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GEOMETRY OF q-HYPERGEOMETRIC FUNCTIONS, QUANTUM AFFINE ALGEBRAS AND ELLIPTIC QUANTUM GROUPS

V. Tarasov, A. Varchenko

Abstract. — The trigonometric quantized Knizhnik-Zamolodchikov (qKZ) equation associated with the quantum group $U_q(\mathfrak{sl}_2)$ is a system of linear difference equations with values in a tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules. We solve the equation in terms of multidimensional q-hypergeometric functions and define a natural isomorphism of the space of solutions and the tensor product of the corresponding evaluation Verma modules over the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ where parameters ρ and γ are related to the parameter q of the quantum group $U_q(\mathfrak{sl}_2)$ and the step p of the qKZ equation via $p = e^{2\pi i\rho}$ and $q = e^{-2\pi i\gamma}$.

We construct asymptotic solutions associated with suitable asymptotic zones and compute the transition functions between the asymptotic solutions in terms of the dynamical elliptic R-matrices. This description of the transition functions gives a connection between representation theories of the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$ and the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ and is analogous to the Kohno-Drinfeld theorem on the monodromy group of the differential Knizhnik-Zamolodchikov equation.

In order to establish these results we construct a discrete Gauss-Manin connection, in particular, a suitable discrete local system, discrete homology and cohomology groups with coefficients in this local system, and identify an associated difference equation with the qKZ equation.

Résumé. — L'équation de Knizhnik-Zamolodchikov (qKZ) trigonométrique quantifiée associée au groupe quantique $U_q(\mathfrak{sl}_2)$ est un système linéaire d'équations aux différences finies à valeurs dans un produit tensoriel de $U_q(\mathfrak{sl}_2)$-modules de Verma. Nous résolvons cette équation en terme de fonctions q-hypergéométriques multidimensionnelles et définissons un isomorphisme naturel entre l'espace des solutions et le produit tensoriel des modules de Verma d'évaluation correspondants sur le groupe quantique elliptique $E_{\rho,\gamma}(\mathfrak{sl}_2)$, les paramètres ρ et γ étant reliés aux paramètres q du groupe quantique elliptique $U_q(\mathfrak{sl}_2)$ et p de l'équation qKZ par les relations $p = e^{2\pi i\rho}$ et $q = e^{-2\pi i\gamma}$.

Nous construisons des solutions asymptotiques associées à des secteurs asymptotiques convenables et calculons les fonctions de transition entre les solutions asymptotiques en fonction des R-matrices elliptiques dynamiques. Cette description des fonctions de transition relie la théorie des représentations de l'algèbre de lacets quantique $U_q(\widetilde{\mathfrak{gl}_2})$ à celle du groupe quantique elliptique $E_{\rho,\gamma}(\mathfrak{sl}_2)$ et est analogue au théorème de Kohno-Drinfeld sur le groupe de monodromie de l'équation différentielle de Knizhnik-Zamolodchikov.

Pour établir ces résultats nous construisons une connexion de Gauss-Manin discrète, en particulier un système local discret convenable, des groupes d'homologie et de cohomologie à coefficients dans ce système local, et identifions une équation aux différences associée à ces données à l'équation qKZ.

Contents

1. Introduction

In this paper we solve the trigonometric quantized Knizhnik-Zamolodchikov (qKZ) equation associated with the quantum group $U_q(\mathfrak{sl}_2)$. The trigonometric qKZ equation associated with $U_q(\mathfrak{sl}_2)$ is a system of difference equations for a function $\Psi(z_1, \dots, z_n)$ with values in a tensor product $V_1 \otimes \dots \otimes V_n$ of $U_q(\mathfrak{sl}_2)$-modules. The system of equations has the form

$$\Psi(z_1, \dots, pz_m, \dots, z_n) = R_{m,m-1}(pz_m/z_{m-1}) \dots R_{m,1}(pz_m/z_1)\, \kappa^{-H_m} \times \\ \times R_{m,n}(z_m/z_n) \dots R_{m,m+1}(z_m/z_{m+1})\, \Psi(z_1, \dots, z_n),$$

$m = 1, \dots, n$, where p and κ are parameters of the qKZ equation, H is a generator of the Cartan subalgebra of $U_q(\mathfrak{sl}_2)$, H_m is H acting in the m-th factor, $R_{l,m}(x)$ is the trigonometric R-matrix $R_{V_l V_m}(x) \in \operatorname{End}(V_l \otimes V_m)$ acting in the l-th and m-th factors of the tensor product of $U_q(\mathfrak{sl}_2)$-modules. In this paper we consider only the steps p with absolute value less than 1.

The qKZ equation is an important system of difference equations. The qKZ equations had been introduced in [FR] as equations for matrix elements of vertex operators of the quantum affine algebra. An important special case of the qKZ equation had been introduced earlier in [S] as equations for form factors in massive integrable models of quantum field theory; relevant solutions of these equations had been given therein. Later the qKZ equations were derived as equations for correlation functions in lattice integrable models, cf. [JM] and references therein.

In the quasiclassical limit the qKZ equation turns into the differential Knizhnik-Zamolodchikov equation for conformal blocks of the Wess-Zumino-Witten model of conformal field theory on the sphere.

Asymptotic solutions of the qKZ equation as the step p tends to 1 are closely related to diagonalization of the transfer-matrix of the corresponding lattice integrable model by the algebraic Bethe ansatz method [TV2].

