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# A TYPE $III_\lambda$ FACTOR WITH CORE ISOMORPHIC TO THE VON NEUMANN ALGEBRA OF A FREE GROUP, TENSOR $B(H)$ .

FLORIN RĂDULESCU

In this paper we obtain a type  $III_\lambda$  factor by using the free product construction from [Vo1,Vo2] and show that its core ([Co]) is  $\mathcal{L}(F_\infty) \otimes B(H)$ . We will prove that

$$M_2(\mathbb{C}) * L^\infty([0, 1], \nu)$$

is a type  $III_\lambda$  factor if  $M_2(\mathbb{C})$  is endowed with a nontracial state. Moreover we will show that the core ([Co]) of this type  $III_\lambda$  factor (when tensorized by  $B(H)$ ) is  $\mathcal{L}(F_\infty) \otimes B(H)$  and we will give an explicit model for the associated (trace scaling) action of  $\mathbb{Z}$  on the core (cf. [Co], [Ta]). Here  $B(H)$  is the space of all linear bounded operators on a separable, infinite dimensional Hilbert space  $H$ .

Recall from [Vo1], that a family  $(A_i)_{i \in I}$  of subalgebras in a von Neumann algebra  $M$  with state  $\phi$ , is free with respect to  $\phi$  if  $\phi(a_1 a_2 \dots a_k) = 0$  whenever

$$\phi(a_i) = 0, a_i \in A_{j_i}, i = 1, 2, \dots, k, j_1 \neq j_2, \dots, j_{k-1} \neq j_k.$$

Reciprocally given a family  $(A_i, \phi_i), i \in I$  of von Neumann algebras with faithful normal states  $\phi_i$ , one may construct (see [Vo1]) the (reduced) free product von Neumann algebra  $*A_i$ , which contains  $A_i, i \in I$  and has a faithful normal state  $\phi$  so that  $\phi|_{A_i} = \phi_i$  and so that the algebras  $(A_i)_{i \in I}$  are free with respect to  $\phi$ .

The aim of this paper is to show the following result.

**Theorem.** *Let  $\mathcal{E} = M_2(\mathbb{C}) * L^\infty([0, 1], \nu)$  be endowed with the free product state  $\phi$  where  $M_2(\mathbb{C})$  is endowed with the state  $\phi_0$  which is subject to the condition*

$$\phi_0(e_{11})/\phi_0(e_{22}) = \lambda \in (0, 1) \text{ and } \phi(e_{12}) = \phi(e_{21}) = 0,$$

*while  $L^\infty([0, 1], \nu)$  has the state given by Lebesgue measure on  $[0, 1]$ . With these hypothesis,  $M_2(\mathbb{C}) * L^\infty([0, 1], \nu)$  is a type  $III_\lambda$  factor and its core is isomorphic to  $\mathcal{L}(F_\infty) \otimes B(H)$ .*

In the proof of the theorem we will also obtain a model for the core of  $\mathcal{E} \otimes B(H)$  and for the corresponding (dual) action on the core, of the modular group of the weight  $\phi \otimes tr$  ( $tr$  is the canonical semifinite trace on  $B(H)$ ). This model will be a submodel of the one parameter action of  $\mathbb{R}_+/\{0\}$  on  $\mathcal{L}(F_\infty) \otimes B(H)$ , that we have constructed in [Ra].

**The model.** Model for the core of  $(M_2(\mathbb{C}) * L^\infty([0, 1], \nu)) \otimes B(H)$  and of the corresponding dual action on the core for the modular group of automorphism for the weight  $\phi \otimes \text{tr}$ :

Let  $\mathcal{A}_0$  be the subalgebra in the algebraic free product

$$L^\infty(\mathbb{R}) * (\mathbb{C}[X] * \mathbb{C}[Y])$$

generated by  $\{pXp, pYp, p\}$   $p$  finite projection in  $L^\infty(\mathbb{R})$  where  $L^\infty(\mathbb{R})$  is endowed with the Lebesgue measure.

Let  $\tau$  be the unique trace on  $\mathcal{A}_0$  defined by the requirement that the restriction  $\tau_p$  to the algebra generated in  $p\mathcal{A}p$  by  $pXp, pYp, pL^\infty(\mathbb{R})$  is subject to the following conditions:

(i) The three algebras generated respectively by  $pXp, pYp, pL^\infty(\mathbb{R})$  are free with respect to  $\tau_p$

(ii)  $\tau(p)^{-1/2}pXp, \tau(p)^{-1/2}pYp$  are semicircular (with respect  $\tau_p$ )(see [Vo1] for the definition of a semicircular element).

