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Improved stability for SK_1 and WMS_d of a non-singular affine algebra

Ravi A. Rao & Wilberd van der Kallen

1 Introduction.

In [MS, Theorem 1], M. P. Murthy–R. G. Swan have shown that a stably free projective module over a two-dimensional affine variety A over an algebraically closed field k is free. (The example of the tangent bundle over the real two sphere shows that the condition that the base field is algebraically closed is necessary.) They showed that every unimodular vector $(a, b, c) \in Um_3(A)$ could be transformed to a unimodular vector of the form (x^2, y, z), and that a unimodular vector of the form (x^2, y, z) over any commutative ring A can always be completed to an invertible matrix. In [Su1] A. Suslin generalised this by showing that a unimodular vector of the form $(a_0, a_1, a_2^2, \ldots, a_r^r)$ over any commutative ring A can always be completed to an invertible matrix; from which he deduced that a stably free projective A-module of rank $\geq \dim(A)$ is free where A is an affine algebra over an algebraically closed field k. In [Su2] Suslin generalised this further by proving that stably free projective A-modules of rank $\geq \dim(A)$ are free when A is an affine algebra over a perfect C_1 field k. (Whenever we speak of "perfect C_1 field", which is admittedly not a very useful combination outside characteristic 0, the more technical conditions in 3.1 actually suffice.) In this note we prove a K_1 analogue of Suslin's result. We prove that:

Theorem 1 (cf. 3.4) *Let A be a non-singular affine algebra of dimension $d \geq 2$ over a perfect C_1-field k. Then*

$$SL_{d+1}(A) \cap E_{d+2}(A) = E_{d+1}(A)$$

i.e. a stably elementary $\sigma \in SL_{d+1}(A)$ belongs to $E_{d+1}(A)$. Consequently, the natural map

$$SL_r(A)/E_r(A) \longrightarrow SK_1(A)$$

is an isomorphism for $r \geq d + 1$.

A beautiful theorem of L. N. Vaserstein [SV, Corollary 7.4], identifies $Um_3(A)/E_3(A)$, the coset space of unimodular 3-vectors $Um_3(A)$ modulo action of the Elementary matrices $E_3(A)$, with the Symplectic Elementary Witt group $W_E(A)$ when $\dim(A) \leq 2$. The correspondence

$$V : Um_3(A)/E_3(A) \to W_E(A)$$

which he has defined for any commutative ring A, is known as the *Vaserstein symbol*. As a consequence of the above theorem we obtain that the Vaserstein symbol is also an isomorphism if A is a three dimensional non-singular affine algebra over a perfect C_1 field k.

We present an example, based on the Hopf map $S^3 \to S^2$, to show that some condition on the field is again necessary.

For higher dimensional rings it was shown by the second author that one still has an (abelian) group structure on the orbit set $Um_d(A)/E_d(A)$, even though one no longer possesses such a nice interpretation as in Vaserstein's theorem. Following Suslin [Su4] one would now like to have a homomorphism from $SL_d(A)$ to this group $WMS_d(A) \approx Um_d(A)/E_d(A)$. It was shown in [vdK2, Prop. 7.10] that this will not work in general, but as another byproduct of the above we show that things improve over perfect C_1 fields.

Theorem 2 *Let A be a non-singular affine algebra of dimension $d \geq 3$ over a perfect C_1-field k. Then the "first row map"*

$$SL_d(A) \to WMS_d(A)$$

is a homomorphism. Taking its cokernel provides a group structure on $Um_d(A)/SL_d(A)$.

If in our C_1 field -1 is a square, one sees from [Ra2, (1.3)] that the row $(a_0, a_1, a_2^2, \ldots, a_{d-1}^{d-1})$—completable by Suslin—represents the $(d-1)!$ power of the class of $(a_0, a_1, a_2, \ldots, a_{d-1})$. So the group $Um_d(A)/SL_d(A)$ in the theorem is a torsion group of exponent at most $(d-1)!$ This makes us hope that it will be manageable in some cases.