We describe the space of solutions of the qKZ equation in terms of representation theory. Namely, we consider the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ with parameters ρ and γ defined by $p = e^{2\pi i\rho}$, $q = e^{-2\pi i\gamma}$ and the $E_{\rho,\gamma}(\mathfrak{sl}_2)$ modules $V_1^e(z_1), \dots, V_n^e(z_n)$ where $V_m^e(z_m)$ is the evaluation Verma module over $E_{\rho,\gamma}(\mathfrak{sl}_2)$ which corresponds to the $U_q(\mathfrak{sl}_2)$-module V_m. Notice that as a vector space the evaluation Verma module $V_m^e(z_m)$ does not depend on z_m. For every permutation $\tau \in \mathbf{S}^n$ we consider the tensor product $V_{\tau_1}^e \otimes \dots \otimes V_{\tau_n}^e$ and establish a natural isomorphism of the space $\mathbb{S}$ of solutions of the qKZ equation with values in $V_1 \otimes \dots \otimes V_n$ and the space $V_{\tau_1}^e \otimes \dots \otimes V_{\tau_n}^e \otimes \mathbb{F}$, where $\mathbb{F}$ is the space of functions of $z_1, \dots, z_n$ which are p-periodic with respect to each of the variables,

$$\mathbf{C}_\tau : V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes \mathbb{F} \to \mathbb{S},$$

cf. (5.32). Notice that if $\Psi(z)$ is a solution of the qKZ equation and $F(z)$ is a p-periodic function, then also $F(z)\Psi(z)$ is a solution of the qKZ equation.

We call the isomorphisms $\mathbf{C}_\tau$ the tensor coordinates on the space of solutions. The compositions of the isomorphisms are linear maps

$$\mathbf{C}_{\tau,\tau'}(z_1,\ldots,z_n) : V^e_{\tau'_1} \otimes \ldots \otimes V^e_{\tau'_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n}$$

depending on $z_1,\ldots,z_n$ and p-periodic with respect to all variables. We call these compositions the transition functions. It turns out that the transition functions are defined in terms of the elliptic R-matrices

$$R^{ell}_{V^e_l V^e_m}(x,\lambda) \in \mathrm{End}\,(V^e_l \otimes V^e_m)$$

acting in tensor products of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules. Namely, for any permutation τ and for any transposition $(m,m+1)$ the transition function

$$\mathbf{C}_{\tau,\tau\cdot(m,m+1)}(z_1,\ldots,z_n) : V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_{m+1}} \otimes V^e_{\tau_m} \otimes \ldots \otimes V^e_{\tau_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n}$$

equals the operator

$$P_{V^e_{\tau_{m+1}} V^e_{\tau_m}} R^{ell}_{V^e_{\tau_{m+1}} V^e_{\tau_m}}\big(z_{\tau_{m+1}}/z_{\tau_m}, (\eta^H \otimes \ldots \otimes \eta^H \otimes \underset{m\text{-th}}{\eta^{-H}} \otimes \ldots \otimes \eta^{-H})\eta^{-1}\kappa\big)$$

twisted by certain adjusting maps, cf. (5.34) and Theorem 4.16. Here $P_{V^e_l V^e_m}$ is the transposition of the tensor factors.

We consider asymptotic zones $|z_{\tau_m}/z_{\tau_{m+1}}| \ll 1$, $m = 1,\ldots,n-1$, labelled by permutations $\tau \in \mathbf{S}^n$. For every asymptotic zone we define a basis of asymptotic solutions of the qKZ equation. We show that for every permutation τ the basis of the corresponding asymptotic solutions is the image of the standard monomial basis in $V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n}$ under the map

$$V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes 1 \hookrightarrow V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes \mathbb{F} \xrightarrow{C_\tau} \mathbb{S},$$

cf. Theorem 6.2. The last two statements express the transition functions between the asymptotic solutions via the elliptic R-matrices.

The trigonometric R-matrix $R_{V_l V_m}(x) \in \mathrm{End}\,(V_l \otimes V_m)$ is defined in terms of the action of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ in the tensor product of $U_q(\mathfrak{sl}_2)$-modules. The quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra which contains the quantum group $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra and has a family of homomorphisms $U'_q(\widetilde{\mathfrak{gl}_2}) \to U_q(\mathfrak{sl}_2)$ depending on a parameter. Therefore, each $U_q(\mathfrak{sl}_2)$-module V_m carries a $U'_q(\widetilde{\mathfrak{gl}_2})$-module structure $V_m(x)$ depending on a parameter. For Verma modules V_l, V_m over $U_q(\mathfrak{sl}_2)$ the quantum loop algebra modules $V_l(x)\otimes V_m(y)$ and $V_m(y)\otimes V_l(x)$ are isomorphic for generic x,y. Moreover, for irreducible $U_q(\mathfrak{sl}_2)$-modules V_l, V_m the quantum loop algebra modules $V_l(x)\otimes V_m(y)$ and $V_m(y)\otimes V_l(x)$ are irreducible and isomorphic for generic x,y. The map

$$P_{V_lV_m}R_{V_lV_m}(x/y) : V_l(x)\otimes V_m(y) \to V_m(y)\otimes V_l(x)$$

is the unique suitably normalized intertwiner [T], [CP].

The elliptic R-matrix $R^{ell}_{V^e_lV^e_m}(x,\lambda)\in \mathrm{End}(V^e_l\otimes V^e_m)$ is defined in terms of the action of the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ in the tensor product of evaluation Verma modules. For evaluation Verma modules $V^e_l(x)$, $V^e_m(y)$ over $E_{\rho,\gamma}(\mathfrak{sl}_2)$, the $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules $V^e_l(x)\otimes V^e_m(y)$ and $V^e_m(y)\otimes V^e_l(x)$ are isomorphic for generic x,y. The map

$$P_{V^e_lV^e_m}R^{ell}_{V^e_lV^e_m}(x/y,\lambda) : V^e_l(x)\otimes V^e_m(y) \to V^e_m(y)\otimes V^e_l(x)$$

is the unique suitably normalized intertwiner [F], [FV].

Our result on the transition functions between asymptotic solutions together with the indicated construction of R-matrices shows that the *qKZ* equation establishes a connection between representation theories of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ and the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$. Our result is analogous to the Kohno-Drinfeld theorem on the monodromy group of the differential Knizhnik-Zamolodchikov equation [K], [D2]. The Kohno-Drinfeld theorem establishes a connection between representation theories of a Lie algebra and the corresponding quantum group, see [D2]. Using the ideas of the Kohno-Drinfeld result it was proved in [KL] that the category of representations of a quantum group is equivalent to a suitably defined fusion category of representations of the corresponding affine Lie algebra. Similarly to the Kazhdan-Lusztig theorem one could expect that our result for the difference *qKZ* equation could be a base for a Kazhdan-Lusztig type result connecting certain categories of representations of quantum affine algebras and elliptic quantum groups, cf. [KS].