Such a construction is possible because of the Theorem 1 in [Ra].

Assume that  $pXp, pYp$  are selfadjoint and let  $\mathcal{A}$  be the weak completion of  $\mathcal{A}_0$  in the G.N.S. representation for  $\tau$ . Then (cf. [Ra]),  $\mathcal{A}$  is a type  $II_\infty$  factor isomorphic to  $\mathcal{L}(F_\infty) \otimes B(H)$  and the trace  $\tau$  extends to a semifinite normal trace on  $\mathcal{A}$  (which we also denote by  $\tau$ ).

Recall (by [Ra]) that in this case, there exists a one parameter group of automorphism  $(\alpha_t)_{t \in \mathbb{R}_+ \setminus \{0\}}$  on  $\mathcal{A}$ , scaling trace by  $t$ , for each  $t \in \mathbb{R}_+ \setminus \{0\}$ , which is induced by  $d_t * M_t$  on  $L^\infty(\mathbb{R}) * (\mathbb{C}[X] * \mathbb{C}[Y])$  where  $d_t$  is dilation by  $t$  on  $L^\infty(\mathbb{R})$ , while  $M_t(X) = t^{-1/2}X; M_t(Y) = t^{-1/2}Y, t > 0$ .

Let  $\mathcal{B}$  the von Neumann subalgebra of  $\mathcal{A}$  generated by

$$q_n = \chi_{[\lambda^{n-1}, \lambda^n]}, n \in \mathbb{Z},$$

the characteristic functions of the intervals  $[\lambda^{n-1}, \lambda^n]$  and by the following subsets of  $\mathcal{A}$ :

$$\tilde{X} = \{q_n X q_m | n, m \in \mathbb{Z}, |n - m| \leq 1\},$$

$$\tilde{Y} = \{q_n Y q_n | n \in \mathbb{Z}\}.$$

Clearly  $\mathcal{B}$  is invariant under  $\{\alpha_{\lambda^n}\}_{n \in \mathbb{Z}}$  and by Lemma 3 in [Ra],  $\mathcal{B}$  is isomorphic to  $\mathcal{L}(F_\infty) \otimes B(H)$ . Let  $\beta_n = \alpha_{\lambda^n}|_{\mathcal{B}}$ .

Let  $\mathcal{D} = \mathcal{B} \rtimes_{\beta} \mathbb{Z}$  be the cross product of  $\mathcal{B}$  by the action  $\mathbb{Z}$  given by  $\beta$ . Then by [Co],  $\mathcal{D}$  is a type  $III_\lambda$  factor. Let  $u \in \mathcal{D}$  be the unitary implementing the cross

product. Moreover let  $\psi$  be the normal semifinite faithful weight on  $\mathcal{D}$  obtained as the composition expectation from  $\mathcal{D}$  onto  $\mathcal{B}$ .

We will prove that  $\mathcal{B}$ , with the action of  $\mathbb{Z}$  given by  $(\beta_n)_{n \in \mathbb{Z}}$  is isomorphic to the core of  $\mathcal{E} \otimes B(H)$ , with the dual action (on the core) for the modular group of automorphisms of the weight  $\phi \otimes tr$  on  $\mathcal{E} \otimes B(H)$ . Our main result will be a consequence of the following proposition:

**Proposition.**

Let  $\mathcal{E}$  be the von Neumann algebra free product  $M_2(\mathbb{C}) * L^{\infty}([0, 1], \nu)$ , with the free product state  $\phi = \phi_0 * \nu$ , where  $M_2(\mathbb{C}) = (e_{ij})_{i,j=1}^2$  is endowed with the normalized state  $\phi_0$  with  $\phi(e_{11})/\phi(e_{22}) = \lambda$  and  $\phi(e_{12}) = \phi(e_{21}) = 0$ . Then, with the above notation  $\mathcal{E}$  is isomorphic to  $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$ .

Moreover the state  $\phi$  on  $\mathcal{E}$  is (via this identification) the (normalized) restriction of  $\psi$  to  $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$ .

(Here  $\mathcal{D} = \mathcal{B} \rtimes_{\beta} \mathbb{Z}$ , where  $\mathcal{B}$  is the von Neumann subalgebra in  $\mathcal{A}$  generated by  $\tilde{X} = \{q_n X q_m | n, m \in \mathbb{Z}, |n - m| \leq 1\}$ ,  $\tilde{Y} = \{q_n Y q_n | n \in \mathbb{Z}\}$  and the characteristic functions  $q_n = \chi_{[\lambda^{n-1}, \lambda^n]}$ ,  $n \in \mathbb{Z}$ ,  $q_n \in L^{\infty}(\mathbb{R}) \subseteq \mathcal{A}$ . Moreover  $\beta_n = \alpha_{\lambda^n}$ ,  $n \in \mathbb{Z}$ .)