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2 Preliminaries on the Vaserstein symbol.

All rings considered are commutative with 1. The set $M_{r,s}(A)$ consists of all matrices of size $r \times s$ over A. Write $M_r(A) = M_{r,r}(A)$.

A vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_r) \in A^r$ is said to be *unimodular* if there are elements $w_1, \ldots, w_r$ in A such that $v_1w_1 + \ldots + v_rw_r = 1$. The set of all unimodular vectors $\boldsymbol{v} \in A^r$ will be denoted $Um_r(A)$. The standard basis of A^r is written $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$.

The group $SL_r(A)$ of invertible matrices of determinant 1 acts on A^r in a natural way:

$$\sigma : \boldsymbol{v} \mapsto \boldsymbol{v}\sigma,$$

if $\boldsymbol{v} \in A^r$, $\sigma \in SL_r(A)$. This map preserves $Um_r(A)$, so $SL_r(A)$ acts on $Um_r(A)$.

Let $E_r(A)$ denote the subgroup of $SL_r(A)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the elementary generators $E_{ij}(\lambda) = I_r + e_{ij}(\lambda)$, $1 \leq i \neq j \leq r$, $\lambda \in A$, where $e_{ij}(\lambda) \in M_r(A)$ has entry λ in its (i,j)-th position, and all other entries zero. Thus $E_r(A)$ acts on $Um_r(A)$; if $\boldsymbol{v}, \boldsymbol{w} \in Um_r(A)$, let $\boldsymbol{v} \sim_E \boldsymbol{w}$ mean that $\boldsymbol{v} = \boldsymbol{w}\epsilon$ for some $\epsilon \in E_r(A)$. Let $Um_r(A)/E_r(A)$ be the set of equivalence classes of vectors $\boldsymbol{v} \in Um_r(A)$ under the equivalence $\sim_E$; and let $[\boldsymbol{v}] = [v_1, \ldots, v_r]$ denote the equivalence class of $\boldsymbol{v} = (v_1, \ldots, v_r)$.

If $\alpha \in M_r(A)$, $\beta \in M_s(A)$, then $\alpha \perp \beta$ denotes $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_{r+s}(A)$. An alternating matrix ϕ has diagonal entries 0 and is skew-symmetric, i.e. transpose$(\phi) = -\phi$. We can define inductively an alternating matrix $\psi_r \in E_{2r}(\mathbb{Z})$, by setting, for $r \geq 2$,

$$\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \psi_r = \psi_{r-1} \perp \psi_1.$$

For an alternating matrix $\phi \in M_{2r}(A)$ its determinant $\det(\phi)$ is a square $(pf(\phi))^2$ of a polynomial $pf(\phi)$ (called the Pfaffian) in the matrix entries with coefficients ± 1. An odd sized alternating matrix has Pfaffian 0, and, clearly, on even sized alternating matrices it is defined up to a sign, to fix which we insist that $pf(\psi_r) = 1$, for all r. For instance, if $\boldsymbol{v} = (v_0, v_1, v_2)$, $\boldsymbol{w} = (w_0, w_1, w_2)$, then

$$V(\boldsymbol{v}, \boldsymbol{w}) = \begin{pmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{pmatrix}$$

has Pfaffian $v_0w_0 + v_1w_1 + v_2w_2$. In particular, if $v_0w_0 + v_1w_1 + v_2w_2 = 1$, then $V(\boldsymbol{v}, \boldsymbol{w})$ has Pfaffian 1.

If $\alpha \in M_{2r}(A)$, and ϕ, θ are alternating matrices, then it can be checked that $pf(\alpha^t\phi\alpha) = pf(\phi)\det(\alpha)$, and that $pf(\phi \perp \theta) = pf(\phi)\, pf(\theta)$.

