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WEN-CHING WINNIE LI

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Number Theoretic Constructions of Ramanujan Graphs

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§1. Introduction

A fundamental problem in communication network is to construct efficient networks at a cost not exceeding a fixed amount. By interpreting a network as a finite graph, one may formulate the problem as constructing graphs with good magnifying constant while the number of vertices and degree are left fixed. It was proved by Tanner [Ta] and Alon-Milman [AM] that the magnifying constant is intimately related to the spectrum of the graph, as explained below.

Given a (finite) graph X, its adjacency matrix $A = A(X)$ may be regarded as a linear operator on the space of functions on (vertices of) X, which sends a function f to Af defined by

$$(Af)(x) = \sum_{x \to y} f(y),$$

where y runs through all outneighbors of x. The eigenvalues of A are called the spectrum of X. If X is k-regular, that is, we have indegree = outdegree = k at each vertex, then k is an eigenvalue of A and all eigenvalues λ of A satisfy

$$|\lambda| \leq k.$$

If, in addition, X is r-partite, that is, the vertices of X are partitioned into r parts,

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and the outneighbors of the vertices in the ith part are in the $i+1st$ part for all $i \bmod r$, then ζk is also an eigenvalue of A for all rth roots of unity ζ. Call the ζk's trivial eigenvalues of X, and the remaining ones nontrivial. Let $\lambda = \lambda(X)$ be the maximum nontrivial eigenvalue of X in absolute value. It was shown in [Ta] that the magnifying constant c has a lower bound

$$c \geq 1 - \frac{k}{3k - 2\lambda},$$

and on the other hand an upper bound of λ was given in terms of c in [AM]:

$$k - \lambda \geq \frac{1}{2} - \frac{1}{2 + c^2}.$$

Thus the smaller λ is, the larger c is, and vice versa.

How small can λ be? If X is a connected k-regular r-partite graph with $A(X)$ diagonalizable by a unitary matrix, then a trivial lower bound for $\lambda(X)$ is

$$\lambda(X) \geq \sqrt{\frac{n - rk}{n - r}}\sqrt{k}.$$

A nontrivial lower bound was given by Alon and Boppana (see [LPS]), which asserts that for k-regular undirected graphs X, $\liminf \lambda(X)$ is at least $2\sqrt{k-1}$ as the size of X tends to infinity. The above statement also holds for k-regular directed graphs with adjacency matrices diagonalizable by unitary matrices. Following Lubotzky, Phillips and Sarnak [LPS], we call a finite (directed or undirected) graph X a *Ramanujan graph* if

(i) X is k-regular,

(ii) $\lambda(X) \leq 2\sqrt{k-1}$,

(iii) the adjacency matrix $A(X)$ is diagonalizable by a unitary matrix.

Here the third condition is automatically satisfied if X is an undirected graph, for its adjacency matrix is then symmetric and hence diagonalizable by an orthogonal matrix. In view of the above analysis, a Ramanujan graph, roughly speaking, is a regular graph with small nontrivial eigenvalues, and hence has large magnifying constant. It is the kind of graph one is looking for in communication theory. Ramanujan graphs often have other good properties, such as small diameter (cf.[C]), large girth and small chromatic number (cf.[LPS]). They also have wide applications in combinatorics and computational complexity. The reader is referred to [C] and [B] for their connections with other fields.

To date, there are three systematic ways to construct Ramanujan graphs explicitly, they are all number-theoretic. We shall survey these methods chronologically in the next three sections. The first construction is due to Margulis [M] and indepentently, Lubotzky, Phillips and Sarnak [LPS], their graphs are defined on double cosets of adelic points of definite quaternion groups over $\mathbb{Q}$. Such graphs are Ramanujan because the Ramanujan-Petersson conjecture for classical cusp forms of weight 2 is proved to be true by Deligne [D2]. The second one is worked out by Chung [C] and Li [L1], where graphs are defined on finite abelian groups; they are shown to be Ramanujan using certain character sum estimates resulting from the Riemann hypothesis for curves over finite fields proved by Weil [W1]. Terras and her students [Te and references therein] came up with the third construction of Ramanujan graphs, which are defined on right cosets of GL_2 over finite fields. The eigenvalues of such graphs are character sums arising from irreducible representaions of GL_2 over finite fields, and one uses the Riemann hypothesis for curves over finite fields to derive the desired bound on these character sums. It is an interesting combination of GL_2 theory with GL_1 theory.