Recall from above that the von Neumann algebra  $\mathcal{A}$  is a type II $_{\infty}$  factor isomorphic to  $\mathcal{L}(F_{\infty}) \otimes B(H)$  and  $\mathcal{A}$  is generated by

$$\{pXp, pYp, p | p \text{ finite projection in } L^{\infty}(\mathbb{R})\}.$$

Here  $\alpha_t, t > 0$  acts as dilation by  $t$  on  $L^{\infty}(\mathbb{R})$  and multiplies  $X, Y$  by  $t^{-1/2}$ . The trace on  $\mathcal{A}$  is subject to the above conditions (i), (ii) and it is scaled by the automorphisms  $\alpha_t, t > 0$ .

This proposition will be a consequence of the following two lemmas.

**Lemma 1.**

With  $\mathcal{A}, \mathcal{B}, \mathcal{D}, \psi, \tau$  and  $u$  as before let

$$e_{11} = q_1 u = u q_0; e_{11} = q_0; e_{22} = q_1$$

Let  $a = x + y$ , where

$$\begin{aligned} x &= (q_0 + q_1)X(q_0 + q_1) - q_0 X q_0 \\ y &= q_0 Y q_0. \end{aligned}$$

Then  $M_2(\mathbb{C}) = (e_{ij})_{i,j=1}^2$  is free with respect to

$$\psi_1 = (\psi(q_0 + q_1))^{-1} \psi|_{(q_0 + q_1)\mathcal{D}(q_0 + q_1)},$$

to the semicircular element  $a$ , in the algebra  $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$  with unit  $q_0 + q_1$ .

Proof. We have to check freeness, which means that the value of  $\psi_1$  on certain monomials in  $a, u, e_{11}, e_{22}$  is null. Since by definition,  $\psi_1$  vanishes the monomials containing a different number of  $u$ 's and  $u^*$ 's, we have only to check this if the number of occurrences for  $u$  is equal to the one for  $u^*$ .

Let  $p_n = q_n + q_{n+1} = \chi_{[\lambda^{n-1}, \lambda^{n+1}]}$ .

Using the fact that  $u$  implements  $\beta_1$  on  $\mathcal{D}$  it follows that we only have to check  $\psi_1(m) = 0$  if

$$m = p_0 f_1 q_{i_1} f_2 q_{i_2} f_3 \dots q_{i_n} f_{n+1} p_0$$

where the following conditions are fulfilled:

- (a)  $i_{j+1}$  is either  $i_j$  or  $i_j \pm 1$ .
- (b) Card  $\{s | i_j = s, j = 1, 2, \dots, n\}$  is even for every  $s$ .
- (c)  $f_k$  is a product

$$f_1^k a_1^k \dots f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k, \quad n_k \geq 1$$

where  $f_s^k$ ,  $s = 1, 2, 3 \dots n_k$ , is an element of null value under the state  $\psi_1$  in the algebra generated by  $\alpha_j(a)$  while  $a_s^k$  is an element of null trace in the algebra generated by  $q_j, q_{j+1}$ . Here  $j$  is an integer which is completely determined, for each  $k$ . If  $i_k \neq i_{k+1}$  then  $j$  is the minimum of the  $i_k$  and  $i_{k+1}$ . If  $i_k = i_{k+1}$  then  $j$  is either  $i_k$  if  $i_{k-1} \leq i_k$  or either  $i_k - 1$  if  $i_{k-1} > i_k$ .

To see that those are all the monomials of null state that may appear in the algebra generated by  $M_2(\mathbb{C})$  and  $a$  it is sufficient to note that any string

$$\begin{aligned} & f_1 e_{21} f_2 e_{21} \dots f_p e_{21} f_{p+1} e_{12} f_{p+2} e_{12} \dots e_{12} f_{2p+1} = \\ & = f_1(q_1 u) f_2 q_1 u \dots f_p q_1 u f_{p+1}(u^* q_1) \dots (u^* q_1) f_{2p+1}, \end{aligned}$$