In this paper we consider the trigonometric *qKZ* equation. There are other types of the *qKZ* equation: the rational *qKZ* equation [FR] and the elliptic *qKZB* equation [F]. Here *KZB* stands for Knizhnik-Zamolodchikov-Bernard, and the difference *qKZB* equation is a discretization of the differential *KZB* equation for conformal blocks on the torus.

The rational *qKZ* equation was considered in [TV3]. It is a system of difference equations analogous to the trigonometric *qKZ* equation in which the role of the trigonometric R-matrix is played by the rational R-matrix defined in terms of the Yangian representations. In [TV3] we solved the rational *qKZ* equation in terms of multidimensional hypergeometric integrals of Mellin-Barnes type, introduced asymptotic zones and described transition functions between asymptotic solutions in terms of trigonometric R-matrices, thus showing that the rational *qKZ* equation gives a connection between representations of Yangians and quantum affine algebras.

The elliptic *qKZB* equation is an analogue of the trigonometric *qKZ* equation in which the role of the trigonometric R-matrix is played by the dynamical elliptic R-matrix [F]. The dynamical elliptic R-matrix is a matrix acting in the tensor product of two evaluation modules over the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$, and the elliptic quantum group is an elliptic analog of the quantum affine algebra $U_q(\widetilde{\mathfrak{sl}_2})$ associated with $\mathfrak{sl}_2$ [F], [FV]. The elliptic quantum group depends on two parameters: an elliptic curve $\mathbb{C}/(\mathbb{Z}+\rho\mathbb{Z})$ and Planck's constant γ. The elliptic *qKZB* equation is considered in [FTV1], [FTV2] where we solve it in terms of multidimensional elliptic q-hypergeometric integrals. We formulate and solve in [FTV2] a "monodromy" problem for the *qKZB* equation analogous to the problem of description of the transition functions for the rational and trigonometric *qKZ* equations. We describe in [FTV2] the "monodromy" matrices of the elliptic *qKZB* equation associated with the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ in terms of the elliptic R-matrix associated with the elliptic quantum group $E_{\alpha,\gamma}(\mathfrak{sl}_2)$ where α is the step of the initial *qKZB* difference equation, thus showing the symmetric role of the elliptic curves $\mathbb{C}/(\mathbb{Z}+\rho\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z}+\alpha\mathbb{Z})$ in the story.

In this paper, in order to establish a connection between representation theories of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ and the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ we define a discrete analogue of a locally trivial bundle and a local system on the space of bundle. We define a discrete analogue of the Gauss-Manin connection for the discrete locally trivial bundle with a discrete local system and consider the corresponding difference equation. We identify that difference Gauss-Manin equation with the difference *qKZ* equation. To realize this idea we introduce a suitable discrete de Rham complex and its cohomology group in the spirit of [A], then we define the homology group as the dual space to the cohomology group and construct a family of discrete cycles, elements of the discrete homology group, using ideas of [S]. We construct the space of discrete cycles as a certain space of functions. Having a representative of a discrete cohomology class (a function) and a discrete cycle (a function again) we define the pairing (the hypergeometric pairing) between the cohomology class and the cycle as an integral of their product with a certain fixed "hypergeometric phase function" over a certain fixed contour of the middle dimension. We show that there are enough discrete cycles and they form the space dual to the quotient space of the space of our discrete closed forms modulo discrete coboundaries. To prove this we compute the determinant of the period matrix and get an explicit formula (5.9) for the determinant analogous to the determinant formulae for the hypergeometric functions of Mellin-Barnes type [TV3] and for the "continuous" hypergeometric functions [V1], cf. Loeser's determinant formula for the Frobenius transformation [L].

The form of our discrete cycles suggests a natural identification of the space of our discrete cycles with a tensor product of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules and this identification allows us to prove the result on transition functions between asymptotic solutions.

The paper is organized as follows. Chapters 2 – 6 contain constructions and statements and Chapter 7 contains proofs. We give necessary prelimimnaries, proofs of technical results and some applications in Appendices.

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2. Discrete flat connections and local systems

In this chapter we recall basic notions introduced in [TV3].

Discrete flat connections

Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Consider a complex vector space $\mathbb{C}^n$ called the *base space*. Let $\mathbb{C}^{\times n}$ be the complement of the coordinate hyperplanes in the base space. Fix a complex number p, such that $p \neq 0, 1$, which is called the *step*. The lattice $\mathbb{Z}^n$ acts on the base space by dilations:

$$l : (z_1, \dots, z_n) \mapsto (p^{l_1} z_1, \dots, p^{l_n} z_n), \qquad l \in \mathbb{Z}^n .$$

Let $\mathbb{B} \subset \mathbb{C}^{\times n}$ be an invariant subset of the base space. Say that there is a *bundle with a discrete connection* over $\mathbb{B}$ if for any $z \in \mathbb{B}$ there are a vector space $V(z)$ and linear isomorphisms

$$A_m(z_1, \dots, z_n) : V(z_1, \dots, pz_m, \dots, z_n) \to V(z_1, \dots, z_n), \qquad m = 1, \dots, n .$$

The connection is called *flat* (or *integrable*) if the isomorphisms $A_1, \dots, A_n$ commute:

$$(2.1) \qquad A_l(z_1, \dots, z_n)\, A_m(z_1, \dots, pz_l, \dots, z_n) = \\ = A_m(z_1, \dots, z_n)\, A_l(z_1, \dots, pz_m, \dots, z_n) .$$

Say that a *discrete subbundle* in $\mathbb{B}$ is given if a subspace in every fiber is distinguished and the family of subspaces is invariant with respect to the connection.

