 *describes R(L) in terms of this labelled decomposition is the quadruple $(\nu_{ijkl}, 1, 1, \lambda_{ij})$, where ν_{ijkl} is the ZG_i–valued Čech* 3*–cocycle locally defined by the Eilenberg–MacLane cocycle formula* (6.2.4).

This discussion may be carried even further by observing that, since the isomorphisms (6.3.7) satisfy the 1-cocycle condition (2.10.2), the abelian sheaves ZG_i glue to a sheaf of abelian groups ZL on X, which Giraud calls the center of the lien L. In particular, when L is a G–lien for some group G on X, ZL is the twisted form $(ZG)^L$ of the center ZG of G appearing in formula (6.2.7). Returning to the general situation, let us observe that the coefficient strict Picard stacks $\mathcal{G}_i = Tors(ZG_i)$ by which we have labeled our 2-gerbe $R(L)$ are simply the restrictions to the open sets U_i of the strict Picard stack $\mathcal{G} = Tors(ZL)$ defined on all of X. This observation allows us to restate proposition 6.4 in the following strenghtened form (for a related, but less precise assertion, see [Gi] VI theorem 2.3):

Corollary 6.5: *Let $\mathcal{G}$ be the Picard stack Tors(ZL) associated to the center ZL of a lien L. With the same notation as in proposition* 6.4, *the local sections x_i of the* 2*–gerbe R(L) and the labelings η_i determine on R(L) a structure of $\mathcal{G}$–*2*–gerbe . Its cohomology class in $H^3(X,ZL)$ is that of the ZL–valued Čech* 3*–cocycle ν_{ijkl} locally defined by the Eilenberg–MacLane cocycle formulas* (6.2.2)–(6.2.4).

7. The classification of stacks, group extensions and *gr*–stacks

We will now review several topics in preparation for the classification of 2-stacks to be given in § 8. The first of these topics is the classification theory, or Postnikov decomposition, for 1-stacks on a space X. We will then construct the "Schreier gerbe" associated to a short exact sequence of sheaves of groups. The Schreier theory for such extensions of groups is of course classical [MacL-1], and we gave a cohomological interpretation for it, in a sheaf theoretic context, in [Br 2] §8. We return to this topic here in order to reinterpret this cohomological data in geometric terms. A final subject discussed in the present section is a review the Postnikov decomposition for 1-stacks endowed with a group structure. Just as a *gr*–category determines a bi-category with a single object, so the theory of 1-stacks with group structure is a special case of the theory of 2- (or rather bi-) stacks, so that we are in effect beginning here our classification of 2-stacks. It was shown in [Br 3] that the classification of *gr*-stacks and Schreier's theory of group extensions are both special cases of a more general extension problem. While the latter problem could also be given a geometric interpretation, this will not be discussed here.

7.1 Let $\mathscr{C}$ be a non-empty stack in groupoids on a site $\mathscr{S}$. To $\mathscr{C}$ is associated the presheaf whose values on an open $U \in \mathscr{S}$ is the set of connected components of the fibre category $\mathscr{C}_U$. This non-empty presheaf does not in general satisfy the sheaf axioms, so that one is led to introduce the associated sheaf, which is denoted by $\pi_0(\mathscr{C})$, or even simply π_0. Consider the map

(7.1.1) $$\mathscr{C} \longrightarrow \pi_0(\mathscr{C}),$$

which associates to each object $X \in \mathscr{C}_U$ the section of $\pi_0(\mathscr{C})$ on U determined by the connected component of $\mathscr{C}_U$ in which X lives. It is a Cartesian functor from the stack $\mathscr{C}$ to the discrete stack whose sheaf of objects is the sheaf $\pi_0(\mathscr{C})$. The axiom for arrows in a stack implies that, for any object $X \in \mathscr{C}_U$, the maps from X to itself determine a sheaf of groups $\underline{Aut}$ (X) on U, which will be denoted by $\pi_1(\mathscr{C}, X)$. Let us denote by $\mathscr{C} \,|\, \pi_0$ the localization of $\mathscr{C}$ above π_0 consisting of $\mathscr{C}$ together with its canonical map (7.1.1) to π_0. This may be viewed as stack in the localized topos T_{π_0}, whose fiber on an open set $f: U \longrightarrow \pi_0$ which lives above π_0 is the full subcategory of $\mathscr{C}_U$ whose objects map by (7.1.1) to the section f of π_0 . The knowledge of $\mathscr{C} \,|\, \pi_0$ determines that of $\mathscr{C}$, since one may simply forget the map from $\mathscr{C}$ to π_0 . We will suppose for simplicity that the sheaf π_0 is represented by an element of the site $\mathscr{S}$ which defines the topos T. The following observation is apparently due to Giraud.

Lemma 7.2: *The localization $\mathscr{C} \,|\, \pi_0$ of the stack in groupoids $\mathscr{C}$ is a gerbe on π_0 .*

Indeed, by definition of π_0, there exists a refinement U of the tautologous section $\xi: \pi_0 \longrightarrow \pi_0$ of π_0 for which the pullback of ξ to U describes a connected component of $\mathscr{C}_U$. Any object in this component is therefore an element of the fibre of $\mathscr{C} \,|\, \pi_0$ above U, so that the first axiom for gerbes is satisfied by $\mathscr{C} \,|\, \pi_0$. On the other hand, any pair of objects x and y in a fiber of $\mathscr{C} \,|\, \pi_0$ above an object V of the site $\mathscr{S}_{|\pi_0}$ determine the same class $[x]=[y]$ in $\Gamma(V, \pi_0)$. It therefore follows that these objects are locally in the same component of $\mathscr{C}$, so that they may locally be connected by an arrow in

$\mathscr{C}$. The second gerbe axiom is therefore also satisfied by the stack $\mathscr{C} \mid \pi_0$.

Remark 7.3: *i)* Let us choose a covering map $U \longrightarrow \pi_0$ for which there exists an object $x \in \mathscr{C}_U$ which realizes the tautological section of $\pi_0(\mathscr{C})$ over π_0. An isomorphism $\eta: \pi_1(\mathscr{C}, x) \longrightarrow \pi_1$ with values in a given sheaf of groups π_1 on U then defines a labeling of the gerbe $\mathscr{C} \mid \pi_0$, whose equivalence class is an element k_0 in the non-abelian cohomology set denoted by $H(\pi_0, \{\pi_1\})$ at the end of 2.4. This element k_0 might be termed the zeroth Postnikov invariant of $\mathscr{C}$. The previous discussion asserts that the stack $\mathscr{C}$ is entirely determined, up to equivalence, by the sheaf $\pi_0(\mathscr{C})$ associated to the presheaf of its connected components, by the locally defined groups $\pi_1(\mathscr{C}, X)$, and finally the π_1 - valued degree two cohomology class k_0.

ii) The class k_0 is neutral if and only if there exists an object x in the fibre category $\mathscr{C}_{\pi_0}$ realizing the tautological section of π_0. Loosely speaking, this means that in that case one can choose, in a consistent manner, for every section a of π_0 above an open set U, an object X_a in the connected component of $\mathscr{C}_U$ which a determines. A gerbe in the punctual topos is always neutral, so this is always the case for the gerbe associated to $\mathscr{C}$ when $\mathscr{C}$ is an ordinary groupoid, rather than a stack of groupoids. It is then an elementary fact that the choice of an object x in each connected component $\mathscr{C}[x]$ of $\mathscr{C}$ and, for each object y in the same connected component as x, of a path γ_{xy} from x to y, determines a retraction of $\mathscr{C}[x]$ onto the groupoid with one object defined by the group $\pi_1(\mathscr{C}, x)$. The vanishing of the k-invariant k_0 in this situation thus simply reflects the split Postnikoff decomposition

$$X = \coprod_{[x]\in\pi_0(X)} K(\pi_1(X,x),1) \tag{7.3.1}$$

of the nerve X of the groupoid $\mathscr{C}$.