Two alternating matrices $\alpha \in M_{2r}(A)$, $\beta \in M_{2s}(A)$ are said to be *equivalent w.r.t.* $E(A)$ if there is a $\epsilon \in E_{2(r+s+l)}(A)$, for some l, such that

$$\alpha \perp \psi_{s+l} = \epsilon(\beta \perp \psi_{s+l})\epsilon^t.$$

It can be seen, cf. [SV, p. 945], that $\perp$ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with Pfaffian 1. This group is called the Symplectic Elementary Witt group and is denoted by $W_E(A)$.

The *Vaserstein symbol* $V = V_A : Um_3(A)/E_3(A) \longrightarrow W_E(A)$ is the map

$$[(a,b,c)] \mapsto [V(\boldsymbol{v},\boldsymbol{w})],$$

where $\boldsymbol{v} = (a,b,c)$, $\boldsymbol{w} = (a',b',c')$ with $aa' + bb' + cc' = 1$. In [SV, Theorem 5.2], L. N. Vaserstein has shown that this map is well defined (i.e. it is independent of both the choice of representative $\boldsymbol{v}$ in $[(a,b,c)]$, as well as the choice of a', b', c' such that $aa' + bb' + cc' = 1$).

Recall that $\mathrm{sdim}(A)$ stands for the stable range dimension of A, i.e. one less than the stable rank $sr(A)$ of [Va2]. For noetherian A it does not exceed the Krull dimension $\dim(A)$.

Lemma 2.1 *Let A be a commutative ring for which $Um_r(A) = e_1E_r(A)$, for $r \geq 5$. Then V_A is surjective. If, moreover, $SL_4(A) \cap E(A) = E_4(A)$ then V_A is bijective. In particular, if* $\mathrm{sdim}(A) \leq 3$, *and $SL_4(A) \cap E_5(A) = E_4(A)$, then V_A is bijective.*

Proof: L. N. Vaserstein in [SV, Theorem 5.2(c)] has shown that V_A is surjective. Apply [SV, Lemma 5.1] to conclude that V_A is injective if $SL_4(A) \cap E(A) = E_4(A)$. L. N. Vaserstein's stability estimate for the linear group in [Va1] settles the last assertion. □

3 Decrease in the injective stability estimate for K_1 of a regular affine algebra over a C_1 field

In [Va1] L. N. Vaserstein shows that

$$\frac{SL_r(A)}{E_r(A)} = \frac{SL_{r+1}(A)}{E_{r+1}(A)} = \cdots = SK_1(A),$$

where $SK_1(A)$ is the Whitehead group of A, when $r \geq \max\{3, d+2\}$, $d = \mathrm{sdim}(A)$.

The reader may construct or find examples (cf. [vdK2, Prop. 7.10]) of regular affine algebras A over the *real* numbers $\mathbb{R}$ for which $SL_{d+1}(A) \cap E_{d+2}(A) \neq E_{d+1}(A)$. This means that the natural map $SL_{d+1}(A)/E_{d+1}(A) \longrightarrow SK_1(A)$ is *not* injective for such rings A.

In this section we show that if A is a regular affine algebra over a perfect C_1-field then

$$\frac{SL_{d+1}(A)}{E_{d+1}(A)} \longrightarrow SK_1(A)$$

is an isomorphism, where $d =$ Krull dimension of A.

We start with a result of Suslin, slightly modified to suit our needs. Namely, we use the observation by P. Raman that one may bypass Suslin's hypothesis that the ring is an affine algebra of an *irreducible* affine variety that is *nonsingular in codimension 1.*

Proposition 3.1 *(cf. [Su3]) Let A be an affine algebra of dimension d over a field k satisfying: For any prime $p \leq d$ one of the following conditions is satisfied:*
(a) $p \neq \mathrm{char}\, k$, $c.d._p k \leq 1$.
(b) $p = \mathrm{char}\, k$ and k is perfect.
Then $Um_{d+1}(B) = e_1 SL_{d+1}(B)$.