A section $s : z \mapsto s(z)$ is called *periodic* (or *horizontal*) if its values are invariant with respect to the connection:

$$(2.2) \qquad A_m(z_1, \dots, z_n)\, s(z_1, \dots, pz_m, \dots, z_n) = s(z_1, \dots, z_n),$$

$m = 1, \dots, n$. A function $f(z_1, \dots, z_n)$ on the base space is called a *quasiconstant* if

$$f(z_1, \dots, pz_m, \dots, z_n) = f(z_1, \dots, z_n), \qquad m = 1, \dots, n .$$

Periodic sections form a module over the ring of quasiconstants.

The *dual bundle* with the *dual connection* has fibers $V^*(z)$ and isomorphisms

$$A^*_m(z_1, \dots, z_n) : V^*(z_1, \dots, z_n) \to V^*(z_1, \dots, pz_m, \dots, z_n) .$$

Let $s_1, \dots, s_N$ be a basis of sections of the initial bundle. Then the isomorphisms A_m of the connection are given by matrices $\mathrm{A}^{(m)}$:

$$A_m(z_1, \dots, z_n)\, s_k(z_1, \dots, pz_m, \dots, z_n) \;=\; \sum_{l=1}^{N} \mathrm{A}^{(m)}_{kl}(z_1, \dots, z_n)\, s_l(z_1, \dots, z_n)\,.$$

For any section $\psi : z \mapsto \psi(z)$ of the dual bundle denote by $\Psi : z \mapsto \Psi(z)$ its coordinate vector, $\Psi_k(z) = \langle \psi(z), s_k(z) \rangle$. The section ψ is periodic if and only if its coordinate vector satisfies the system of difference equations

$$\Psi(z_1, \dots, pz_m, \dots, z_n) \;=\; \mathrm{A}^{(m)}(z_1, \dots, z_n)\, \Psi(z_1, \dots, z_n)\,, \qquad m = 1, \dots, n\,.$$

This system of difference equations is called the *periodic section equation.*

Say that functions $\varphi_1, \dots, \varphi_n$ in variables $z_1, \dots, z_n$ form a *system of connection coefficients* if

$$\varphi_l(z_1, \dots, pz_m, \dots, z_n)\, \varphi_m(z_1, \dots, z_n) \;=\; \\ =\; \varphi_m(z_1, \dots, pz_l, \dots, z_n)\, \varphi_l(z_1, \dots, z_n)$$

for all $l, m \;=\; 1, \dots, n$. These functions define a connection on the trivial complex one-dimensional vector bundle.

The system of connection coefficients is called *decomposable* if it has the form

$$\varphi_m(z_1, \dots, z_n) \;=\; \kappa_m \Big[\prod_{1 \leqslant l < m} \phi_{lm}(p^{-1} z_l / z_m) \Big]^{-1} \prod_{m < l \leqslant n} \phi_{ml}(z_m / z_l)\,,$$

$m = 1, \dots, n$, for certain functions ϕ_{lm}, $l < m$, in one variable and nonzero complex numbers κ_m. The functions ϕ_{lm} are called *primitive factors* and κ_m are called *scaling parameters*.

A function $\bar{\Phi}(z_1, \dots, z_n)$ is called a *phase function* of a system of connection coefficients if

$$\bar{\Phi}(z_1, \dots, pz_m, \dots, z_n) \;=\; \varphi_m(z_1, \dots, z_n)\, \bar{\Phi}(z_1, \dots, z_n)\,, \qquad m = 1, \dots, n\,.$$

Similarly, a function $\varPhi(x)$ is called a *phase function* of a function $\phi(x)$ in one variable if $\varPhi(px) = \phi(x)\varPhi(x)$. Notice that the phase functions are not unique.

For any function $f(z_1,\dots,z_n)$ define new functions $Q_1 f,\dots,Q_n f$ and $D_1 f,\dots,D_n f$ by the rule:

$$(Q_m f)(z_1,\dots,z_n) = \varphi_m(z_1,\dots,z_n)\, f(z_1,\dots,pz_m,\dots,z_n)\,,$$

and $D_m f = Q_m f - f$, $m = 1,\dots,n$. The functions $D_1 f,\dots,D_n f$ are the *discrete partial derivatives* of the function f. We have that $D_l D_m f = D_m D_l f$ for any $l, m = 1,\dots,n$.

Let F be a vector space of functions of $z_1,\dots,z_n$ such that the operators $Q_1,\dots,Q_n$ induce linear isomorphisms of F:

$$Q_m : \mathrm{F} \to \mathrm{F}\,.$$

Say that the space F and the connection coefficients $\varphi_1,\dots,\varphi_n$ form a one-dimensional *discrete local system* on $\mathbb{C}^{\times n}$. F is called the *functional space* of the local system.

Define the *de Rham complex* $\big(\Omega^\bullet(\mathrm{F}), D\big)$ of the local system in a standard way. Namely, set

$$\Omega^a = \Big\{\omega = \sum_{k_1,\dots,k_a} f_{k_1,\dots,k_a}\, Dz_{k_1} \wedge \dots \wedge Dz_{k_a}\Big\}$$

where $Dz_1,\dots,Dz_n$ are formal symbols and the coefficients $f_{k_1,\dots,k_a}$ belong to F. Define the differential of a function by $Df = \sum_{m=1}^{n} D_m f\, Dz_m$, and the differential of a form by

$$D\omega = \sum_{k_1,\dots,k_a} Df_{k_1,\dots,k_a} \wedge Dz_{k_1} \wedge \dots \wedge Dz_{k_a}\,.$$

The cohomology groups $H^1,\dots,H^n$ of this complex are called the *cohomology groups of $\mathbb{C}^{\times n}$ with coefficients in the discrete local system.* In particular, the top cohomology group is $H^n = \mathrm{F}/D\mathrm{F}$ where $D\mathrm{F} = \sum_{m=1}^{n} D_m \mathrm{F}$. The dual spaces $H_a = (H^a)^*$ are called the *homology groups.*

Discrete Gauss-Manin connection

There is a geometric construction of bundles with discrete flat connections, a discrete version of the Gauss-Manin connection construction.