7.4 The oldest example of a non-trivial element in a non-abelian H^2 set is the Schreier cocycle associated to an extension of abstract groups

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1. \tag{7.4.1}$$

This is generalized in [Br 2], where it is shown how to associate to such an exact sequence of sheaves of groups an element in the set $H^2(BK, G \longrightarrow Aut(G))$ which describes it. A geometrical interpretation of the element in question is given by Giraud in [Gi] VIII 7.3, who introduces there the notion of an extension of the topos BK. We prefer here to show how the cohomological data associated to a short exact sequence of sheaves of groups (7.4.1) may be encoded in an appropriately defined G–gerbe on the classifying space BK of the group K. Let us begin by recalling that to any group K in a topos T is associated its classifying space BK. This simplicial object of T is the nerve of the groupoid $K[1]$ determined by the group K. Its component in degree n is the n–fold product K^n, endowed with the usual face and degeneracy maps $d_i : K^n \longrightarrow K^{n-1}$ and $s_i : K^n \longrightarrow K^{n+1}$. Above BK lives the universal bundle EK. This is another simplicial object of T whose nth degree component is K^{n+1}, and which we endow with a left action of K defined by the formula

$$k(k_0, \ldots, k_n) = (k_0 k^{-1}, \cdots, k_n k^{-1}).$$

The projection

$$\begin{aligned} \pi:\ EK &\longrightarrow BK \\ (k_0, \ldots, k_n) &\longmapsto (k_0 k_1^{-1}, k_1 k_2^{-1}, \ldots, k_{n-1} k_n^{-1}) \end{aligned}$$

defines on EK a left K–torsor structure above BK. Recall that to any simplicial object X in T is associated a topos $Top(X)$ of sheaves on X, whose objects consist in families $F^{\cdot}$ of sheaves F^n on X_n, together with morphisms of sheaves

$$\delta^i : d_i^* F^{n-1} \longrightarrow F^n \tag{7.4.2}$$

$$\sigma^i : s_i^* F^{n+1} \longrightarrow F^n$$

on X_n satisfying the standard simplicial identities up to coherent isomorphisms [Il] VI 5.2.1. In fact, when T is the topos associated to a site $\mathcal{S}$, and the components of X are representable, the topos $Top(X)$ may be viewed the topos associated to an appropriately defined site (*see* [Del 1] 5.1.8). To any sheaf F of T, endowed with a right action of the sheaf of groups K corresponds, as explained in *op. cit.*, an object of the topos associated to BK. This is obtained as follows. One begins by pulling back F to the "constant" sheaf p^*F on the simplicial object EK, whose value on the nth component K^{n+1} of EK is the pullback $p_n^*F = F \times K^{n+1}$ of F by the projection $p_n : K^{n+1} \longrightarrow e$ to the final object of T. The group K now acts diagonally on the left on (each component of) the sheaf p^*F, by the rule

$$k(f, k_0, \ldots, k_n) = (fk^{-1}, k_0\, k^{-1}, \ldots, k_n\, k^{-1}). \tag{7.4.3}$$

Since this action is equivariant with respect to the action of K on EK, it determines descent data for the projection π of EK on BK and it therefore defines a sheaf on BK whose nth component will be denoted $F \wedge^K K^{n+1}$. The sheaf in question on BK might be thus be called, as is often the case in topology in a somewhat similar context, the "Borel construction" associated to F. It will be denoted by $F \wedge^K EK$, but the notation $F//K$ would also be appropriate, since it is the homotopy theoretic quotient of the sheaf F by the action of K. When K acts trivially on F, we retrieve in this manner the constant simplicial sheaf $F \times BK$ on BK, whose nth component is simply the pullback of F under the projection from BK to e.

Let K be a group in T, and let $G \longrightarrow Aut(G)$ be the crossed module in T defined by another group G of T. It may be viewed as a (constant) crossed module in the topos $Top(BK)$. The class in $H^1(BK, G \longrightarrow Aut(G))$ which classifies an extension (7.4.1) (*see* [Br 2] propositions 8.2) therefore describes the class of a G–gerbe in the topos $Top(BK)$, which we will call the Schreier gerbe on BK associated to the exact sequence (7.4.1). We will now describe this gerbe explicitly. We observe that the Borel construction reviewed above merely depended on the effectivity of descent for sheaves from EK to BK. It therefore may be extended from a construction involving sheaves to one involving stacks (or even 2–stacks) in T. To be more specific, let us suppose that the topos T is associated to a site $\mathcal{S}$, and that $\mathcal{C}$ is a stack in T, on which a group K of T acts on the right, by a morphism of fibered categories

$$\mathcal{C} \times K \longrightarrow \mathcal{C}$$

(with K viewed as a discrete fibered category on $\mathcal{S}$), which is associative, up to a coherent natural transformation as in [Br 2] 6.1.3. The pullback $\mathcal{C} \times EK$ of $\mathcal{C}$ by the projection $EK \longrightarrow e$ is a stack on EK, consisting of a family of stacks $\mathcal{C} \times K^{n+1}$ on the components K^{n+1} of EK, together with a coherent family of face and degeneracy 1-arrows, as in (7.4.2). The group K acts diagonally on $\mathcal{C} \times EK$ as in (7.4.3),and this action is coherently equivariant with respect to the right action of K on EK, as in [Bry] 7.3.1. It therefore determines a set of 2-descent data on $\mathcal{C} \times EK$ relative to the projection of EK on BK, to which corresponds a descended stack on BK, which will be termed the "Borel construction" stack for the action of K on $\mathcal{C}$, and will be denoted $\mathcal{C} \wedge^K EK$.

We apply this contruction to the following particular situation. Lemma 8.3 of [Br 2] asserts that the information given by an exact sequence (7.4.1) is encoded in the morphism of gr–stacks

$$\begin{aligned} K &\longrightarrow Bitors(G) \\ k &\longmapsto H_k \end{aligned} \tag{7.4.4}$$

from the discrete stack K to the stack of G–bitorsors of T determined by H. The Morita theorem (proposition 1.6.*ii*)) asserts that the group of sections of the stack *Bitors* (G) is opposite to the group of self-equivalences of the stack *Tors* (G), so that (7.4.4) defines a right action of K on the stack *Tors* (G). This action therefore defines a Borel construction stack in groupoids

$$\mathscr{C} = Tors(G) \wedge^K EK \tag{7.4.5}$$

on BK. Its pullback by the covering morphism $EK \longrightarrow BK$ is, by construction, the constant stack *Tors* (G) on EK, and the gluing data is defined by the G–bitorsor on K which H determines. The discussion in §8 of [Br 2] now translates into the assertion that an extension (7.4.1) may be described by the associated Schreier G–gerbe $\mathscr{C}$ (7.4.5). This may be restated by considering the constant stack $\mathscr{T}_G = p^* Tors(G)$ on BK, which is defined by pulling back the stack *Tors* (G) to BK by the projection map $p{:}BK \longrightarrow e$. The stack $\mathscr{T}_G$ is simply the trivial G–gerbe on BK, and it is endowed, as we have just seen, with a right action of K defined by H. Since definition (7.4.5) describes $\mathscr{C}$ as the gerbe obtained from $\mathscr{T}_G$ by twisting it by the universal left K–torsor EK, it is therefore consistent with the notation of [Gi] III 2.3, and somewhat suggestive, to rewrite (7.4.5) as

$$\mathscr{C} = (\mathscr{T}_G)^{EK}.$$

7.5 It is possible to give an even more direct description of the Schreier gerbe $\mathscr{C}$, and one which is closer to Giraud's notion of extension of a topos, if we are willing to consider gerbes defined over stacks (rather than simply over sites). It is not appropriate to launch here into a detailed study of this concept, and we will simply discuss this in an informal manner. The main observation is that to any extension of groups (7.4.1) is associated a locally essentially surjective morphism of stacks

$$\pi{:}\, Tors(H) \longrightarrow Tors(K), \tag{7.5.1}$$

which pushes out an H-torsor to a K-torsor by the group homomorphism $H \longrightarrow K$. It is tempting to consider π as defining an object in the 2-category of "stacks above the stack $Tors(K)$". Since it makes sense to talk about products of fibered categories ([L-M] §1), we may certainly think of $Tors(H)$ as being fibered over the 2-category of categories above $Tors(K)$, by associating to any object $\mathcal{E} \longrightarrow Tors(K)$ in this 2-category, the fibre category

$$Tors(H)_{\mathcal{E}} = \mathcal{E} \times_{Tors(K)} Tors(H). \tag{7.5.2}$$

For the present discussion to be complete, it would be necessary to discuss here which functors $\mathcal{E} \longrightarrow Tors(K)$ are "covering functors", and what sort of topology (2-topology ?) they determine on $Tors(K)$, in other words what are the gluing properties on the fiber categories $Tors(H)_{\mathcal{E}}$. It will, however, not be necessary to go this far. Instead, we may view BK as the (nerve of the) groupoid $K[1]$ defined by the group K, and consider the associated stack map

$$a: K[1] \longrightarrow Tors(K).$$

Let us pull back the fibered category (7.5.1) by the functor α, and observe that this yields a fibered category $Tors(H)_{BK}$ on BK. Since α sends the unique object of $K[1]$ to the trivial K–torsor, the fiber $Tors(H)_e$ of π above this unique object e consists in the category of H–torsors P, together with trivialisations of their pushouts to $Tors(K)$. It is well known that this category is equivalent to the category $Tors(G)$. Furthermore, any automorphism u of e in $K[1]$ induces a functor u^* from the fiber category $Tors(H)_e$ to itself. This is simply, in another guise, the action determined by H of a section u of K on the category $Tors(G)$, so that we connect in this manner with the description of the Schreier gerbe given in (7.4.5).