§2. Ramanujan graphs based on adelic quaternionic groups

Graphs constructed using this method will have valency $k = p + 1$, where p

is a prime. The general method is as follows. Take a definite quaternion algebra H defined over $\mathbb{Q}$ unramified at p and ramified at ∞. Let D be the multiplicative group $H^\times$ divided by its center. Let X be the double coset space of the adelic points of D :

$$X = D(\mathbb{Q})\backslash D(A_{\mathbb{Q}})/D(\mathbb{R}) \prod_{q \text{ prime}} D(Z_q),$$

where Z_q is the ring of integers of the q-adic field $\mathbb{Q}_q$. By the strong approximation theorem, the above (global) double coset space can be expressed locally :

$$\begin{aligned} X &= D\big(Z[\tfrac{1}{p}]\big)\backslash D(\mathbb{Q}_p)/D(Z_p) \\ &= D\big(Z[\tfrac{1}{p}]\big)\backslash PGL_2(\mathbb{Q}_p)/PGL_2(Z_p). \end{aligned}$$

Here $D(\mathbb{Q}_p)$ is isomorphic to $PGL_2(\mathbb{Q}_p)$ since H is unramified at p. The right coset space $PGL_2(\mathbb{Q}_p)/PGL_2(Z_p)$ has a natural structure as a $(p+1)$-regular infinite tree $\mathcal{T}$ (see [S]), and the discrete group $\Gamma = D\big(Z[\frac{1}{p}]\big)$ acts on $\mathcal{T}$. Since H is ramified at ∞, $\Gamma\backslash\mathcal{T}$ is a finite $(p+1)$-regular graph, which is the graph structure on X. We may replace Γ by a congruence subgroup Γ' and thus obtain a finite cover X' of X.

To study the eigenvalues of X', we first note that the functions on X' are automorphic forms for the quaternion group D over $\mathbb{Q}$ which are trivial on $D(\mathbb{R})$ and on an open compact subgroup of the product of the standard maximal compact subgroups at nonarchimedean places. Included in such automorphic forms are the constant functions, which are eigenfunctions of the adjacency matrix $A = A(X')$ with eigenvalue $p+1$. The orthogonal complement V of the constant functions is invariant under A, and the automorphic forms there, by the results in representation theory explained in [JL] and [GJ], correspond to a certain kind of classical cusp forms of weight 2 and trivial character. When interpreted as classical cusp forms, the action of A on V is nothing but the action of the classical Hecke operator T_p acting on weight 2 cusp forms. Hence the eigenvalues λ of A on V, which are also eigenvalues of T_p on weight 2 cusp forms, satisfy the Ramanujan-Petersson conjecture

$$|\lambda| \leq 2\sqrt{p} = 2\sqrt{k-1}.$$

This conjecture was established by Deligne, who first showed that Ramanujan conjecture follows from the Weil conjectures, and then proved the Weil conjectures [D2]. Therefore we have shown

Theorem 1. *The graph X' is a $(p+1)$-regular Ramanujan graph.*

The $(p+1)$-regular Ramanujan graphs studied in [Me] by Mestre and Oesterlé were constructed by choosing $H = H_\ell$ ramified only at ∞ and at a prime $\ell \neq p$, with the double coset space always modulo the product of real points and the standard maximal compact subgroups at nonarchimedean places on the right; and by letting ℓ tend to infinity, they obtained infinitely many such graphs. On the other hand, Margulis [M] and, independently, Lubotzky, Phillips and Sarnak [LPS] (see also [Ch]) took H to be the Hamiltonian quaternion, they obtained an infinite family of $(p+1)$-regular Ramanujan graphs by taking congruence subgroups of Γ. In general, one may both vary quaternion algebras and take congruence subgroups to construct infinitely many Ramanujan graphs.

The same argument works when the base field $\mathbb{Q}$ is replaced by a function field of one variable over a finite field. In that case the resulting graphs are Ramanujan because Drinfeld [Dr] has proved the Ramanujan conjecture for GL_2 over function fields. This is done in Morgenstern's thesis. Note that the graphs so constructed have valency $k = q + 1$ with q a power of a prime. In [P] Pizer constructed $(p+1)$-regular Ramanujan graphs allowing multiple edges by using the action of the classical Hecke operator at p on spaces of certain theta series of weight 2.