Let $\pi : \mathbb{C}^{\ell+n} \to \mathbb{C}^n$ be an affine projection onto the base with fiber $\mathbb{C}^\ell$. $\mathbb{C}^{\ell+n}$ will be called the *total space*. Let $z_1,\dots,z_n$ be coordinates on the base, $t_1,\dots,t_\ell$ coordinates on the fiber, so that $t_1,\dots,t_\ell$, $z_1,\dots,z_n$ are coordinates on the total space. When it is convenient, we denote the coordinates $z_1, \dots,$ z_n by $t_{\ell+1}, \dots, t_{\ell+n}$.

Let F, $\varphi_1, \dots, \varphi_{\ell+n}$ be a local system on $\mathbb{C}^{\times(\ell+n)}$. For a point $z \in \mathbb{C}^{\times n}$ define a local system $\mathrm{F}(z)$, $\varphi_a(\cdot; z)$, $a = 1, \dots, \ell$, on the fiber over z. Set

$$\mathrm{F}(z) = \{f\big|_{\pi^{-1}(z)} \mid f \in \mathrm{F}\} \qquad \text{and} \qquad \varphi_a(\cdot; z) = \varphi_a\big|_{\pi^{-1}(z)}.$$

The de Rham complex, cohomology and homology groups of the fiber are denoted by $\big(\Omega^\bullet(z), D(z)\big)$, $H^a(z)$ and $H_a(z)$, respectively.

There is a natural homomorphism of the de Rham complexes

$$(\Omega^\bullet(\mathbb{C}^{\times(\ell+n)}, \mathrm{F}), D) \to (\Omega^\bullet(z), D(z)), \qquad \omega \mapsto \omega|_{\pi^{-1}(z)},$$

where the restriction of a form is defined in a standard way: all symbols Dz_1, $\dots, Dz_n$ are replaced by zero and all coefficients of the remaining monomials $Dt_{k_1} \wedge \dots \wedge Dt_{k_a}$ are restricted to the fiber.

For a fixed a the vector spaces $H^a(z)$ form a bundle with a discrete flat connection. The linear maps

$$A_m(z_1, \dots, z_n) : H^a(z_1, \dots, pz_m, \dots, z_n) \to H^a(z_1, \dots, z_n)$$

are defined as follows. Define $Q_m : \Omega^a(\mathbb{C}^{\times(\ell+n)}, \mathrm{F}) \to \Omega^a(\mathbb{C}^{\times(\ell+n)}, \mathrm{F})$ by

$$\omega \mapsto \sum_{k_1, \dots, k_a} Q_m f_{k_1, \dots, k_a} Dz_{k_1} \wedge \dots \wedge Dz_{k_a}.$$

Then Q_m induces a homomorphism of the de Rham complexes

$$\big(\Omega^\bullet(z_1, \dots, pz_m, \dots, z_n), D(z_1, \dots, pz_m, \dots, z_n)\big) \to$$
$$\to \big(\Omega^\bullet(z_1, \dots, z_n), D(z_1, \dots, z_n)\big).$$

We set $A_m(z_1, \dots, z_n)$ to be equal to the induced map of the cohomology spaces. This connection is called the *discrete Gauss-Manin connection*.

The Gauss-Manin connection on the cohomological bundle induces the dual flat connection on the homological bundle:

$$A_m^*(z_1, \dots, z_n) : H_a(z_1, \dots, z_n) \to H_a(z_1, \dots, pz_m, \dots, z_n).$$

Connection coefficients of local systems

In this paper we study the Gauss-Manin connection for a class of local systems with decomposable connection coefficients, namely for trigonometric $\mathfrak{sl}_2$-type local systems [TV3], for the rational case see [TV3] and for the elliptic case see [FTV1], [FTV2].

Primitive factors and scaling parameters of a trigonometric $\mathfrak{sl}_2$-type local system have the following form:

$$\begin{aligned}
\phi_{ab}(x) &= \frac{x-\eta}{\eta x - 1} && \text{for} \quad a < b \leqslant \ell\,, \\
\phi_{ab}(x) &= \frac{\xi_{b-\ell} x - 1}{x - \xi_{b-\ell}} && \text{for} \quad a \leqslant \ell < b\,, \\
\phi_{ab}(x) &= 1 && \text{for} \quad \ell < a < b\,. \\
\kappa_a &= \kappa && \text{for} \quad a \leqslant \ell\,, \\
\kappa_a &= 1 && \text{for} \quad \ell < a\,.
\end{aligned}$$

Such a system of connection coefficients depends on $n+2$ nonzero complex numbers $\xi_1, \dots, \xi_n, \eta, \kappa$. The connection coefficients of a trigonometric $\mathfrak{sl}_2$ type local system have the form

$$(2.3) \qquad \varphi_a(t,z) = \kappa \prod_{m=1}^{n} \frac{\xi_m t_a - z_m}{t_a - \xi_m z_m} \prod_{a<b\leqslant \ell} \frac{t_a - \eta t_b}{\eta t_a - t_b} \prod_{1\leqslant b<a} \frac{p t_a - \eta t_b}{p\eta t_a - t_b}\,, \qquad a = 1, \dots, \ell\,,$$

$$\varphi_{\ell+m}(t,z) = \prod_{a=1}^{\ell} \frac{t_a - p\xi_m z_m}{\xi_m t_a - p z_m}\,, \qquad m = 1, \dots, n\,.$$

Let $\Phi(x;\alpha)$ be a phase function of the function $(x\alpha - 1)/(x\alpha^{-1} - 1)$. Then a phase function of the system of connection coefficients is given by

$$(2.4) \qquad \bar{\Phi}(t_1, \dots, t_\ell, z_1, \dots, z_n) = \\ = f(t_1, \dots, t_\ell, z_1, \dots, z_n)\, \Phi(t_1, \dots, t_\ell, z_1, \dots, z_n)$$

where

$$(2.5) \qquad \Phi(t_1, \dots, t_\ell, z_1, \dots, z_n) = \prod_{m=1}^{n} \prod_{a=1}^{\ell} \Phi(t_a/z_m; \xi_m) \prod_{1\leqslant a<b\leqslant \ell} \Phi(t_a/t_b; \eta^{-1})$$

and $f(t_1,\dots,t_\ell,z_1,\dots,z_n)$ is an arbitrary function such that

$$f(t_1,\dots,pt_a,\dots,t_\ell,z_1,\dots,z_n) = \tag{2.6}$$

$$= \kappa\eta^{\ell-2a+1}\prod_{m=1}^{n}\xi_m^{-1}\, f(t_1,\dots,t_\ell,z_1,\dots,z_n)\,,$$

$$f(t_1,\dots,t_\ell,z_1,\dots,pz_m,\dots,z_n) = \xi_m^{\ell}\, f(t_1,\dots,t_\ell,z_1,\dots,z_n)\,.$$

Later in this chapter we will describe a convenient space of such functions called the elliptic hypergeometric space.