7.6 We now abandon this discussion of Schreier gerbes, and return to the classification of stacks, as discussed in 7.1-7.3. Let us make the additional assumption that the stack $\mathcal{C}$ of T under consideration is endowed with a multiplication law which defines a gr–stack structure on it. The classification of $\mathscr{C}$ in terms of the sheaves π_0, π_1, and the Postnikov invariant $k_0 \in H(\pi_0, \{\pi_1\})$ may now be strenghtened, in order to take into account the group structure on $\mathscr{C}$. We set $\pi_1 =: \underline{Aut}(I)$. For any object $X \in \mathscr{C}_U$, left multiplication by X defines an isomorphism

$$\eta_X : \pi_1 |_U \longrightarrow \underline{Aut}(X) \tag{7.6.1}$$

of sheaves of groups on U, so that the gerbe $\mathscr{C} \,|\, \pi_0$ is now a π_1-gerbe on π_0. The functoriality of the group law on $\mathscr{C}$ implies that for any arrow $X \longrightarrow Y$ in $\mathscr{C}_U$, the corresponding diagram (2.9.1) commutes. The group law on π_1 is therefore abelian, and $\mathscr{C} \,|\, \pi_0$ is now an abelian π_1-gerbe, whose k–invariant k_0 lives in the traditional cohomology group $H^2(\pi_0, \pi_1)$ of the underlying sheaf of sets of π_0, with values in the sheaf of abelian groups π_1.

Much of the structure on $\mathscr{C}$ defined by the group law on the stack $\mathscr{C}$ is lost when one simply considers the class k_0 of $\mathscr{C}$ in $H^2(\pi_0, \pi_1)$. The problem of retrieving in cohomological terms the full *gr*–stack structure on $\mathscr{C}$ is analogous to the Schreier problem of classifying group extensions, and indeed these two problems have, as we have already mentioned, a common generalisation (*see* [Br 3] theorem 3.2.2). We can therefore deal with the classification of group laws on $\mathscr{C}$ (in other words with the description of the Postnikov decomposition of $\mathscr{C}$ as *gr*–stack) by the same techniques as were used in 7.4-7.5 in the construction of the Schreier gerbe of an extension of groups.

We have already observed that when $\mathscr{C}$ is a *gr*–stack, the sheaf π_1 abelian. The stack $\mathscr{A} = Tors(\pi_1)$ is therefore endowed with a monoidal structure defined by the usual contracted product of torsors, and it is in fact a Picard stack. The fiber of the functor $\mathscr{C} \longrightarrow \pi_0$ above the neutral element $e \in \pi_0$ is the pullback of the π_1 - gerbe $\mathscr{C} \,|\, \pi_0$ by e. By remark 7.3, it is a neutral gerbe on e, since it contains the unit object I of $\mathscr{C}$. This fiber of the projection from $\mathscr{C}$ to π_0 is therefore equivalent to the neutral π_1– gerbe $\mathscr{A} = Tors(\pi_1)$, and the inclusion $\mathscr{A} \longrightarrow \mathscr{C}$ defined by the unit element I is compatible with the group structures. To any gr–stack $\mathscr{C}$ is therefore associated an extension of gr–stacks

$$I \longrightarrow \mathscr{A} \longrightarrow \mathscr{C} \longrightarrow \pi_0 \longrightarrow I \tag{7.6.2}$$

as defined in [Br 3] 2.1. Forgetting part of the group law on $\mathscr{C}$, we observe that $\mathscr{C}$ is in any case an $\mathscr{A}$-bitorsor (as in *op. cit.*, definition 3.1.8) under left and right multiplication by $\mathscr{A}$ in $\mathscr{C}$. The higher level Morita theorem of *op.cit.*, proposition 3.1.12 asserts that this bitorsor defines a section of the 2-stack of self-equivalences of the pullback over π_0 of the 2-stack $Tors(\mathscr{A})$. The associativity data for the full group law on $\mathscr{C}$ then asserts that this section defines a right action of the sheaf of groups π_0 on the Picard 2-stack $Tors(\mathscr{A})$ and this is associative up to a coherent natural transformation. The Borel construction at the 2-stack level then yields by 2-descent, as in (7.4.5), a 2-stack $Tors(\mathscr{A}) \wedge^{\pi_0} E\pi_0$ above $B\pi_0$. This is in fact an $\mathscr{A}$–2–gerbe on $B\pi_0$, since its pullback above $E\pi_0$ is by construction the trivial $\mathscr{A}$–2–gerbe $Tors(\mathscr{A})$. We obtain in this manner a geometrical description of the cohomology class $k_0' \in H^1(B\pi_0, \mathscr{A} \longrightarrow \mathscr{E}q(\mathscr{A}))$ which, according to [Br 3] corollary 3.2.4, describes the extension (7.6.2).

The class k_0' can in fact be described in a somewhat more elementary manner. For any abelian group A of T, we saw in 1.7 that the gr–stack $\mathscr{E}q(\mathscr{A})$ of self-equivalences[7] of the gr–stack $\mathscr{A}$ is equivalent to the discrete gr–stack defined by the sheaf $Aut(A)$. The somewhat loosely described "crossed module of gr–stacks" $\mathscr{A} \longrightarrow Eq(\mathscr{A})$ in which this cohomology class takes its value may therefore be replaced by the crossed square

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[7] For the distinction between these self-equivalences, which preserve the group structure of $\mathscr{A} = Tors(A)$, and the sections of the gr–stack $Eq(\mathscr{A})$ of arbitrary self-equivalences of $\mathscr{A}$, *see* 1.7.

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & Aut(A) \end{array} \tag{7.6.3}$$

(with the obvious action of $Aut(A)$ on the abelian group $A = \pi_1(\mathscr{C})$). It follows that the cocycle quadruple which determines this cohomology class may be taken to be of the form $(\nu_{ijkl}, 1, 1, \lambda_{ij})$, with the terms λ_{ij} satisfying the strict cocycle condition (6.3.9). The terms in question are a priori also endowed with an upper index, but a degenerate form of (6.2.2) implies, as in (2.7.6), that they glue to a globally defined homomorphism

$$\lambda : \pi_0 \longrightarrow Aut(\pi_1) \tag{7.6.4}$$

which determines a π_0-module structure on the sheaf π_1. The class k_0' is therefore entirely described by the class of the cocycle ν_{ijkl} , viewed now as an element of cohomology set $H^3(B\pi_0, \pi_1)$ with coefficients in the abelian sheaf π_1 twisted by this action of π_0. In the case of the punctual topos, we recover here the classification of a *gr*-category by [Si] in terms of the group cohomology of the group π_0 with values in the π_0-module π_1. Returning to the general situation, it should be noted that the forgetful map from the *gr*-stack $\mathscr{C}$ to its underlying stack above π_0 now yields a map

$$\begin{array}{c} H^3(B\pi_0,\, \pi_1) \longrightarrow H^2(\pi_0, \pi_1) \\ k_0' \longmapsto k_0 \end{array} \tag{7.6.5}$$

which is a twisted version of the map induced in cohomology by the suspension map $S^1 \wedge K(\pi_0, 0) \longrightarrow K(\pi_1, 1)$. The class k_0' may thus be viewed as a delooping of the k-invariant $k_0 \in H^2(\pi_0, \pi_1)$, so that one might suggestively set $k_0' = Bk_0$.

Remarks 7.7: *i)* Any *gr*–stack $\mathscr{C}$ with homotopy invariants π_0 and π_1 may be replaced by an equivalent *gr*–stack $\mathscr{C}'$ in which the associativity isomorphisms are the identity, and such that the set of objects of $\mathscr{C}'$ form a group. To such a stack corresponds a crossed module $G_1 \longrightarrow G_0$, which lives in a 4-term non-abelian exact sequence of sheaves of groups

$$1 \longrightarrow \pi_1 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow \pi_0 \longrightarrow 1. \tag{7.7.1}$$

The description of $\mathscr{C}$ (and hence of the exact sequence (7.7.1)) in terms of its k–invariant k_0' given above is a very non-abelian version of the description by cocycles of an element in the group $Ext^2(\pi_0, \pi_1)$ of Yoneda 2-extensions of abelian sheaves obtained in [Br 1] (*see* also exact sequence (7.9.2) below).

ii) The present discussion is related to the discussion in proposition 6.4, where we came across a similar set of cocycles. The relation between these two situations is easiest to state in the case of a G–lien L defined, for a group G of T, by a class in the set $H^1(Out(G))$. Since the *gr*–stack $\mathscr{C} = Bitors(G)$ of G–bitorsors of T is associated to the prestack defined by the crossed module $G \longrightarrow Aut(G)$, its k-invariant is an element in the set $H^3(BOut(G), ZG)$, for the action of $Out(G)$ on the center ZG of G induced by the natural action of $Aut(G)$ on the characteristic subgroup ZG of G. The functor (6.1.2) from a lien L to the 2-gerbe $R(L)$ of its realizations may therefore be thought of as embodying, for any object X in T, the map $H^1(X, Out(G)) \longrightarrow H^3(X, ZG)$ defined, so long as the action of L on ZG is ignored, by cupping a class in $H^1(X, Out(G))$ under the pairing

$$H^1(X, Out(G)) \times H^3(BOut(G), ZG) \longrightarrow H^3(X, ZG)\,,$$

with the class in $H^3(BOut(G), ZG)$ of the k-invariant of the *gr*–stack $Bitors(G)$.