§3. Ramanujan graphs based on finite abelian groups

We start by giving a general recipe. Let G be a finite abelian group and S a k-element subset of G. Define two k-regular graphs on G, called the sum graph $X_s(G,S)$ and the difference graph $X_d(G,S)$ as follows : the out-neighbors of $x \in G$ in the sum graph are elements in $\{y \in G : x+y \in S\} = -x+S$, and those in the difference graph are in $\{y \in G : y-x \in S\} = x+S$, respectively. For X_s and X_d so constructed, one can show that their adjacency matrices are diagonalizable by unitary matrices and the eigenvalues of their adjacency matrices, although different, have the same absolute value given explicitly by

$$\left|\sum_{s\in S}\psi(s)\right| \qquad \text{for characters } \psi \text{ of } G.$$

Hence if we can find a suitable group G and a subset S such that

$$\left|\sum_{s\in S}\psi(s)\right| \le 2\sqrt{k-1}$$

for all nontrivial characters ψ of G, then we will have constructed two Ramanujan graphs $X_s(G,S)$ and $X_d(G,S)$. This observation converts a combinatorial problem to a number-theoretic problem.

In what follows, F denotes a finite field with q elements, and F_n denotes a degree n field extension of F. Let N_n be the set of elements in F_n with norm to F equal to 1. In [D1], Deligne proved the following estimate of the generalized Kloosterman sum :

<u>Theorem</u> 2. (Deligne) *For all nontrivial ψ of F_n, we have*

$$\left|\sum_{s\in N_n}\psi(s)\right| \le n\,q^{\frac{n-1}{2}}.$$

When $n=2$, N_2 has cardinality $q+1=k$. The above theorem implies that $X_s(F_2,N_2)$ and $X_d(F_2,N_2)$ are Ramanujan graphs.

Deligne proved the above theorem using étale cohomology and algebraic geometry. We exhibit more character sum estimates below using the Riemann hypothesis for curves over finite fields which was proved by Weil [W1]. The reader is referred to [L1] for more detail.

Let t be an element in F_n such that $F_n = F(t)$. Assume $n \geq 2$. Let

$$S = \left\{ \frac{t^q - a}{t - a} : a \in F \cup \{\infty\} \right\}.$$

When $a = \infty$, the quotient is understood to be 1. Note that $t^q - a$ is the image of $t - a$ under the Frobenius automorphism $x \mapsto x^q$, hence the set S is contained in N_n.

Theorem 3. *For all nontrivial characters χ of N_n, we have*

$$\left| \sum_{s \in S} \chi(s) \right| \leq (n-2)\sqrt{q}.$$

The set S has cardinality $q+1 = k$. Thus we get interesting Ramanujan graphs by taking $G = N_n$ for $n = 3, 4$ and S as above.

Let

$$S' = \left\{ \frac{1}{t - a} : a \in F \cup \{\infty\} \right\}.$$

Here the value of $\frac{1}{t-a}$ for $a = \infty$ is zero.

Theorem 4. *For all nontrivial characters ψ of F_n, we have*

$$\left| \sum_{s \in S'} \psi(s) \right| \leq (2n-2)\sqrt{q}.$$

Hence we obtain Ramanujan graphs by taking $G = F_2$ and $S = S'$.

Observe that S and S' are affine transformations of each other, thus Theorem 4 can be restated as

Theorem 4'.

$$\left|\sum_{s\in S}\psi(s)\right| \le (2n-2)\sqrt{q}$$

for all nontrivial characters ψ of F_n.

When $n = 2$, we find $S = N_2$, hence the above statement yields immediately

Corollary 1. *Deligne's theorem (Theorem 2) for $n = 2$ follows from the Riemann hypothesis for curves over finite fields.*

Theorem 5. *For all nontrivial characters (χ, ψ) of $N_n \times F_n$, we have*

$$\left|\sum_{s\in S}\chi(s)\,\psi(s)\right| \le (2n-2)\sqrt{q}.$$

Again we get Ramanujan graphs by taking $G = N_2 \times F_2$, and S the diagonal imbedding of the above set S in $N_2 \times F_2$.