The function $\Phi(t_1,\dots,t_\ell,z_1,\dots,z_n)$ will be called a *short phase function.*

Example. Let $0<|p|<1$. Let $(u)_\infty = \prod_{k=0}^{\infty}(1-p^k u)$. We can take

$$\Phi(x;\alpha) = \frac{(x\alpha^{-1})_\infty}{(x\alpha)_\infty}\,.$$

Then the phase function $\Phi(x;\alpha)$ has a symmetry property

$$\Phi(-x;\alpha) = \Phi(x;\alpha)\,\frac{(x\alpha-1)}{(x-\alpha)}\,\frac{\theta(x\alpha)}{\alpha\theta(x/\alpha)}$$

where $\theta(u)=(u)_\infty(pu^{-1})_\infty(p)_\infty$ is the Jacobi theta-function. The symmetry property for $\Phi(x;\alpha)$ leads to a symmetry property

$$\Phi(t_1,\dots,t_{a+1},t_a,\dots,t_\ell,z_1,\dots,z_n) = \tag{2.7}$$

$$= \Phi(t_1,\dots,t_\ell,z_1,\dots,z_n)\,\frac{(t_a-\eta t_{a+1})}{(\eta t_a - t_{a+1})}\,\frac{\eta\theta(\eta^{-1}t_a/t_{a+1})}{\theta(\eta t_a/t_{a+1})}$$

of the short phase function of the system of connection coefficients. This property motivates definitions (2.9) and (2.29) of certain actions of the symmetric group.

The functional space of a trigonometric $\mathfrak{sl}_2$-type local system

Define the functional space $\widehat{\mathcal{F}}$ of a trigonometric $\mathfrak{sl}_2$-type local system as the space of rational functions on the total space $\mathbb{C}^{\ell+n}$ with at most simple poles at the following hyperplanes:

$$t_a = p^{s+1}\xi_m^{-1}z_m\,,\qquad t_a = p^{-s}\xi_m z_m\,,\qquad t_a = p^{s+1}\eta t_b\,,\qquad t_b = p^s\eta t_a\,, \tag{2.8}$$

$1 \leqslant a < b \leqslant \ell$, $m = 1,\dots,n$, $s \in \mathbb{Z}_{\geqslant 0}$, and any poles at the coordinate hyperplanes $t_a = 0$, $a = 1,\dots,\ell$, and $z_m = 0$, $m = 1,\dots,n$. It is easy to check that the functional space is invariant with respect to all the shift operators $Q_1^{\pm 1},\dots,Q_n^{\pm 1}$.

Define an action of the symmetric group $\mathbf{S}^\ell$ on the functional space:

$$(2.9)\qquad \sigma : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}, \qquad f \mapsto [f]_\sigma, \qquad \sigma \in \mathbf{S}^\ell,$$

$$[f]_\sigma(t_1,\dots,t_\ell,z_1,\dots,z_n) = f(t_{\sigma_1},\dots,t_{\sigma_\ell},z_1,\dots,z_n) \prod_{\substack{1\leqslant a<b\leqslant \ell \\ \sigma_a>\sigma_b}} \frac{t_{\sigma_b} - \eta t_{\sigma_a}}{\eta t_{\sigma_b} - t_{\sigma_a}}\,.$$

The operators $Q_1,\dots,Q_{\ell+n}$ and $D_1,\dots,D_{\ell+n}$ commute with the action of the symmetric group.

We extend the $\mathbf{S}^\ell$-action to the de Rham complex assuming that it respects the exterior product and

$$\sigma : Dt_a \mapsto Dt_{\sigma_a}\,, \qquad \sigma : Dz_m \mapsto Dz_m\,, \qquad \sigma \in \mathbf{S}^\ell.$$

The same formulae define an action of the symmetric group on the de Rham complex of a fiber. The homomorphism of the restriction of the de Rham complex of the total space to the de Rham complex of a fiber commutes with the action of the symmetric group. The action of the symmetric group induces an action of the symmetric group on the homology and cohomology groups. The Gauss-Manin connection commutes with this action.

If a symmetric group acts on a vector space V, we denote by V_Σ the subspace of invariant vectors and by V_A the subspace of skew-invariant vectors.

In this paper we are interested in the skew-invariant part $H^\ell_A(z)$ of the top cohomology group of a fiber. This subspace is generated by forms $f\,Dt_1 \wedge \dots \wedge Dt_\ell$ where f runs through the space $\widehat{\mathcal{F}}_\Sigma(z)$ of invariant functions.