iii) Since the Postnikov decompostion of a *gr*–stack may be viewed as a Schreier type problem for the extension of *gr*–stacks (7.6.2), the action λ (7.6.4) of π_0 on π_1 is induced by that of π_0 on $\mathcal{A} = Tors(\pi_1)$ defined by inner conjugation in $\mathcal{C}$. This action is essentially the identity on the trivial π_1–torsor T, and is therefore characterized by its effect on the group π_1 of automorphisms of T. For any section $p \in \pi_0$, and any object P in $\mathcal{C}$ which locally determines p, the action λ is therefore locally described in the following manner. For any section $u \in \pi_1 = Aut(I)$, $\lambda(p)(u)$ is the automorphism of I determined by the diagram

(7.7.2)
$$\begin{array}{ccc} PIP^* & \xrightarrow{PuP^*} & PIP^* \\ \downarrow & & \downarrow \\ I & \xrightarrow[\lambda(p)(u)]{} & I \end{array}$$

for a chosen inverse object P^* of P. This definition of the action π_0 *on* π_1 by conjugation in $\mathcal{G}$ is consistent with the one given in [Si].

iv) Just as in the case of ordinary group extensions in a topos T which we examined in (7.5.1), it can be shown that the $\mathcal{A}$–2-gerbe $Tors(\mathcal{A}) \wedge^{\pi_0} E\pi_0$ which represents the extension of *gr*–stacks (7.6.1) is equivalent to the $\mathcal{A}$–2-gerbe on $Tors(\pi_0)$ defined by the natural projection

(7.7.3)
$$Tors(\mathcal{C}) \longrightarrow Tors(\pi_0)$$

7.8 There exist two intermediate problems between the (non-abelian) classification of *gr*–stacks $\mathcal{C}$ just discussed and the fully abelian problem of describing by cocycles, for any pair of of abelian group π_0 and π_1 of T, an element in the group $Ext^2(\pi_0, \pi_1)$ of degree 2 extensions of sheaves of abelian groups. These intermediate problems were discussed at the cocycle level, in the case of the punctual topos, in [Br 3] 2.4.6-2.4.8 and we will limit ourselves in this section to an examination of their geometric content. We will be returning to this topic at the beginning of § 8.3 in preparation for the classification of group laws on 2-stacks.

The first of these intermediate problems is the classification of braided stacks $\mathscr{C}$ with given associated sheaves $\pi_0 = \pi_0(\mathscr{C})$ and $\pi_1 = \pi_1(\mathscr{C})$. It follows from (1.8.1) that the associated group π_0 is then necessarily abelian, and that conjugation in $\mathscr{C}$ is essentially trivial, so that the action (7.6.4) of π_0 on π_1 is trivial, and the class k'_0 introduced in 7.6 now lives in the ordinary cohomology group $H^3(B\pi_0, \pi_1)$, with the (untwisted) abelian group π_1 as its group of coefficients. As stated in [Br 2], the full structure of the braided stack $\mathscr{C}$ is now encoded in a k-invariant k''_0 which is a delooping in the group $H^4(K(\pi_0, 2), \pi_1)$ of the invariant $k'_0 \in H^3(B\pi_0, \pi_1)$. We will not work this out in detail, and refer to [Br 1] for a description, for any abelian group π_0 of T, of the associated Eilenberg-MacLane simplicial abelian group $K(\pi_0, 2)$ and of its cohomology. One method for viewing the class k''_0 is the following. When π_0 is an abelian group, the stack $\Pi_0 = Tors(\pi_0)$ associated to the groupoid determined by π_0 is itself a gr–stack (and indeed it is even Picard stack), so that it would make sense to consider as a geometric model for $K(\pi_0, 2)$ the simplicial stack $B\Pi_0$ or its associated gr–2–stack $Tors(\Pi_0)$. While we have already defined the 2-stack $Tors(\Pi_0)$, it is more delicate to define such a simplicial stack $B\Pi_0$ along the same lines as in 7.4, since the group law on Π_0 is no longer strictly associative. One option here is to replace Π_0 by an equivalent gr–stack Π'_0 in which the group law is strictly associative, and to consider instead the stack $B\Pi'_0$ but this is somewhat unsatisfactory since the construction of such a Π'_0 is not very natural from a geometric point of view. Another possible approach would rest on the observation that Stasheff's definition of the classifying space of an A_∞-space, or one of its variants defined in terms of A_∞-operads, is sufficiently functorial to carry over to the sheaf context. We will not explore either of these options, since we have an even simpler model for

$K(\pi_0, 2)$ at our disposal. This is the (nerve of) the 2-prestack $\pi_0[2]$ (as described in [Mo-Sv] §2). This prestack has a unique object and a unique 1-arrow in each fibre, and the 2-arrows are the sections of abelian group π_0, the horizontal and vertical (compatible) composition laws for the 2-arrows in $\pi_0[2]$ being both defined by the group law of π_0. This 2-prestack may also be viewed as the classifying 2-prestack $B(\pi_0[1])$ of the *gr*–prestack $\pi_0[1]$, since its sheaf of objects is trivial, and it has the *gr*–prestack $\pi_0[1]$ as prestack of arrows. Since the associated stack functor $i\colon \pi_0[1] \longrightarrow \Pi_0$ (1.1.9) respects the group structure, it induces a morphism of 2-prestacks

$$Bi\colon B(\pi_0[1]) \longrightarrow B\Pi_0 ,$$

so that there exist natural functors

$$\pi_0[2] \longrightarrow B\Pi_0 \longrightarrow Tors(\Pi_0) \tag{7.8.1}$$

between these three progressively more sheafified representatives of the 2-stack determined by $K(\pi_0, 2)$.

With this in mind, we now return to the question of describing geometrically the k-invariant k_0'' of a braided stack $\mathscr{C}$. We have seen in 2.13 that whenever $\mathscr{C}$ is braided, the corresponding 2-stack $\mathbb{C} = Tors(\mathscr{C})$ is itself endowed with a multiplication

$$Tors(\mathscr{C}) \times Tors(\mathscr{C}) \longrightarrow Tors(\mathscr{C} \times \mathscr{C}) \longrightarrow Tors(\mathscr{C}). \tag{7.8.2}$$

This is associative up to a coherent homotopy (in the sense of definition 8.4 below), and it therefore defines a 2-*gr*–stack structure on $\mathbb{C}$. Since the projection (7.7.3) is compatible with this group law, it defines a "central" extension of 2–*gr*-stacks

$$1 \longrightarrow \mathbb{A} \longrightarrow \mathbb{C} \longrightarrow \Pi_0 \longrightarrow 1. \tag{7.8.3}$$

where $\mathbb{A} = Tors(\mathscr{A})$. A higher Schreier theory for such central extension (7.8.3) would then yield the sought-after description of the class k_0'' in the group $H^2(B\Pi_0 \mathbb{A}) = H^2(B\Pi_0, \pi_1[2]) = H^4(K(\pi_0, 2), \pi_1)$. The simplest recipe for defining this class geometrically is to follow the approach of 7.4, 7.6, and to view the underlying $\mathbb{A}$-bitorsor of $\mathbb{C}$ as determining an action of the *gr*–stack Π_0 on the 3-stack *Tors* ($\mathbb{A}$). Since this action is locally defined by conjugation in $\mathbb{C}$, its pullback to $B(\pi_0[1])$ has the following concrete description. First of all, the unique object e in each fiber of $\pi_0[1]$ acts trivially on *Tors* ($\mathbb{A}$). It therefore suffices to determine the action on *Tors* ($\mathbb{A}$) of a section u of π_0 , viewed as an arrow in $\pi_0[1]$. Let $U: I \longrightarrow I$ be an arrow in $\mathbb{C}$ which locally lifts the automorphism of e determined by such a section u. Conjugation by U in $\mathbb{C}$ defines a natural transformation

$$\mathbb{A} \underset{1_{\mathbb{A}}}{\overset{1_{\mathbb{A}}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\scriptstyle U_* \Downarrow}}}} \mathbb{A} \tag{7.8.4}$$

from the identity functor $1_{\mathbb{A}} : \mathbb{A} \dashrightarrow \mathbb{A}$ to itself. The induced transformations $Tors\,(U_*)$

$$Tors\,(\mathbb{A}) \underset{1_{Tors(\mathbb{A})}}{\overset{1_{Tors(\mathbb{A})}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\scriptstyle Tors(U_*)\Downarrow}}}} Tors\,(\mathbb{A})$$

glue together, in a manner reminiscent of the discussion in (5.8.7)-(5.8.9), to the sought-after action of the arrows in $\pi_0[1]$ on the 3-stack *Tors* ($\mathbb{A}$). A Borel construction at the 3-stack level for this action may now in principle be carried out by a descent argument similar to that which was used in the construction of the stack (7.4.5). Note, however, that we in doing so, one is two stages further than in (7.4.5). It is no longer 2-descent data for 1-stacks which is required for this construction, as in (7.4.5), or even 3-descent data for 2-stacks, as discussed in 1.12 and in 5.5 - 5.7 above. We must here descend a 3-stack endowed with 4-descent data provided by the action of $\pi_0[1]$. We will not discuss this in detail here, and simply remark that the precise form which such 4-descent data takes is spellt out in diagram O_5 of [St]. The effectivity of such descent is then provided by the assertion that the fibered 4-category of 3-stacks is endowed with a 4-stack structure. We will not attempt to justify this assertion here, which is of a formal nature, and is analogous to the corresponding statements for 1- and 2-stacks discussed earlier (example 1.11 and remark 1.12).