after cancelation, is equal to

$$\begin{aligned} & f_1(q_1 u) f_2 \dots q_1 u f_p q_1 \beta_1(f_{p+1}) q_1 f_{p+2}(u^* q_1) \dots (u^* q_1) f_{2p+1} = \\ & = f_1(q_1 u) f_2 \dots q_1 \beta_1(f_p) q_2 \beta_2(f_{p+1}) q_2 \alpha_1(f_{p+2}) q_1 \dots (u^* q_1) f_{2p+1} = \\ & = f_1 q_1 \alpha_1(f_2) q_2 \dots \beta_{p-1}(f_p) q_p \beta_p(f_{p+1}) q_p \beta_{p-1}(f_{p+2}) \dots q_1 f_{2p+1} \end{aligned}$$

and similarly for a string in which each  $q_1 u$  is replaced by  $u^* q_1$  and conversely.

Here the  $f_i$ 's are products of the form  $f_1^i a_1^i f_2^i a_2^i \dots f_n^i$  where  $f_j^i$  are elements of null trace in the algebra generated by  $a = (q_0 + q_1)a(q_0 + q_1)$ , while  $a_j^i$  are elements of null trace in the algebra generated by  $q_0, q_1$ .

The monomials in the algebra generated by  $M_2(\mathbb{C})$  and  $a$  that are to be checked for having zero value under  $\psi_1$  are obtained by replacing certain  $f_j$  by other strings of this form, or by putting together such strings.

To show that the value of  $\psi_1(m)$  is zero we will use the following observation which is a consequence of Lemma 3.1 in [Vo2]. This observation will be used to replace the elements  $f_1, \dots, f_{n+1}$  in the monomial  $m$  by elements of null trace.

**Observation.** *Let  $B$  be a  $W^*$ -algebra with trace  $\tau$ , let  $X$  be a semicircular element and  $p$  a nontrivial projection that is free with  $X$ . Then any element of null trace in the algebra (with unit  $p$ ) generated by  $pXp$  is a sum of monomials which are products either of elements of null trace in the algebra generated by  $pXp$  or either of the form  $p - \tau(p)$ , but no such monomial is  $p - \tau(p)$  itself.*

*Proof.* Indeed if  $x$  is such an element then  $pxp = x$ , and moreover any other such monomial, which is different from  $p - \tau(p)$ , when multiplied with  $p$ , preserves the property of having null trace.

On the other hand

$$\tau(p(p - \tau(p))) = 1 - \tau(p) \neq 0.$$

This ends the proof of the observation.

To conclude the proof of Lemma 1 we let  $p$  a projection which is greater than the supremum of all the projections  $\{q_i | i \in I_m\}$  that are involved in  $m$ .

We may then assume by construction that we are given a finite family of semicircular elements  $z^j$  so that  $z^j = pz^j p$  and so that (modulo a multiplicative constant)  $\alpha_j(a) = (q_j + q_{j+1})z^j(q_j + q_{j+1})$  for  $j$  in  $I_m$ .

Using the above observation we may express  $f_k = f_1^k a_1^k \dots f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k$  as a sum of products of null trace in the algebras generated by  $\{q_j\}$  and  $\{z_j\}$  (adjacent elements are always in different algebras).

(Note that  $(q_j + q_{j+1})(q_j - \tau(q_j)(\tau(q_j + q_{j+1})))^{-1}(q_j + q_{j+1})$  has always null trace).

Again the above observation shows that each of these monomials must contain at least one term in  $z^j$ . Since consecutive  $f^i$  involve different elements in the set  $\{z^j\}$  it follows that  $\psi_1(m) = 0$ .

This ends the proof of Lemma 1.

**Lemma 2.** *With  $\mathcal{B}, \mathcal{D}$  as before we have that  $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$  coincides with the von Neumann subalgebra  $\mathcal{C} \subseteq (q_0 + q_1)\mathcal{D}(q_0 + q_1)$  (with unit  $q_0 + q_1$ ) that is generated by the monomials with an equal number of  $e_{12}$ 's and  $e_{21}$ 's.*

Proof. We have to show that the subalgebra  $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$  coincides with the subalgebra  $\mathcal{C} \subseteq (q_0 + q_1)(\mathcal{B} \rtimes_{\beta} \mathbb{Z})(q_0 + q_1) = (q_0 + q_1)\{\mathcal{B}, u\}''(q_0 + q_1)$  that is generated by monomials in  $a$  and  $(e_{ij})_{i,j=1}^2$  containing an equal number of  $e_{12}$  's and  $e_{21}$  's.