Proof: We still have to explain how P. Raman reduces this to the proof given by Suslin. Here is her argument: First assume k is perfect. We may assume A is reduced, because a row which is completable modulo the nilradical is completable. If k is finite, one uses Vaserstein's result ([SV, Ch. III]) that $\mathrm{sdim}(A) < d$. Now let k be infinite and let $\boldsymbol{u}$ be a unimodular row over A of length $d+1$. Let J be the ideal defining the singular locus of A. Since A/J has dimension at most $d-1$, by general stability, $\boldsymbol{u}$ may be elementarily brought to $\boldsymbol{e}_1$ mod J. Modifying $\boldsymbol{u}$ we may thus assume that $\boldsymbol{u} = (u_1, \dots u_{d+1})$ such that u_1 is 1 mod J and other u_i are 0 mod J. Then applying Swan's version of Bertini as quoted in [KM], a general linear combination u_1+ suitable multiples of u_j added on gives a new u_1 which is still 1 mod J and so that $A/(u_1)$ is smooth outside the singular set of A. But since u_1 is congruent to 1 mod J, $A/(u_1)$ is smooth of dimension at most $d-1$. Now use standard Bertini and complete with Suslin's argument. Note that Suslin never needs that by repeated application of Bertini's theorem one gets down to an *irreducible* smooth curve; just a smooth curve C will do. As nilpotents do not matter for $SK_1(C)$, it would in fact be enough to have a curve which is smooth by nilpotent, i.e. whose underlying reduced curve is smooth. If k is not perfect and a curve C is geometrically smooth by nilpotent, then one may choose a finite purely inseparable field extension l of k so that $l \otimes C$ is smooth by nilpotent. Under our hypotheses this extension has degree prime to $d!$, so by a transfer argument vanishing of $SK_1(l \otimes C)/d!SK_1(l \otimes C)$ implies vanishing of $SK_1(C)/d!SK_1(C)$. □

Remark 3.2 By [Se, Ch. II Section 3.2] any C_1 field k satisfies the conditions $c.d._p k \leq 1$.

Proposition 3.3 *Let A be an affine algebra of dimension d over a field k satisfying the conditions mentioned in Proposition 3.1. If $\boldsymbol{v} \in Um_{d+1}(A)$ is congruent to $\boldsymbol{e}_1$ modulo (t) for some $t \in A$, then $\boldsymbol{v}$ can be completed to a $\sigma \in SL_{d+1}(A)$ with $\sigma \equiv I_{d+1}$ modulo (t).*

Proof: Put $B = A[T]/(T^2 - Tt)$. Then B is an affine algebra of dimension d over the field k. Write $\boldsymbol{v} = \boldsymbol{e}_1 + t\boldsymbol{w}$ with $\boldsymbol{w} \in A^{d+1}$. Lift it to $\boldsymbol{u}(T) = \boldsymbol{e}_1 + T\boldsymbol{w} \in Um_{d+1}(B)$. (Yes, it is unimodular!) So $\boldsymbol{u}(t) = \boldsymbol{v}$ and $\boldsymbol{u}(0) = \boldsymbol{e}_1$. Now 3.1 allows us to find $\alpha(T) \in SL_{d+1}(B)$ with $\boldsymbol{u}(T) = \boldsymbol{e}_1\alpha(T)$. The matrix $\alpha(0)^{-1}\alpha(t)$ does the trick. □

Theorem 3.4 *Let B be an affine algebra of dimension d over a field k satisfying the conditions mentioned in Proposition 3.1, and let $\sigma \in SL_{d+1}(B)$ be a stably elementary matrix. Then σ is* isotopic *to the Identity. Moreover, if B is regular and $d > 1$ then $\sigma \in E_{d+1}(B)$.*