With this proviso, the previous construction yields a twisted form

$$E(\pi_0[1]) \wedge^{\pi_0[1]} Tors(\mathbb{A})$$

of the 3-stack $Tors(\mathbb{A})$, which lives above $B\pi_0[1] = K(\pi_0, 2)$. This is an $\mathbb{A}$–3–gerbe, since it is locally equivalent to $Tors(\mathbb{A})$, and in fact an abelian $\mathbb{A}$–3–gerbe in as sense analogous to that of definition 4.13. Its class in $H^2(K(\pi_0,2), \mathbb{A})$ is that of k_0''.

7.9 We may now turn to the problem of classifying stacks $\mathcal{C}$ which are Picard, instead of simply braided. We have seen that in that case the induced 2-category $\mathbb{C}$ is braided in the sense of [K-V]. The extension (7.8.3) is no longer simply central, but even compatible with the commutativity laws on $\mathbb{A}$ and $\mathbb{C}$. The morphism of 3-stacks $Tors(\mathbb{C}) \longrightarrow K(\pi_0, 2)$ which k_0'' determines now lives in a (commutative) extension of 3-stacks

$$1 \longrightarrow Tors(\mathbb{A}) \longrightarrow Tors(\mathbb{C}) \longrightarrow K(\pi_0, 2) \longrightarrow 1 \tag{7.9.1}$$

whose class in $H^2(BK(\pi_0, 2),\ Tors(\mathbb{A})) = H^5(K(\pi_0, 3),\ \pi_1)$ is a further delooping k_0''' of k_0''.This new cohomology class lives in the stable range for the cohomology of the Eilenberg-MacLane simplicial sheaf $K(\pi_0, n)$, since it is in a cohomology group of the form $H^{n+i}(K(\pi_0, n),\ \pi_1)$ with $i<n$. It follows that $\mathbb{C}$ is not simply braided, but is in fact homotopy commutative in the strongest possible sense (and so deserves to be called a Picard 2-stack), so that no further conditions can be imposed on the category $\mathcal{C}$ by iterating this delooping process. We refer to 8.5 below for a formal definition of such Picard 2-stacks. In topological terms, the nerve of the Picard category $\mathcal{C}$ is now a (two-stage) infinite loop object in T and cohomology with values in $\mathcal{C}$ is a $\mathcal{C}$-valued extraordinary sheaf cohomology theory, with values in the

spectrum which the nerve of $\mathscr{C}$ determines (for a discussion of such theories, *see* [Th]).

There does however exist one last condition, of a somewhat different nature, which can be imposed on the stack $\mathscr{C}$. This is the so-called strict Picard condition. It follows from an explicit description of the homology in low degree of the simplicial object $K(\pi_0, 3)$, as given in [E-M], that the latter condition translates into the requirement that the invariant k_0''' just defined live in the subgroup $Ext^2(\pi_0, \pi_1)$ of the cohomology group $H^5(K(\pi_0, 3), \pi_1)$ determined by the universal coefficient short exact sequence

(7.9.2)

$$0 \longrightarrow Ext^2(\pi_0, \pi_1) \longrightarrow H^5(K(\pi_0, 3), \pi_1) \longrightarrow Hom(\pi_0/2\pi_0, \pi_1) \longrightarrow Ext^3(\pi_0, \pi_1)$$

for the simplicial sheaf $K(\pi_0, 3)$. If we replace $\mathscr{C}$ by an equivalent Picard stack $\mathscr{C}'$ for which the group law is strict on objects, in other words by one associated to a simplicial abelian group, the corresponding Moore complex $C_1 \longrightarrow C_0$ of the nerve of $\mathscr{C}'$ now lives in a short exact sequence of abelian sheaves

(7.9.3) $$0 \longrightarrow \pi_1 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \pi_0 \longrightarrow 0$$

of T. One retrieves in this manner from the previous discussion the interpretation à la Yoneda of the elements of the group $Ext^2(\pi_0, \pi_1)$. The Ext^2 and Ext^3 groups appearing in the exact sequence (7.9.2) are always trivial when T is the punctual topos since every abelian group has a length one free resolution. It follows that the middle arrow in exact sequence (7.9.2) is an isomorphism in ordinary topology, so that any strict Picard category is equivalent, under the decomposition described in (7.3.1) for the underlying groupoid, to the trivial Picard category $\pi_1[1] \times \pi_0$ which has π_0 as group of objects, π_1 as group of automorphisms of each object, and no other arrows. This is commonly stated as the assertion that a simplicial abelian group (which in our case is merely a 2-stage Postnikov system) is always a product of Eilenberg-Mac Lane spaces. This is no longer the case in a general topos T, since a strict Picard stack then need no longer be equivalent to the trivial Picard stack. We refer to [Br 1] and to the author's subsequent papers on this topic for a fuller discussion of non-vanishing higher *Ext* groups.

8. The classification of 2-stacks, and beyond

Let $\mathscr{C}$ be a 2-stack in 2-groupoids in a topos T, in other words a 2-stack which statisfies conditions $(G3)$ and $(G4)$ of definition 3.1. As in 7.1, we may consider the sheaf π_0 associated to the presheaf of its connected components and the morphism of 2-stacks

$$\mathscr{C} \longrightarrow \pi_0$$

with values in the discrete 2-stack (*i.e*, with no non-trivial 1- or 2-arrows) defined by π_0. We may therefore view $\mathscr{C}$ as a 2-stack in the localized topos $T_{|\pi_0}$, which we will denote by $\mathscr{C} \,|\, \pi_0$. Suppose that π_0 is representable by an object of the site defining T. The tautological section of π_0 is therefore liftable, over some refinement U of π_0, to an object $x \in \mathscr{C}_U$. The gr–stack $\underline{Aut}(x)$ of self-arrows of x is then defined over U. The following proposition is proved in exactly the same manner as lemma 7.2.

Proposition 8.1: *Let $\mathscr{C}$ be a 2–stack in 2–groupoids in T, and let π_0 be the sheaf associated to the presheaf of its connected components. The localized stack $\mathscr{C} \,|\, \pi_0$ is a 2–gerbe on π_0.*

In the notation which has just been introduced, the choice of such an object $x \in \mathscr{C}_U$, and of an equivalence of gr–stacks $\eta\colon \underline{Aut}\,(x) \longrightarrow \mathscr{G}$ provides a labeling of this 2-gerbe by the locally defined gr–stack $\mathscr{G}$. Since we know how to describe such a gr–stack in cohomological terms, we can now give a fully cohomological description of the 2-gerbe $\mathscr{C} \mid \pi_0$, and hence of the 2-stack $\mathscr{C}$ itself. This is carried out by adjoining to the sheaf of sets π_0 the homotopy sheaves of the gr–stack $\underline{Aut}(x)$ defined by

$$\pi_1(\mathscr{C},x) =: \pi_0(Aut(x)) \tag{8.1.1}$$
$$\pi_2(\mathscr{C},x) =: \pi_1(Aut(x)).$$

We will also denote these sheaves by the shorthand expressions $\pi_i(\mathscr{C})$ or even π_i for $i=1,2$. They are defined above the open set U, and the abelian sheaf π_2 is endowed by (7.6.4) with a π_1–module structure. With this new notation, the invariant k_0' describing as in 7.6 the gr–stack $\mathscr{G}$ now lives in the twisted cohomology class $H^3(B\pi_1, \pi_2)$. It will be denoted by $k_1(\mathscr{C},x)$, (or simply by k_1) and called the first Postnikov invariant of the 2-stack $\mathscr{C}$. The gr–stack $\mathscr{G}$ on U is entirely determined by k_1 up to equivalence, and the remaining element of structure of $\mathscr{C}$ is the class of $\mathscr{C} \mid \pi_0$ in the cohomology set denoted by $H(\pi_0, \{\mathscr{G}\})$ in proposition 4.6. This $\mathscr{G}$-valued cohomology set may roughly be thought of as a mixture of the non-abelian 2- and 3-cohomology of the object π_0 , with values in twisted versions of the sheaves π_1 and π_2. We will denote the class of $\mathscr{C} \mid \pi_0$ in the set in question by $k_0(\mathscr{C})$ (or even by k_0 when there is no risk of confusion with the corresponding element associated in 7.3 to a 1-stack) and call it the zeroth k–invariant of the 2-stack $\mathscr{C}$. It follows from the previous discussion that the three homotopy sheaves $\pi_i(\mathscr{C})$, together with the two associated Postnikov invariants $k_1' \in H^2(B\pi_1, \pi_2)$ and $k_0 \in H(\pi_0, \{\mathscr{G}\})$ constitute a full set of invariants for the 2-stack $\mathscr{C}$.