As noted before, $S = N_2$ when $n = 2$, thus we get

Corollary 2. *For all nontrivial characters (χ, ψ) of $N_2 \times F_2$, we have*

$$\left|\sum_{s\in N_2}\chi(s)\,\psi(s)\right| \le 2\sqrt{q}.$$

In view of the analogous character sum estimate for split degree 2 algebra extension of F proved by Mordell (Theorem 6 below), Deligne in [D1] conjectured that the same bound should hold for twisted generalized Kloosterman sums, namely,

Conjecture (Deligne) *For all nontrivial characters (χ, ψ) of $N_n \times F_n$, we have*

$$\left|\sum_{s\in N_n}\chi(s)\,\psi(s)\right| \le nq^{\frac{n-1}{2}}.$$

Corollary 2 above establishes Deligne's conjecture for the case $n = 2$. In fact, the conjecture for $n = 2$ follows from the Riemann hypothesis for projective curves over finite fields.

Using the same method, one can also re-establish the following known character sum estimates:

Theorem 6. (Mordell [Mo]) *For each nontrivial character (χ, ψ) of $F^\times \times F$, we have*

$$\left| \sum_{x \in F, x \neq 0} \chi(x)\psi(x + x^{-1}) \right| \leq 2q^{1/2}.$$

Theorem 7. (Katz [K1]) *Let B be an étale algebra over F of degree n and let x be a regular element of B. Then for every nontrivial character χ of $B^\times$, we have*

$$\left| \sum_{\substack{a \in F \\ x-a \in B^\times}} \chi(x - a) \right| \leq (n-1)q^{1/2}.$$

Here in writing B as a product $K_1 \times \cdots \times K_r$ of finite field extensions K_i of F with total degree n, an element $x = (x_1, \cdots, x_r)$ of B is called regular if the field $F(x_i) = K_i$ for $i = 1, \cdots, r$ and no two components of x are conjugate.

By taking B to be F_2 and x an element in F_2 but not in F, one gets q-regular Ramanujan graphs $X_s(F_2^\times, F)$ and $X_d(F_2^\times, F)$ constructed in [C]. In fact, by taking $B = F \times F$, $x = (a, b)$ with $a \neq b$, one obtains $(q-2)$-regular Ramanujan graphs $X_s(F \times F, S)$ and $X_d(F \times F, S)$ with $S = F - \{a, b\}$ as well.

The way to prove Theorems 3-7 is to construct a nontrivial character ξ of the idèle class group of the rational function field $F(t)$ such that the nonzero terms in the given character sum are precisely the values of ξ at uniformizers of the places of $F(t)$ of degree one where ξ is unramified. Hence the associated L-function $L(u, \xi)$

is a polynomial in u of finite degree, say, d, and the coefficient of u is the character sum in question. It then follows from the Riemann hypothesis for curves over finite fields that the character sum has absolute value majorized by $d\,q^{1/2}$.

§4. Ramanujan graphs based on finite nonabelian groups

In this section we shall construct Ramanujan graphs based on finite nonabelian groups, which enjoy certain properties already seen in the previous two methods. Again, we start with a general approach.

Let G be a finite group and let K be a subgroup of G. Denote by $L(G)$ the space of complex-valued functions on G and by $L(G/K)$ the subspace of functions invariant under right translations by K. For a K-double coset $S = KsK$, define the operator A_S on $L(G)$ by sending a function f on G to

$$(A_S f)(x) = \sum_{y \in S} f(xy), \qquad x \in G.$$

Let λ be a nonzero eigenvalue of A_K, and let f be a nonzero eigenfunction with eigenvalue λ. Define h on G by

$$h(x) = \sum_{k \in K} f(xk), \qquad x \in G.$$

Then for $k' \in K,\ x \in G$, we have

$$h(xk') = h(x) = (A_K f)(x) = \lambda f(x) = \lambda f(xk').$$

From $\lambda \neq 0$ we get $f(x) = f(xk')$ for all $k' \in K$ and hence

$$h(x) = |K| f(x) = \lambda f(x) \qquad \text{for all } \ x \in G.$$

As $f(x) \neq 0$ for some x, this implies $\lambda = |K|$. Denote by L_0 the 0-eigenspace of A_K. We have shown that

$$L(G) = L(G/K) \oplus L_0$$

and each space is invariant under the left translation by G. Further, A_S acts on L_0 as 0 operator, and $L(G/K)$ is A_S-invariant.