Proof: By stability for $K_1(B)$, $\sigma \in E_{d+2}(B)$. Therefore, there is an isotopy $\tau(T) \in E_{d+2}(B[T])$ with $\tau(0) = I_{d+2}$, $\tau(1) = \{1\} \perp \sigma$. Take $A = B[T]$, $t = T^2 - T$, and $\boldsymbol{v} = \boldsymbol{e}_1\tau(T)$, in Proposition 3.3 to get a $\chi(T) \in SL_{d+2}(B[T],(t))$, with $\boldsymbol{v} = \boldsymbol{e}_1\chi(T)$. Therefore, $\boldsymbol{e}_1\tau(T)\chi(T)^{-1} = \boldsymbol{e}_1$, and so

$$\tau(T)\chi(T)^{-1} = (\{1\} \perp \rho(T))\prod_{i=2}^{d+2} E_{i1}(\lambda_i),$$

for some $\lambda_i \in A$, $\rho(T) \in SL_{d+1}(B[T])$. Clearly, $\rho(T)$ is an isotopy of σ with I_{d+1}.

Now let B be regular, and $d > 1$. By T. Vorst's K_1-analogue in [Vo] of H. Lindel's theorem in [Li], $SL_{d+1}(B_{\mathfrak{p}}[T]) = E_{d+1}(B_{\mathfrak{p}}[T])$ for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(B)$. By the "Local–Global Principle" for $E_r(B[T])$ $(r \geq 3)$ in [Ra1], $\rho(T) \in \rho(0)E_{d+1}(B[T])$, and so $\rho(T) \in E_{d+1}(B[T])$ as $\rho(0) = I_{d+1}$. But then $\sigma = \rho(1) \in E_{d+1}(A)$. □

Corollary 3.5 *Let A be a regular affine algebra of Krull dimension 3 over a C_1 field k which is perfect if its characteristic is 2 or 3. Then the Vaserstein symbol $V : Um_3(A)/E_3(A) \longrightarrow W_E(A)$ is an isomorphism. Further, if $\boldsymbol{v} \in Um_3(A)$ has a completion which is stably elementary then it also has a completion which is elementary.*

Proof: We are in the situation that lemma 2.1 applies. If $\boldsymbol{v} \in Um_3(A)$ has a completion which is stably elementary, then this completion is in $E_4(A)$, so Vaserstein's lemma [Su3, 2.2] applies. □

4 An example.

The purpose of this section is to show that the first part of corollary 3.5 fails when the C_1 field is replaced by $\mathbb{R}$.

Question Are there also examples of 3 dimensional algebras where the second half of 3.5 fails?

4.1 Let S^3 be the unit three–sphere embedded in 4 space in the standard way. Let C_3 be the ring of real valued continuous functions on S^3 and let P_3 be the subring of polynomial functions on S^3. Thus $P_3 = \mathbb{R}[x, y, z, t]$ with x, y, z, t satisfying the relation $x^2 + y^2 + z^2 + t^2 = 1$. Observe that $\mathrm{sdim}(C_3) = 3$ by [Va2] and observe that P_3 has Krull dimension 3, so that $\mathrm{sdim}(P_3) \leq 3$. (In fact one must have $\mathrm{sdim}(P_3) = 3$ too, as otherwise one could not have proposition 4.2 below). We want to show that the Vaserstein symbol is not injective, so we must be able to distinguish two orbits. To show that orbits $[\boldsymbol{v}], [\boldsymbol{w}] \in Um_3(C_3)/E_3(C_3)$ are not equal, it suffices to show that the corresponding homotopy classes of maps from S^3 to S^2 differ. (See for instance [vdK2, 7.6]). Now this can be decided by computing a Hopf invariant, as is explained in detail in [BT].

Proposition 4.2 *The ring P_3 is a 3 dimensional ring for which the Vaserstein symbol*

$$V : Um_3(P_3)/E_3(P_3) \to W_E(P_3)$$

is not injective. For C_3 even the universal weak Mennicke symbol is not injective.

Proof: For rings with sdim at most 3, the universal weak Mennicke symbol is surjective (cf. [vdK2, 4.29, 4.30]). The failure of injectivity for C_3 is thus just the last example of [vdK2]. (The indication of proof in that example is rather sketchy. We will compensate for that by being more explicit for the other ring). The word "even" in the proposition is explained by the following lemma, which was part of the motivation for introducing weak Mennicke symbols.