Introduce an important *trigonometric hypergeometric space* $\mathcal{F} \subset \widehat{\mathcal{F}}_\Sigma$ as the subspace of functions of the form

$$(2.10)\quad P(t_1,\dots,t_\ell,z_1,\dots,z_n) \prod_{a=1}^{\ell} t_a \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{1}{t_a - \xi_m z_m} \prod_{1\leqslant a<b\leqslant \ell} \frac{t_a - t_b}{\eta t_a - t_b}$$

where P is a polynomial with complex coefficients which is symmetric in variables $t_1,\dots,t_\ell$ and has degree less than n in each of the variables t_1, $\dots,t_\ell$. The restriction of the trigonometric hypergeometric space to a fiber defines the *trigonometric hypergeometric space* $\mathcal{F}(z) \subset \widehat{\mathcal{F}}_\Sigma(z)$ of the fiber which is a complex finite-dimensional vector space. If required, we will write down explicitly dependence of the trigonometric hypergeometric spaces on z_1, $\dots, z_n$, $\xi_1, \dots, \xi_n$ and ℓ, that is

$$\mathcal{F} = \mathcal{F}[z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell]$$

and

$$\mathcal{F}(z) = \mathcal{F}[z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell](z) .$$

A form $f\,Dt_1 \wedge \ldots \wedge Dt_\ell$ with the coefficient f in the trigonometric hypergeometric space of a fiber is called a *hypergeometric form*. The subspace $\mathcal{H}(z) \subset H^\ell_A(z)$ of the top cohomology group of a fiber generated by the hypergeometric forms is called the *hypergeometric cohomology group*.

The union of the hyperplanes

$$\xi_l\,\xi_m z_l/z_m = p^s\eta^r , \qquad r = 0, \dots, \ell-1 , \qquad s \in \mathbb{Z} , \tag{2.11}$$

$l, m = 1, \dots, n$, $l \neq m$, in the base space $\mathbb{C}^n$ is called the *discriminant*. The complement to the discriminant in $\mathbb{C}^{\times n}$ will be denoted by $\mathbb{B}$.

(2.12) Theorem. [V3], [TV1] *The family of subspaces $\{\mathcal{H}(z)\}_{z\in\mathbb{B}}$ is invariant with respect to the Gauss-Manin connection and, therefore, defines a discrete subbundle.*

This subbundle is called the *hypergeometric subbundle*.

Later on we make the following assumptions. We always assume that the step p is such that $0 < |p| < 1$, and the parameters η, κ, $\xi_1, \dots, \xi_n$, z_1, $\dots, z_n$ are nonzero. We often assume that the parameter η is such that

$$\eta^r \neq p^s , \qquad r = 1, \dots, \ell , \qquad s \in \mathbb{Z} , \tag{2.13}$$

the parameters $\xi_1, \dots, \xi_n$ are such that

$$\xi_m^2 \neq p^s\eta^r , \qquad m = 1, \dots, n , \qquad r = 1-\ell, \dots, \ell-1 , \qquad s \in \mathbb{Z} , \tag{2.14}$$

and the coordinates $z_1, \dots, z_n$ obey the condition

$$\xi_l^{\pm 1}\xi_m^{\pm 1} z_l/z_m \neq p^s\eta^r , \qquad l, m = 1, \dots, n , \quad l \neq m , \qquad s \in \mathbb{Z} , \tag{2.15}$$

for any $r = 1-\ell, \dots, \ell-1$ and for an arbitrary combination of signs.

(2.16) Theorem. *Let* $\kappa \neq p^s \eta^{-r} \prod_{m=1}^{n} \xi_m$ *and* $\kappa \neq p^{-s-1} \eta^r \prod_{m=1}^{n} \xi_m^{-1}$, $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}_{\geqslant 0}$. *Let* $0 < |p| < 1$. *Let* (2.13) – (2.15) *hold. Then*

$$\dim \mathcal{H}(z) = \dim \mathcal{F}(z) = \binom{n+\ell-1}{n-1}.$$

This means that

$$\mathcal{H}(z) \simeq \mathcal{F}(z). \tag{2.17}$$

In what follows we will consider in detail only two of the special values of the scaling parameter κ mentioned in Theorem 2.16, namely $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$ and $\kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$, which correspond to the values $r = \ell - 1$ and $s = 0$. In principle, all other special values of the scaling parameter κ can be considered similarly.

(2.18) Theorem. *Let either* $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$ *or* $\kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$. *Let* $0 < |p| < 1$. *Let* (2.13) – (2.15) *hold. If* $\prod_{m=1}^{n} \xi_m^2 \neq p^s \eta^r$ *for all* $r = \ell - 1, \dots, 2\ell - 2$ *and* $s \in \mathbb{Z}_{<0}$, *then* $\dim \mathcal{H}(z) = \binom{n+\ell-2}{n-2}$.

Theorems 2.16 and 2.18 follow from Theorems 5.9 and 5.10, 5.11, respectively, and Lemma 7.7.

Theorem 2.16 means that if the value of the scaling parameter κ is not special, then every nonzero hypergeometric form defines a nonzero cohomology class. If either $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$ or $\kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$, then Theorem 2.18 says that there exist exact hypergeometric forms. We describe them in Lemma 2.23.

Bases in the trigonometric hypergeometric space of a fiber

The finite-dimensional trigonometric hypergeometric space $\mathcal{F}(z)$ of a fiber has $n!$ remarkable bases. These bases will allow us to identify the geometry of an $\mathfrak{sl}_2$-type local system with representation theory. The bases are labelled by elements of the symmetric group $\mathbf{S}^n$. First we define the basis corresponding to the unit element of the symmetric group.

Let

$$\mathcal{Z}^n_\ell = \{\mathfrak{l} \in \mathbb{Z}^n_{\geqslant 0} \mid \sum_{m=1}^{n} \mathfrak{l}_m = \ell\}. \tag{2.19}$$

Set $\mathfrak{l}^m = \sum_{k=1}^{m} \mathfrak{l}_k$. In particular, $\mathfrak{l}^0 = 0$, $\mathfrak{l}^n = \ell$. For any $\mathfrak{l} \in \mathcal{Z}^n_\ell$ define a rational function $w_\mathfrak{l} \in \mathcal{F}$ as follows:

$$\begin{aligned} w_\mathfrak{l}(t_1, \dots, t_\ell, z_1, \dots, z_n) &= \\ &= \prod_{k=1}^{n} \prod_{s=1}^{\mathfrak{l}_k} \frac{1-\eta}{1-\eta^s} \sum_{\sigma \in \mathbf{S}^\ell} \Bigl[\prod_{m=1}^{n} \prod_{a \in \Gamma_m} \Bigl(\frac{t_a}{t_a - \xi_m z_m} \prod_{1 \leqslant l < m} \frac{\xi_l t_a - z_l}{t_a - \xi_l z_l} \Bigr) \Bigr]_\sigma \end{aligned} \tag{2.20}$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \dots, \mathfrak{l}^m\}$, $m = 1, \dots, n$. The functions $w_\mathfrak{l}$ are called the *trigonometric weight functions.*