Remark 8.2: What we have obtained here is an upside-down Postnikov decomposition for $\mathscr{C}$, in which the simplest invariant involves the highest degree homotopy groups. In order to obtain instead an ordinary Postnikov decomposition, it would have been necessary to replace the projection of $\mathscr{C}$ on the discrete 2-stack π_0 by the projection of $\mathscr{C}$ on the 1-stack $\mathscr{C}\langle 1\rangle$ with same objects as $\mathscr{C}$, but whose sheaf of arrows from x to y is the sheaf $\pi_0(\mathscr{A}r(x,y))$. This is by definition the sheaf associated to the presheaf of connected components of the stack of arrows in $\mathscr{C}$ from x to y. The two homotopy groups of $\mathscr{C}\langle 1\rangle$ are $\pi_0(\mathscr{C})$ and the locally defined group $\pi_1(\mathscr{C}, x) = \pi_0(\mathscr{A}r\,(x,x))$, so that there is a Postnikov invariant $k_0(\mathscr{C}\langle 1\rangle) \in H^2(\pi_0, \{\pi_1(\mathscr{C},x)\})$. In order to define the second invariant for $\mathscr{C}$, it would then have been necessary to come to terms with the cohomological invariant determined by the projection $\mathscr{C} \longrightarrow \mathscr{C}\langle 1\rangle$, as an element in an appropriately twisted π_2-valued cohomology class of the stack $\mathscr{C}\langle 1\rangle$. Since $\mathscr{C}\langle 1\rangle$ is a stack, not a sheaf, one encounters here the sort of difficulties which were already apparent in 7.5, and it is for this reason that the upside-down Postnikov approach has been preferred. The invariant $k_0(\mathscr{C}\langle 1\rangle)$ of our first approach may however be retrieved from the invariant $k_0(\mathscr{C})$ by introducing the morphism of gr-stacks above the open set U

$$\mathscr{G} \longrightarrow \pi_1(\mathscr{C}) \tag{8.2.1}$$

defined by (7.1.1). The map $H(\pi_0, \mathscr{G}) \longrightarrow H^2(\pi_0, \pi_1)$ induced by this arrow sends $k_0(\mathscr{C})$ to $k_0(\mathscr{C}\langle 1\rangle)$.

8.3 We now briefly discuss the classification of group laws on 2-stacks, in a manner analogous to the classification of gr–stacks in 7.7-7.9. In order not to be swamped in a morass of definitions, it will be useful to begin by reviewing in parallel terms both this classification of gr–stacks and the even simpler classification of extensions of groups in a topos T. The latter may be described, for a pair of groups A and B in T, by the maps

$$
\begin{array}{ccc}
H^3(K(A,2),B) \longrightarrow H^2(BA,B) & \longrightarrow & H^1(A,B) \\
\downarrow & & \downarrow \\
H^1(BA, B \longrightarrow Aut(B)) & \longrightarrow & H^0(A,B \longrightarrow Aut(B))
\end{array}
\tag{8.3.1}
$$

The horizontal maps are those induced in cohomology, for n=0,1, by the suspension maps $S^1 \wedge K(A,n) \longrightarrow K(A,n+1)$. The two left-hand terms of the top line only makes sense when B is abelian, and they are abelian groups, whereas the bottom line and the right-hand term of the top line make sense for any group B. On the other hand, both terms in the right-hand column are defined for an arbitrary sheaf of sets A, but when we move to the left, it successively becomes necessary to assume that the sheaf of sets A is endowed with the structure of a group (*resp.*, of an abelian group). The left-hand term in the top line lies in the stable range, so that it is in fact isomophic to any of the group $H^{n+1}(K(A,n),B)$ for any $n \geq 2$. It is therefore unecessary to prolong the upper line to the left by additional suspension maps. The universal coefficient theorem (together with the vanishing of the integral homology groups $H_{n+1}(K(A,n), \mathbb{Z})$ for $n>1$) identify this stable term with the group $Ext^1(A,B)$ of abelian extensions of A by B. The first horizontal suspension map is the forgetful map from this group of abelian extension to the group central extension of A by B. The second horizontal suspension map sends the class of such an extension to the class of B–torsor on A which this extension determines. Passing to the lower line, we see that the left-hand term in it is a pointed set, whereas the right-hand one is a group. This line is related to the upper one by the map which sends the set of classes of central extensions of A by B to the set of classes of arbitrary extensions (*resp.* a torsor on A under an abelian group B to the associated B–bitorsor). Finally, the horizontal "suspension" map at the lower level is a non-abelian generalization of the one immediately above it: it sends an arbitrary extension $B \longrightarrow E \longrightarrow A$ of A by B to the underlying B-bitorsor on A.

If we now pass to the classification of stacks with given homotopy sheaves $\pi_0 = A$ and $\pi_1 = B$, the cohomology sets in which their various k–invariants live may be assembled in a similar manner:

(8.3.2)

$$\begin{array}{ccccccc} H^5(K(A,3),B) & \longrightarrow & H^4(K(A,2),B) & \longrightarrow & H^3(BA,B) & \longrightarrow & H^2(A,B) \\ & & & & & & \downarrow \\ & & & & & & H^1(A, B \longrightarrow Aut(B)) \end{array}$$

Once more, the top line is only defined for B abelian. Its left-hand term is in the stable range $H^{n+2}(K(A,n), B)$ for $n \geq 3$ and we have seen in 7.9 that it is where the Postknikov invariants k_0''' for Picard stacks with homotopy invariants A and B lives. The universal coefficient theorem now merely yields an inclusion, described by the left-hand arrow of exact sequence (7.9.2), of the group $Ext^2(A,B)$ of strict Picard stacks into this cohomology group. The horizontal suspension maps of diagram (8.3.2) are now the successive forgetful maps , which send the invariant k_0''' of a given Picard stack to the invariants k_0'', k_0' respectively associated in (7.6)-(7.8) to the underlying braided (*resp.*, *gr*–) stack, and finally to the class k_0 of the underlying abelian B-gerbe on A. The vertical map is the inclusion of the class of such an abelian B–gerbe into the set of classes of arbitrary B–gerbes on X, a set which is defined without the assumption that B is abelian. If we replace the group B by a crossed module $(B_1 \longrightarrow B_0)$, or by its associated *gr*–stack $\mathcal{B}$, diagram (8.3.2) will be replaced by the diagram

(8.3.3)

$$\begin{array}{ccccccc} H^4(K(A,3),\mathcal{B}) & \longrightarrow H^3(K(A,2),\mathcal{B}) & \longrightarrow & H^2(BA,\mathcal{B}) & \longrightarrow & H^1(A,\mathcal{B}) \\ & & & \downarrow & & \downarrow \\ & & & H^1(BA,\mathcal{B}\longrightarrow\mathcal{E}q(\mathcal{B})) & \longrightarrow & H^0(A,\mathcal{B}\longrightarrow\mathcal{E}q(\mathcal{B})) \end{array}$$

which is more reminiscent of (8.3.1), to which it in fact reduces when $\mathcal{B}$ is the discrete stack associated to the sheaf B. Diagram (8.3.2) may also be deduced from (8.3.3), by setting instead $\mathcal{B} = Tors(B) = (B[1])^{\sim}$. Once more, it is necessary to impose successive commutativity conditions on $\mathcal{B}$ and on A as one passes from each column to the one to its left. The conditions which A must satisfy are the same ones as in (8.3.1), so that we needn't restate them. We know that no additional condition on the *gr*–stack $\mathcal{B}$ is required in order for the set $H^1(A,\mathcal{B})$ to be defined, but as we move from the right to the left along the suspension maps, it will in the first instance be necessary for $\mathcal{B}$ to be braided, and then for it to be Picard in the two subsequent columns. The top left hand term once more lives in the stable range, as can be for example be seen by a dévissage to the corresponding terms in the diagram (8.3.1) and (8.3.2). It was shown in [Br 3] (2.4.7)-(2.4.8) in the case of the discrete topos (but the discussion carries over to the general case) that this stable term classified commutative (*i.e.*, Picard) extensions $\mathcal{K}$ of the discrete stack A by a Picard stack $\mathcal{B}$, and that the horizontal maps are the successive forgetful maps from Picard to braided, to arbitrary central extensions, and finally to the underlying $\mathcal{B}$-gerbe of the extension. Once more, the lower line is defined for an arbitrary *gr*–stack $\mathcal{B}$, and its terms respectively classify arbitrary extensions of A by $\mathcal{B}$ and the underlying $\mathcal{B}$–bitorsors. The left hand vertical inclusion sends a central extension to an arbitrary extension of A by $\mathcal{B}$, and the right-hand one sends a torsor under a braided stack $\mathcal{B}$ to the associated $\mathcal{B}$–bitorsor.