Lemma 4.3 *For any commutative ring A the Vaserstein symbol*

$$V : Um_3(A)/E_3(A) \to W_E(A)$$

is a weak Mennicke symbol.

Proof We have to show that

$$V[q,a,b] = V[r,a,b] + V[1+q,a,b]$$

whenever the unimodular rows are such that $r(1+q) \equiv q \pmod{(a,b)}$. By Vaserstein's rule [SV, 5.2(a_2)] , we can compute the right hand side as follows. Choose c, d with $ac + bd + (1-r)(1+q) = 1$. Then the right hand side equals

$$V[\begin{pmatrix} r & a \end{pmatrix}\begin{pmatrix} 1+q \\ -c \end{pmatrix}, \begin{pmatrix} r & a \end{pmatrix}\begin{pmatrix} a \\ 1-r \end{pmatrix}, b] =$$

$$V[r(1+q) - ac, a, b] = V[q,a,b].$$

□

Remark 4.4 Recall that the definition of the universal Mennicke symbol $Um_n(A)/E_n(A) \to MS_n(A)$ is built on relations that are valid in SK_1 of any commutative ring. In the same vein one may build the definition of $Um_n(A)/E_n(A) \to WMS_n(A)$ on the relations given in Lemma 5.4 of [Va3], valid in the symplectic K_1 of any commutative ring.

4.5 Two orbits. To proceed with the proof of the proposition, we give two orbits $[v], [w] \in Um_3(P_3)/E_3(P_3)$ that differ in their Hopf invariant (by a sign), but have identical image under the Vaserstein symbol V.

Recall that the Hopf map may be obtained as follows. The action by conjugation of the unit quaternion $q = x + yi + zj + tk$ on the real vector space of pure quaternions, i.e. the map $p \mapsto qpq^{-1}$ is given by the matrix

$$\begin{pmatrix} -t^2+x^2+y^2-z^2 & -2tx+2yz & 2ty+2xz \\ 2tx+2yz & -t^2+x^2-y^2+z^2 & -2xy+2tz \\ 2ty-2xz & 2xy+2tz & t^2+x^2-y^2-z^2 \end{pmatrix}$$

with respect to the real basis i, j, k. That matrix may be viewed as an element of $SL_3(P_3)$. Its first row $\boldsymbol{v} = (-t^2 + x^2 + y^2 - z^2, -2tx + 2yz, 2ty + 2xz)$ may be viewed as an element of $Um_3(P_3)$, or as a map $h : S^3 \to S^2$. That is the Hopf map, i.e. it generates $\pi_3(S^2)$. (We do not care whether the map we just described has Hopf invariant 1 or -1.) Indeed one easily checks that $(1, 0, 0)$ is a regular value whose inverse image under h is a circle. Similarly $(-1, 0, 0)$ is a regular value whose inverse image is a circle, and these two circles are linked (simply) in S^3. This proves h generates $\pi_3(S^2)$ (see [BT]).

If we send z to $-z$, thus reversing the orientation on the 3-sphere, we replace the Hopf map by its negative in $\pi_3(S^2)$, which is different. That means that if $\boldsymbol{w} = (-t^2 + x^2 + y^2 - z^2, -2tx - 2yz, 2ty - 2xz)$, then $[\boldsymbol{v}] \neq [\boldsymbol{w}]$. Now let us show that nevertheless the images of $\boldsymbol{v}$, $\boldsymbol{w}$ in $W_E(P_3)$ are the same.