We assume that the operators A_S are diagonalizable and mutually commutative. As $A_S f$ is nothing but the convolution of f with the characteristic function of S^{-1}, our assumption implies that the convolution algebra $L(K\backslash G/K)$ of the K-bi-invariant functions on G is commutative. Note that the assumption is satisfied if each double coset S is symmetric, that is, $S = S^{-1}$. Then $L(G/K)$ is a direct sum of common eigenspaces of A_S, and each common eigenspace is invariant under left translations by G, that is, $g \in G$ acts on f by $(g \cdot f)(x) = f(g^{-1}x)$ for $x \in G$. Hence $L(G/K)$ decomposes as a direct sum of irreducible representations (V, π) of G such that on each space V the operators A_S act by multiplication by scalars λ_S. We want to find a way to compute the λ_S. Denote by e the identity of G.

Proposition 1. *Let (V, π) be an irreducible representation of G occurring in $L(G/K)$. Then*

(i) If h is a K-bi-invariant function in V with $h(e) = 0$, then $h = 0$.

(ii) There exists a unique K-bi-invariant function h in V with $h(e) = 1$.

(iii) The function h in (ii) satisfies

$$\frac{1}{|K|}\sum_{k\in K} h(xky) = h(x)h(y) \qquad \text{for all } x, y \in G$$

and for any K-double coset $S = KsK$,

$$\lambda_S = \frac{|K|^2}{|Stab\ s|}h(s),$$

where Stab s *consists of $k \in K$ such that $ksK = sK$.*

Proof. Let f be a nonzero function in V with $f(g) = 1$ for some $g \in G$. Replacing

f by $\pi(g^{-1})f$ if necessary, we may assume $f(e) = 1$. Define

$$h(x) = \frac{1}{|K|} \sum_{k \in K} f(kx) = \frac{1}{|K|} \sum_{k \in K} \left(\pi(k)f\right)(x) \qquad \text{for } x \in G.$$

Then h lies in V, is K-bi-invariant, and satisfies $h(e) = f(e) = 1$. This proves the existence part of (ii). The uniqueness part of (ii) will follow from (i).

Let h be a K-bi-invariant function in V. Then $A_S h = \lambda_S h$ for all K-double cosets $S = KsK$. Fix $s \in G$. We have, for all $x \in G$,

$$\frac{1}{|K|} \sum_{k \in K} h(xks) = \frac{|Stab\ s|}{|K|^2} \sum_{y \in S} h(xy) = \frac{|Stab\ s|}{|K|^2} (A_S h)(x) = \frac{|Stab\ s|}{|K|^2} \lambda_S h(x).$$

Setting $x = e$, we obtain

$$h(s) = \frac{|Stab\ s|}{|K|^2} \lambda_S h(e).$$

Hence $h(s) = 0$ for all $s \in G$ if $h(e) = 0$. This proves (i). Suppose $h(e) = 1$. The above equation yields

$$h(s) = \frac{|Stab\ s|}{|K|^2} \lambda_S$$

and hence

$$\frac{1}{|K|} \sum_{k \in K} h(xks) = h(x)h(s) \qquad \text{for all } x, s \in G$$

and λ_S is as asserted.

__Remark.__ If we take a K-bi-invariant function h in L_0, then the same proof shows that $h = 0$. Hence L_0 contains no nontrivial K-bi-invariant function. This proves that representations occurring in L_0 have no K-invariant vectors.

The proposition above implies that in V, the $\pi(K)$-invariant subspace is 1-dimensional, generated by h in (ii); and further, as remarked above, the representations occurring in $L(G/K)$ are characterized by having a nonzero K-invariant vector. Fix $x \in G$ and consider the operator $\sum_{k \in K} \pi(kx^{-1}) =: \rho(x)$ on V. Let $f \in V$.