After this preparation, we can now study the corresponding group laws on 2-stacks, by examining the corresponding diagram of cohomology classes

(8.3.4)

$$H^6(K(A,4),\mathcal{B})\to H^5(K(A,3),\mathcal{B}))\to H^4(K(A,2),\mathcal{B})\to H^3(BA,\mathcal{B})\to H^2(A,\mathcal{B})$$

$$\downarrow$$

$$H^1(A,\mathcal{B}\longrightarrow \mathcal{E}q(\mathcal{B}))$$

The top right-hand set is defined for any braided gr–stack $\mathcal{B}$, and we have seen in 4.12 that it classifies the set of abelian $\mathcal{B}$–2-gerbes on A. The vertical map is the inclusion of this set in the set of all $\mathcal{B}$–2-gerbes on A. An interpretation along the lines of proposition 2.14 of such an abelian $\mathcal{B}$-2-gerbe $\mathcal{C}$ as a torsor under the 2-gr–stack $\mathbb{B} = Tors(\mathcal{B})$ shows that this is a group (and in fact an abelian group) whenever $\mathcal{B}$ is Picard. The other terms only make sense under this hypothesis on $\mathcal{B}$. They correspond to additional structures on the 2-stack $\mathcal{C}$. The most basic one is that of a gr–structure on a 2-stack. Let us recall the following definition:

Definition 8.4: *Let $\mathcal{C}$ be a 2–stack in 2–groupoids in a topos T. A gr–structure on $\mathcal{C}$ consists in a composition law $m\colon \mathcal{C}\times\mathcal{C}\longrightarrow\mathcal{C}$ on $\mathcal{C}$, which is coherently associative, and of unit objects which are compatible with the composition law . It is required that this law be group–like, in other words that the functors of left or right multiplication by any object X in $\mathcal{C}$ be equivalences.*

We refer to [Br 3] (4.1.8) for the definition of the coherence condition on the group law of $\mathcal{C}$ (we also recommend the much more explicit discussion in [K-V] §4, where, in particular, the compatibility beween the associativity and the unity conditions in a 2-category is worked out very carefully). To the unit element I in such a gr–2–stack is associated the gr–stack $\mathcal{B} = \underline{Aut}(I)$, which is defined above the final object of T. Functoriality of the group law on $\mathcal{C}$ implies that multiplication by varying objects X defines on $\mathcal{C}\,|\,\pi_0$ an abelian $\mathcal{B}$–2–gerbe structure on π_0

$$\eta_X\colon \mathcal{B}\longrightarrow \underline{Aut}(X)\,, \qquad (8.4.1)$$

as defined in 4.13. It is shown there that the gr–stack $\mathscr{B}$ is then necessarily braided. Furthermore, compatibility of the left and the right multiplication by X now determines an additive functor

$$\lambda: A \longrightarrow \mathscr{E}q(\mathscr{B}), \tag{8.4.2}$$

from the discrete gr–stack defined by the sheaf $A = \pi_0(\mathscr{C})$ to the gr–stack of equivalences of $\mathscr{B}$ compatible with the braided structure. This is the analog of (7.6.4) in the 2-stack situation, and determines what we may refer to as an A–module structure on the braided stack $\mathscr{B}$. To the braided stack $\mathscr{B}$ is associated the 2-gr–stack $\mathbb{B} = Tors(\mathscr{B})$, the group stucture on $\mathbb{B}$ being defined by the rule (7.8.2). The unit element in $\mathscr{C}$ determines an additive 2-functor from $\mathbb{B}$ to $\mathscr{C}$, which, together with the canonical projection from $\mathscr{C}$ to A, determines an extension of 2-gr–stacks

$$I \longrightarrow \mathbb{B} \longrightarrow \mathscr{C} \longrightarrow A \longrightarrow I \tag{8.4.3}$$

analogous to (7.6.2). We refer to *op.cit.*, definition 4.1.10 for the definition of the objects in the 3–stack $Tors(\mathscr{C})$ of right torsors under a 2-gr–stack $\mathscr{C}$ (noting however that the obvious condition that the induced map $\mathscr{C} \times \mathscr{P} \longrightarrow \mathscr{C} \times \mathscr{C}$ be an equivalence was inadvertently omitted there). To the exact sequence (8.4.3) corresponds an action of the group A on the 3-stack $Tors(\mathbb{B})$. If we invoke, as we have already done in 7.8, the descent properties for 3-stacks, we see that this action of A yields, by a Borel construction analogous to that discussed in 7.6, a cohomology class in $H^3(BA, \mathscr{B})$, where $\mathscr{B}$ is endowed with the A-module structure defined by (8.4.2). When the action of A on $\mathscr{B}$ is equivalent to the trivial one, one may say that the extension (8.4.3) is central. This is the situation discussed in [Del 5] 5.5, under the additional assumption that $\mathscr{B}$ is the gr–stack $Tors(B)$ determined by an abelian sheaf B. Another approach to the description of this class, in the general case, would be the following. One could consider, as in (7.7.3), the projection

$$Tors(\mathscr{C}) \longrightarrow Tors(A)$$

induced by the projection (8.4.3) as a $\mathcal{B}$–3-gerbe on $Tors\,(A)$, or on its pullback BA by the inclusion i (1.1.10).

There is one last approach to the problem of obtaining the invariant describing a 2–gr–stack. It is more concrete, though somewhat less geometrical than either of the ones outlined above. This consists in mimicking the description of an extension of gr–stacks given in *op.cit.*, 3.2.2 (we have seen in (7.6.2) that a gr–stack determines such an extension). It is a somewhat formal argument, once the definition of a 2-gr–stack is given, so that we will no work it out in detail. One begins by observing that, as we have seen above, such a stack $\mathcal{C}$ may automatically be viewed as an abelian $\mathcal{B}$–2-gerbe $\mathcal{H}$ on A. One then systematically unwinds in terms of this 2–gerbe $\mathcal{H}$ the additional structure provided by the group law of $\mathcal{C}$. First of all, the composition law on $\mathcal{C}$ defines as morphism of $\mathcal{B}$–2-gerbes

$$\mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$$

above the multiplication in A. The associativity isomorphism then provides a 2-arrow between the two induced morphisms of 2-$\mathcal{B}$-gerbes $\mathcal{H} \times \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ and the pentagon 2-arrow defines a 3-arrow between the corresponding induced 2-arrows above A^4. Finally, Stasheff's higher pentagon coherence condition provides a compatibility condition between pullbacks to A^5 of these 3-arrows. Taking into account the definition as a simplicial object of the classifying space BA of the group A, it then follows that this data defines the required cohomology class in the set $H^3(BA, \mathcal{B})$.

The advantage of this final approach to the classification problem for 2-`gr–stacks is that it will easily extend to the description of the various commutativity laws on such a 2-gr–stack $\mathcal{C}$, without forcing upon us a discussion of n–stacks for unreasonably large n. Observe in this context that the description in 7.8 of increasingly commutative group laws on a gr–stack $\mathcal{C}$ already implicitly involved higher gerbes: the extension (7.9.1) by which we described a braided stack structure on $\mathscr{C}$ is effect determined a gr–3–gerbe on $K(\pi_0, 2)$, so that its Picard delooping k_0''' therefore corresponds to a 4-gerbe on $K(\pi_0, 3)$. Our point of view here will be that a possible commutativity law will only be considered to be significant when the associated invariant lies in one of the cohomology groups appearing in the upper line of diagram (8.3.4). In particularly, we will not consider here additional structures such as ribbon structures.

The first cohomology group to be considered here is the group $H^4(K(A, 2), \mathscr{B})$, and for this group the situation is quite satisfactory, since a familiar commutativity law occurs. The author has verified explicitly in the punctual case, but the formal argument just outlined will carry it through to the general situation, that in order for the class of $\mathscr{C}$ in $H^3(BA, \mathscr{B})$ to lift to the group $H^4(K(A,2), \mathscr{B})$, the group law on the 2-stack $\mathscr{C}$ must be endowed with a functorial commutativity law natural transformation

$$R_{X,Y}: XY \longrightarrow YX \tag{8.4.4}$$

which satisfy the 2-braiding axioms of Kapranov and Voevodsky [K-V] §6, together with the additional condition that, in their terminology, the pair of 2-arrows defining the induced Z–systems of *op.cit.*, (6.10) coincide. We will say that such a 2-stack is Z-braided, to distinguish it from the 2-braided stacks in the sense of *op.cit.*

Passing to the next cohomology group, one now introduces the following additional commutativity condition on a Z–braided 2-stack $\mathscr{C}$. For each pair of objects X and Y in a fibre category, we now give ourselves, functorially in the objects, a 2-arrow

$$XY \;\underset{R_{YX}\circ R_{XY}}{\overset{1_{XY}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow S_{XY}}}}}\; XY \tag{8.4.5}$$

Such a 2-arrow S_{XY} implies that R_{YX} is a weak inverse for the braiding arrow R_{XY}. Consider the two possible hexagonal compatibility 2-arrows for the group law on $\mathscr{C}$ which occur in the definition of a braided 2-category (these are respectively denoted by $R_{X_1,X_2\,|\,Y}$ and $R_{X\,|\,Y_1,Y_2}$ in *op.cit* §6.1, where they appear under the guise of triangles since the associativity is taken to be strict). These two hexagonal 2-arrows are comparable whenever the commutativity law R_{XY} is invertible. An additional condition on $\mathscr{C}$ is the requirement that these two hexagons be compatible with each other under this comparison. When it is satisfied, the explicit definition given in [E-M] of chains on the Eilenberg-MacLane space $K(A,3)$ implies that the associated k–invariant of $\mathscr{C}$ deloops from the group $H^4(K(A,2), \mathscr{B})$ to the group $H^5(K(A, 3), \mathscr{B})$