The first entries of $\boldsymbol{v}$, $\boldsymbol{w}$ agree, so we may use Vaserstein's rule [SV, 5.2(a_2)] for multiplying (or dividing rather) the images of two rows with an equal entry. We first complete $\begin{pmatrix} -2tx + 2yz & 2ty + 2xz \end{pmatrix}$ to an element of SL_2 modulo $-t^2 + x^2 + y^2 - z^2$. Note that we are computing in P_3, so that $-t^2 + x^2 + y^2 - z^2 = 1 - 2t^2 - 2z^2 = 2x^2 + 2y^2 - 1$. As completion we take $\begin{pmatrix} -2tx + 2yz & 2ty + 2xz \\ -2ty - 2xz & -2tx + 2yz \end{pmatrix}$, with inverse $\begin{pmatrix} -2tx + 2yz & -2ty - 2xz \\ 2ty + 2xz & -2tx + 2yz \end{pmatrix}$. In $W_E(P_3)$ we have by Vaserstein's rule

$$V([\boldsymbol{w}]) - V([\boldsymbol{v}]) =$$

$$V[2x^2 + 2y^2 - 1, \begin{pmatrix} -2tx - 2yz & 2ty - 2xz \end{pmatrix} \begin{pmatrix} -2tx + 2yz & -2ty - 2xz \\ 2ty + 2xz & -2tx + 2yz \end{pmatrix}]$$

$$= V[2x^2 + 2y^2 - 1, 4t^2x^2 + 4t^2y^2 - 4x^2z^2 - 4y^2z^2, 8tx^2z + 8ty^2z]$$

$$= V[2x^2 + 2y^2 - 1, 2t^2 - 2z^2, 4tz]$$

$$= V[1 - 2t^2 - 2z^2, 2t^2 - 2z^2, 4tz],$$

which vanishes by an easy computation or by the Suslin–Quillen solution of Serre's conjecture. All in all, the images did not change in $W_E(P_3)$ when we reversed z, so that the Vaserstein symbol is not injective. □

5 Trivial module structure on WMS_d.

In this section we show that the module structure on $WMS_d(A)$, discussed in [vdK2], trivializes entirely for smooth affine algebras A of dimension d over a perfect C_1 field (or a field as in 3.1). In particular this gives a homomorphism $SL_d(A) \to WMS_d(A)$.

Theorem 5.1 *Let A be a smooth affine algebra of dimension $d \geq 3$ over a field as in 3.1. Let $1 \leq k \leq d$. Let $\boldsymbol{v} = (v_1, \ldots, v_d) \in Um_d(A)$ and let T be a k by k matrix over A with first row $\boldsymbol{u} = (u_1, \ldots, u_k)$ such that $\overline{\det(T)}$ is a square of a unit in $A/(v_{k+1}, \ldots, v_d)$. Then*

$$[(v_1, \ldots, v_k)T, v_{k+1}, \ldots, v_d] = [\boldsymbol{v}] + [\boldsymbol{u}, v_{k+1}, \ldots, v_d]$$

In particular, taking $k = d$, we have $[\boldsymbol{v}g] = [\boldsymbol{v}] + [e_1 g]$ for $g \in SL_d(A)$.

Proof: The conclusion is the same as in [vdK2, Thm. 5.9], which is proved by induction on k. We can use the same proof, except for $k = 2$. For $k = 2$ the problem is that the row $(u_2, \ldots, u_k, v_2, \ldots, v_d)$ does not have at least $d + 1$ entries, as would be required to make it easily unimodular. Take $k = 2$. Using the $k = 1$ case we easily may assume that $\overline{\det(T)} = 1$ in $A/(v_3, \ldots, v_d)$. If $d = 3$, we may by 3.5 identify $WMS_3(A)$ with $W_E(A)$ through the Vaserstein symbol. Then the result follows from Vaserstein's rule [SV, 5.2(a_2)]. If $d > 3$, we note that $(\det(T)v_1, \det(T)v_2, v_3, \ldots, v_d)$ is unimodular, so that by adding general position multiples of $\det(T)v_1$, $\det(T)v_2$, v_3 to $(v_4, \ldots, v_d)$ we may achieve (Bertini again!) that $A/(v_4, \ldots, v_d)$ is smooth of dimension 3. From the result in $WMS_3(A/(v_4, \ldots, v_d))$ it now follows that the required relation holds in $WMS_d(A)$. (Use the obvious homomorphism $WMS_3(A/(v_4, \ldots, v_d)) \to WMS_d(A)$.) □

Theorem 2 now follows easily. □

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