Then $\rho(x)f$ is a bi-K-invariant function in V whose value at e is equal to

$$\left(\rho(x)f\right)(e) = \sum_{k \in K} f(xke) = |K| f(x).$$

Hence $\rho(x)f = |K|f(x)h$. Choose a basis $f_1 = h, f_2, \cdots, f_r$ of V. Then the matrix representation of $\rho(x)$ with respect to this basis is

$$\begin{pmatrix} |K|h(x) & |K|f_2(x) & |K|f_3(x) & \cdots & |K|f_r(x) \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

so that

$$\frac{1}{|K|} tr\, \rho(x) = \frac{1}{|K|} \sum_{k \in K} tr\, \pi(kx^{-1}) = h(x).$$

We summarize the above discussion in

<u>**Theorem**</u> **8.**(cf. [Te1]) *The group G acts on the space $L(G/K)$ of right K-invariant functions by left translations. The space $L(G/K)$ can be decomposed into a direct sum of irreducible subspaces (V, π) such that on each subspace V the operators A_S act by scalar multiplications by $\lambda_{S,\pi}$ for all K-double cosets S. Further, on each subspace V, the space of left K-invariant functions is one-dimensional, generated by*

$$h_\pi(x) := \frac{1}{|K|} \sum_{k \in K} tr\, \pi(kx^{-1}). \tag{1}$$

Such a function is K-bi-invariant, satisfies $h_\pi(e) = 1$ and

$$\frac{1}{|K|} h_\pi(xky) = h_\pi(x) h_\pi(y) \qquad \text{for all } x, y \in G.$$

Moreover, the eigenvalues $\lambda_{S,\pi}$ are given by

$$\lambda_{S,\pi} = \frac{|K|^2}{|Stab\ s|} h_\pi(s) \qquad \text{for any } s \in S,$$

where Stab s *consists of $k \in K$ such that $ksK = sK$.*

Now we turn to the construction of graphs. Let T be a finite union of double cosets. Define a graph $X = X(G/K, T/K)$ on cosets G/K such that the outneighbors of xK are xg_iK, $i = 1, \cdots, k$, where $T = \bigsqcup_{i=1}^{k} g_iK$. This is a k-regular directed graph, and it is undirected if $T = T^{-1}$. When G is abelian and K is trivial, this is the difference graph defined in the previous section. When $G = PGL_2(\mathbb{Q}_p)$, $K = PGL_2(Z_p)$ and $T = K \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K$, the resulting graph is the $(p+1)$-regular infinite tree $\mathcal{T}$ associated to $PGL_2(\mathbb{Q}_p)$ in §2.

The adjacency matrix $\overline{A}_T$ of $X = X(G/K, T/K)$ is $\frac{1}{|K|} \sum_{\substack{S\ K-\text{double coset} \\ S \subset T}} A_S$. By the theorem above, the eigenvalues of $\overline{A}_T$ are given by

$$\sum_{S \subset T} \frac{|K|}{|Stab\ s|} h_\pi(s), \qquad \text{for any element } s \text{ in } S,$$

as π runs through the irreducible representations of G occurring in $L(G/K)$.

As remarked before, an irreducible representation π of G occurs in $L(G/K)$ if and only if it has a nonzero K-invariant vector. Thus to construct Ramanujan graphs of the above type, it suffices to find suitable G, K and T such that $L(K\backslash G/K)$ is a commutative algebra, $\overline{A}_T$ is diagonalizable, and for all nontrivial irreducible representations π of G containing a nonzero K-invariant vector, the function h_π defined by (1) satisfies

$$\left| \sum_{\substack{S \subset T \\ S\ K-\text{double coset}}} \frac{|K|}{|Stab\ s|} h_\pi(s) \right| \leq 2\sqrt{k-1},$$

where

$$k = \sum_{S \subset T} \frac{|K|}{|Stab\ s|}.$$

A family of $(q+1)$-regular Ramanujan graphs was constructed and studied by A. Terras and her students ([Te2] and references therein, see also [E]). They took a finite field F of q elements with q odd, $G = GL_2(F)$, and

$$K = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \in G \; : \; a, b \in F \right\}$$

for a fixed nonsquare element δ in F. Then G/K can be represented by

$$H = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \; : \; y \in F^\times \text{ and } x \in F \right\}$$

so that it is analogous to the classical upper half-plane. The group G has q double cosets, of which $K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} K = K$ and $K \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} K$, and each of the remaining $q-2$ double cosets KtK is the union of $(q+1)$ K-cosets. More precisely, each double coset KtK is associated to an ellipse $x^2 = ay + \delta(y-1)^2$, where $a \in F, a \neq 0, 4\delta$, such that $KtK = \bigsqcup \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K$ with (x, y) running through all the F-points of the ellipse. Also denote such a double coset KtK by S_a. Choose $T = S_a$, $a \neq 0, 4\delta$, so that we get a $(q+1)$-regular graph $X = X(G/K, \; T/K)$. Because all K-double cosets S are symmetric, i.e., $S = S^{-1}$, the graph $X = X(G/K, \; T/K)$ is undirected and the algebra $L(K\backslash G/K)$ is commutative. Further, *Stab* t is the subgroup of diagonal matrices in K, hence $|Stab \; t| = q - 1$ and $|K|/|Stab \; t| = q + 1 = k$.