We propose that a Z-braided 2-stack in 2-groupoids (or a 2-groupoid) which is endowed with functorial 2-arrows (8.4.5) satisfying this additional compatibility between the two sets of hexagons diagrams be called a *strongly braided* 2-category. In order to reach the stable range, there is one last condition to be added to the strong 2-braiding condition. This is the condition that for any two objects X and Y in $\mathscr{C}$, the pair of 2-arrows from R_{XY} to $R_{XY} \circ R_{YX} \circ R_{XY}$ defined by each of the two following diagrams coincide:

$$XY \underset{R_{YX}\,\circ\, R_{XY}}{\overset{1_{XY}}{\underset{S_{XY}\Downarrow}{\rightrightarrows}}} XY \xrightarrow{R_{XY}} YX$$

$$XY \xrightarrow{R_{XY}} YX \underset{R_{XY}\,\circ\, R_{YX}}{\overset{1_{YX}}{\underset{S_{YX}\Downarrow}{\rightrightarrows}}} YX \tag{8.4.6}$$

When this is the case, we will say that the 2-stack $\mathcal{C}$ is Picard, since the associated invariant then deloops to an element of the stable cohomology group $H^6(K(A, 3), \mathcal{B})$. The 2-arrow $S_{X,Y}$, together with its coherence condition (8.4.6), provides the natural generalization to 2-categories of the Picard condition for $R_{XY} \circ R_{YX} = 1_{XY}$ which Picard 1-categories satisfies. Similarly, the additional strict Picard category condition $R_{XX} = 1_{XX}$ must be replaced in the present 2-categorical context by a 2-arrow

$$XX \; \underset{R_{XX}}{\overset{1_{XX}}{\underset{\longrightarrow}{\overset{\longrightarrow}{s_X \Downarrow}}}} \; XX \tag{8.4.7}$$

which is functorial in X and satisfies two compatibility conditions. The first one says that the 2-arrow $S_X * S_X$

$$XX \; \underset{R_{XX}}{\overset{1_{XX}}{\underset{\longrightarrow}{\overset{\longrightarrow}{s_X \Downarrow}}}} \; XX \; \underset{R_{XX}}{\overset{1_{XX}}{\underset{\longrightarrow}{\overset{\longrightarrow}{s_X \Downarrow}}}} \; XX \tag{8.4.8}$$

obtained by horizontally composing two copies of (8.4.7) must coincide with the 2-arrow $S_{X,X}$ previously defined in (8.4.5). A further condition which this 2-arrow is must satisfy asserts that the 2-arrow S_X is additive in X. More precisely, it asserts the commutativity of the diagram of 2-arrows which may be built, for any pair of objects X and Y in $\mathcal{C}$, when the 2-arrow S_{XY} is compared to the 2-arrows S_X and S_Y. We do not display this diagram here, simply noting, as a hint to the reader, that it involves the additional 2-arrow $S_{X,Y}$ and the three hexagonal diagrams $R_{X|X,Y}$, $R_{Y|X,Y}$ and $R_{X,Y|XY}$.

Definition 8.5: *A Picard 2-stack in 2-groupoids $\mathcal{C}$ endowed with a functorial 2-arrow S_X (8.4.7) which is additive in X and for which the* *composite* 2–*arrow* (8.4.8) *is equal to the* 2–*arrow* $S_{X,X}$ (8.4.5) *is called a strict Picard* 2–*stack.*

Suppose now that $\mathcal{B}$ is a strict Picard stack, associated to a length one complex of sheaves of abelian groups $B_1 \longrightarrow B_0$. The universal coefficient theorem now provides us with a short exact sequence analogous to the sequence (7.9.2)

$$0 \longrightarrow Ext^2(A, B_1 \longrightarrow B_0) \longrightarrow H^6(K(A,4), \mathcal{B}) \longrightarrow Hom(A/2A, B_1 \longrightarrow B_0)$$

and we recognize in the requirement that the arrow R_{XX}, viewed via (8.4.1) as an object in the stack $\mathcal{B}$, be additive in elements $X \in A$ and that it be killed on elements of the subgroup $2A$, the two separate conditions defining a strict Picard 2-stack. The equivalence classes of such strict Picard 2-stacks with invariants A and $\mathcal{B}$ are therefore classified by the group $Ext^2(A, B_1 \longrightarrow B_0)$ In particular, when $\mathcal{B} = Tors(B) = (B[1])^{\sim}$ is the strict Picard stack associated to a sheaf of abelian groups B, we obtain in this manner a categorical interpretation of the group $Ext^3(A, B)$. In order to prove this in a more direct fashion, one would have to show that any strict Picard 2-stack is associated to a 2-prestack defined by a sheaf of simplicial abelian groups. The corresponding Moore complex would then yield the requisite Yoneda extension

$$0 \longrightarrow B \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0 .$$

8.6 We conclude this text with some general remarks on the relationship between the role played by the loop and classifying space functors in ordinary topology and their role in the sheaf theoretical topology considered here. Recall that there exists, in ordinary topology, a loop space functor Ω, which associates to an arbitrary connected pointed space X a coherently homotopy associative space (*i.e.*, an A_∞-space in Stasheff's terminology) ΩX which is group-like. Conversely, to every such group-like A_∞-space G is associated its (pointed) classifying space BG. The adjunction maps

(8.6.1) $$X \longrightarrow B\Omega X$$

and

(8.6.2) $$\Omega BG \longrightarrow G$$

which the pair of functors B and Ω between the category of group-like A_∞–spaces and that of pointed spaces determine are homotopy equivalences ([Ad] ch 2, [Se]), once the categories involved are appropriately defined. The analog in the sheaf-theoretic context for the classifying space functor B is the functor which associates to a group (*resp.*, *gr*–stack,*resp.* 2*gr*–stack) G the stack (*resp.*, 2–stack, *resp.*, 3-stack) $Tors(G)$. More generally, this construction associates to any *n-gr*–stack G, defined as an n–stack with a group-like composition law satisfying Stasheff's higher associativity conditions, the $(n+1)$–stack of G-torsors $Tors(G)$, pointed by the trivial G–torsor. Conversely, we may associate to any n–stack $\mathscr{C}$ defined on a space S, and to any object x in $\mathscr{C}$ the $(n-1)$-*gr*–stack $\underline{Aut}(x)$, which plays the role here of the space $\Omega(\mathscr{C},x)$ of loops on $\mathscr{C}$ pointed at x. It isn't however in general true that a globally defined object x exists, so that in the sheaf-theoretic context only locally defined loop spaces exist in general. Indeed, we have seen that if $\mathscr{C}$ is a gerbe (*resp.*, a 2-gerbe), such a global object x only exists when $\mathscr{C}$ is neutral. The analog of equivalence (8.6.2) is always satisfied in the present situation, since this is simply the assertion that for any *gr*–*n*-stack G on S, the adjoint (or gauge) *n*–*gr*–stack $\underline{Aut}(Triv_G)$ defined by the trivial G-torsor $Triv_G$ is canonically equivalent to G itself. Conversely, suppose that $\mathscr{C}$ is a locally connected n–stack in n–groupoids, a concept which we have at least defined for $n=1,2$. Equivalence (8.6.1) is only true when $\mathscr{C}$ is pointed by a globally defined object x, since it is then the assertion that the neutral gerbe (*resp.*, 2-gerbe) $\mathscr{C}$ is equivalent to one of the form $Tors(G)$, with G the $(n-1)$-*gr*–stack $\underline{Aut}(x)$ defined by some global object $x \in \mathscr{C}$. All that remains from the assertion (8.6.1) in the general case of a locally connected stack (*resp.* 2–stack) in groupoids, is the statement, which was fundamental for our analysis of such objects, that an arbitrary gerbe (*resp.*, 2–gerbe) is locally of the type *Tors* (*Aut* (x)), for some locally defined object x. As in topology, the loop space and classifying space functors are useful in understanding commutativity conditions on group laws. As we iterate the process of passing from an n–stack to the $(n-1)$–*gr*–stack of automorphisms of one of its objects, we encounter progressively more commutative structures on progressively lower-levelled stacks. Varying the integer n, and assuming that we have the definition of an n-stack in hand for arbitrary n, we obtain in this way all appropriate commutativity laws on an m-stack, m is a fixed integer. In our study of such commutativity laws, we have mainly appealed to the opposite principle, which states that, as we iterate the passage from an n–stack G with appropriate commutativity structure to the *gr*–$(n+1)$–stack *Tors* (G), we encounter progressively less commutative structures on progressively higher-level stacks. There in general comes a point at which all commutativity conditions have disappeared, and the process comes to a halt at the subsequent step since one then obtains a pointed stack without any group law. Only in the stable (or Picard) situation does the process continue indefinitely, since in that case the $(n+1)$–stack of torsors under a Picard n–stack G is once more Picard. In the strict Picard case familiar from ordinary homological algebra, this corresponds to the indefinitely iterable operation $A \longmapsto A[1]$, which shifts by one degree to the left an element A of the derived category of abelian sheaves in a topos.

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