Clearly  $\mathcal{C}$  is invariant under the action of  $\mathbb{R}$  (or  $\mathbb{T}$ ) on  $\mathcal{D}$  given by the modular group of  $\psi$  which acts by  $\sigma_i^{\psi}(u) = \lambda^{it}u, \sigma_i^{\psi}|_{\mathcal{B}} = Id_{\mathcal{B}}$  so that  $\mathcal{C} \subseteq (q_0 + q_1)\mathcal{D}^{\mathbb{R}}(q_0 + q_1) = (q_0 + q_1)\mathcal{B}(q_0 + q_1)$ .

Hence we have to only prove the reverse inclusion. But due to the specific form of the generators in  $\mathcal{B}$ , we obtain that  $\mathcal{B}$  is generated by elements of the form

$$m = f^1 q_{i_1} f^2 q_{i_2} \dots f^n q_{i_n} f^{n+1}$$

where the conditions on  $i_1, \dots, i_n$  are

$$a) i_{j+1} \in \{i_j, i_j - 1, i_j + 1\}, j = 1, 2, \dots, n, i_0, i_n \in \{0, 1\}$$

$$b) \text{card } \{j | i_j = s\} \text{ is even,}$$

while  $f$  is one of the elements

$$\alpha_s(q_0 X q_0); \alpha_s(q_0 X q_1); \alpha(q_1 X q_0) \text{ or } \alpha_s(q_1 Y q_1),$$

where  $s$  is either  $i_j$  or  $i_{j+1}$  if  $i_j \neq i_{j+1}$ . If  $i_j = i_{j+1}$ , then either  $s = i_j$  and  $f^j = \alpha_s(q_0 X q_0)$  or either  $s = i_{j-1}$  and  $f^j = \alpha_s(q_1 Y q_1)$ .

The assumptions we made are sufficient to show that in such a monomial we have some symbols corresponding to  $\alpha_s(a)$  which are then necessary followed by symbols corresponding to  $\alpha_{s+1}(a)$  (or to  $\alpha_{s-1}(a)$ ). Moreover in  $m$  this sets of symbols are always separated by one of the projections  $q_p$  ( $p \in \{s, s \pm 1\}$ ).

If we replace in  $m$  any such  $q_p$  by  $q_1 u$  (or respectively by  $u^* q_1$ ) and we replace the symbols from  $\alpha_s(a)$  by the corresponding symbols in  $a$  we get the same  $m$ , but this time expressed as an element in the subalgebra of  $\mathcal{C}$ , generated by monomials with equal occurrence number of  $e_{12}$  's and  $e_{21}$  's. This ends the proof of Lemma 2.

To conclude the proof, we note the following observation:

**Remark.**

Let  $\mathcal{B}, \mathcal{D} = \mathcal{B} \rtimes_{\beta} \mathbb{Z}$  and  $u, \{q_i\}_{i \in \mathbb{Z}}$  be as before . Then  $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$  coincides with the algebra generated by  $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$  and  $e_{12} = q_1 u = u q_0$ .

Proof. With  $q_0, q_1$  as before we have to show that  $q_0(u^*)^n b q_0 = q_0(u^*)^n q_n b q_0$  is contained in the algebra generated by  $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$ . Assume  $n > 1$ ; we may express  $q_n b q_0$  as

$$q_n b_n q_{n-1} b_{n-1} q_{n-2} \dots q_1 b_1 q_0.$$

Then

$$\begin{aligned} q_0(u^*)^n b q_0 &= q_0(u^*)^n q_n b_n q_{n-1} \dots q_1 b_1 q_0 = \\ &= q_0 u^* \alpha_{n-1}(b_n) q_0 u^* \alpha_{n-2}(b_{n-1}) q_0 \dots q_0 u^* b_1 q_0 \end{aligned}$$

which is an element in the algebra generated by  $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$  and  $uq_0$ .

**Proof of the theorem.**

Clearly the subalgebra generated by  $e_{12}$  and all the elements in

$$M_2(\mathbb{C}) * L^\infty([0, 1], \nu)$$

with equal occurrence number of  $e_{12}$  's and  $e_{21}$  's coincides with the algebra itself. Thus  $M_2(\mathbb{C}) * L^\infty([0, 1], \nu)$  with the free product state  $\phi$  is identified with  $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$  with the restriction of  $\psi$  (which is generated by  $uq_0 = q_1u$ , and  $a$ ). In particular the modular group of  $\phi$  is  $\sigma_t^\phi(e_{ij}) = \lambda^{it} e_{ij}$  and  $\sigma_t^\phi$  is the identity on  $L^\infty([0, 1], \nu)$ .

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