One checks from the known table of representations of G in [PS] that there are two types of nontrivial irreducible representations of G containing a nonzero K-invariant vector. The first type arises from the $(q+1)$-dimensional representation of G induced from the 1-dimensional representation of the Borel subgroup given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \chi(a)\chi^{-1}(d)$$

for a character χ of $F^\times$. If χ has order greater than 2, then the above induced representation is irreducible, denote it by π_χ; while if χ has order 2, denote by π_χ the q-dimensional irreducible subrepresentation of the above induced representation.

The realization of π_χ in $L(G/K)$ is the subspace generated by the left translations of the function f on G/K defined by

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K\right) = \chi(y).$$

The eigenvalue $\lambda_{T,\chi}$ of $\overline{A}_T$ on the space of π_χ can be easily seen to be

$$\lambda_{T,\chi} = \sum_{\substack{y \in F \\ ay+\delta(y-1)^2=x^2}} \chi(y).$$

It was shown by R. Evans and by H. Stark that

$$|\lambda_{T,\chi}| \le 2\sqrt{q} = 2\sqrt{k-1} \quad \text{for } \chi \text{ nontrivial},$$

using Weil's estimate [W2], which follows from the Riemann hypothesis for projective curves over F. Soto-Andrade [SA] computed h_χ using (1) with $\pi = \pi_\chi$ and arrived at the same expression for $\lambda_{T,\chi}$. Here we note that since $T = T^{-1}$, we have

$$\lambda_{T,\pi} = \frac{|K|}{q-1} h_\pi(t^{-1}) = \frac{1}{q-1} \sum_{k \in K} tr\, \pi(kt)$$

for all representations π occurring in $L(G/K)$.

The second type of irreducible representations of G in question, denoted by π_ω, is the representation of G associated to a multiplicative character ω of the quadratic extension $F(\sqrt{\delta})$ of F. Here $\omega \neq \omega^q$, that is, ω is not the lift of a multiplicative character of F by composing with the norm map from $F(\sqrt{\delta})$ to F, and the representation is $(q-1)$–dimensional. The eigenvalue $\lambda_{T,\omega}$ on the space of π_ω was computed by Soto-Andrade [SA] to be

$$\lambda_{T,\omega} = \sum_{\substack{z=x+y\sqrt{\delta} \in F(\sqrt{\delta}) \\ x^2-\delta y^2=1}} \varepsilon\left(\frac{a}{\delta} - 2 + 2x\right)\omega(z),$$

where ε is the function on F equal to 1 on squares in $F^\times$, -1 on nonsquares in $F^\times$ and 0 at 0. Using étale cohomology and algebraic geometry, N. Katz [K2] showed that

$$\left|\lambda_{T,\omega}\right| \leq 2\sqrt{q} = 2\sqrt{k-1}.$$

In fact, the above inequality can also be proved directly by constructing an idèle class character η of the rational function field $F(x)$ such that the attached L-function $L(u, \eta)$ is a polynomial in u of degree at most 2 and $\lambda_{T,\omega}$ is the coefficient of u. The desired inequality then follows from the Riemann hypothesis for curves over finite fields, the same way as we obtained character sum estimates in the previous section. (See [L2], chap 9, for details.) This proves

Theorem 9. *The graphs* $X = X(G/K,\ S_a/K)$, $a \in F, a \neq 0, 4\delta$, *are* $(q+1)$*-regular Ramanujan graphs.*

This is the third explicit construction of Ramanujan graphs, which uses both representations of GL_2 over a finite field and character sum estimates resulting from the Riemann hypothesis for curves over finite fields.

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Wen-Ching Winnie Li
Department of Mathematics
Penn State University
University Park, PA 16802
U.S.A.