(2.21) Lemma. *Let $\mathfrak{l} \in \mathcal{Z}^n_\ell$. Then*

$$w_\mathfrak{l}(t_1, \dots, t_\ell, z_1, \dots, z_n) = \prod_{1 \leqslant a < b \leqslant \ell} \frac{t_a - t_b}{\eta t_a - t_b} \times$$

$$\times \sum_{\Gamma_1, \dots, \Gamma_n} \Bigl\{ \prod_{m=1}^{n} \prod_{a \in \Gamma_m} \Bigl(\frac{t_a}{t_a - \xi_m z_m} \prod_{1 \leqslant l < m} \frac{\xi_l t_a - z_l}{t_a - \xi_l z_l} \Bigr) \prod_{\substack{1 \leqslant l < m \leqslant n \\ a \in \Gamma_l,\, b \in \Gamma_m}} \frac{\eta t_a - t_b}{t_a - t_b} \Bigr\}$$

where the summation is over all n-tuples $\Gamma_1, \dots, \Gamma_n$ of disjoint subsets of $\{1, \dots, \ell\}$ such that Γ_m has $\mathfrak{l}_m$ elements.

The lemma is proved in Appendix A.

Example. For $\ell = 1$ the trigonometric weight functions have the form

$$w_{\mathfrak{e}(m)}(t_1, z_1, \dots, z_n) = \frac{t_1}{t_1 - \xi_m z_m} \prod_{1 \leqslant l < m} \frac{\xi_l t_1 - z_l}{t_1 - \xi_l z_l}$$

where $\mathfrak{e}(m) = (0, \dots, \underset{m\text{-th}}{1}, \dots, 0)$, $m = 1, \dots, n$.

Example. For $n = 1$ the function $w_{(\ell)}$ has the form

$$w_{(\ell)}(t_1, \dots, t_\ell, z_1) = \prod_{a=1}^{\ell} \frac{t_a}{t_a - \xi_1 z_1} \prod_{1 \leqslant a < b \leqslant \ell} \frac{t_a - t_b}{\eta t_a - t_b}.$$

Example. For $\ell = 2$ and $n = 2$ the functions $w_{\mathfrak{l}}$ have the form

$$w_{(2,0)}(t_1, t_2, z_1, z_2) = \frac{t_1 t_2}{(t_1 - \xi_1 z_1)(t_2 - \xi_1 z_1)} \frac{t_1 - t_2}{\eta t_1 - t_2},$$

$$w_{(1,1)}(t_1, t_2, z_1, z_2) = \frac{t_1 t_2}{(t_1 - \xi_1 z_1)(t_2 - \xi_2 z_2)} \frac{\xi_1 t_2 - z_1}{t_2 - \xi_1 z_1} + $$

$$+ \frac{t_1 t_2}{(t_2 - \xi_1 z_1)(t_1 - \xi_2 z_2)} \frac{\xi_1 t_1 - z_1}{t_1 - \xi_1 z_1} \frac{t_1 - \eta t_2}{\eta t_1 - t_2},$$

$$w_{(0,2)}(t_1, t_2, z_1, z_2) = \frac{t_1 t_2}{(t_1 - \xi_2 z_2)(t_2 - \xi_2 z_2)} \frac{(\xi_1 t_1 - z_1)(\xi_1 t_2 - z_1)}{(t_1 - \xi_1 z_1)(t_2 - \xi_1 z_1)} \frac{t_1 - t_2}{\eta t_1 - t_2}.$$

(2.22) Lemma. *The functions $w_{\mathfrak{l}}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, restricted to a fiber over z form a basis in the trigonometric hypergeometric space $\mathcal{F}(z)$ of the fiber provided that $\xi_l \xi_m z_m / z_l \neq \eta^r$, $1 \leqslant l < m \leqslant n$ for any $r = 0, \dots, \ell - 1$.*

Lemma 2.22 is proved in Chapter 7.

Let $\mathfrak{e}(m) = (0, \dots, \underset{m\text{-th}}{1}, \dots, 0)$, $m = 1, \dots, n$.

(2.23) Lemma. *Let $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$. Then for any $\mathfrak{l} \in \mathcal{Z}^n_{\ell-1}$ the following relation holds:*

$$\sum_{m=1}^{n} w_{\mathfrak{l}+\mathfrak{e}(m)} (1 - \eta^{\mathfrak{l}_m + 1}) (\xi_m - \eta^{\mathfrak{l}_m} \xi_m^{-1}) \prod_{1 \leqslant l \leqslant m} \eta^{-\mathfrak{l}_l} \xi_l =$$

$$= (1 - \eta) \sum_{a=1}^{\ell} D_a \big[w_{\mathfrak{l}}(t_2, \dots, t_\ell) \big]_{(1,a)},$$

where $(1, a) \in \mathbf{S}^\ell$ are transpositions. Moreover, if $\mathcal{R}(z)$ is the subspace in $\mathcal{F}(z)$ generated by the elements in the left hand side of the relations, then

$$\dim \mathcal{F}(z) / \mathcal{R}(z) = \binom{n + \ell - 2}{n - 2}$$

provided that $\xi_l \xi_m z_m / z_l \neq \eta^r$, $1 \leqslant l \leqslant m \leqslant n$, for any $r = 0, \dots, \ell - 1$.

(2.24) Lemma. *Let* $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\xi_m^{-1}$. *Then for any* $\mathfrak{l} \in \mathcal{Z}_{\ell-1}^n$ *the following relation holds:*

$$\sum_{m=1}^{n} w_{\mathfrak{l}+\mathfrak{e}(m)}\,(1-\eta^{\mathfrak{l}_m+1})\,(\xi_m-\eta^{\mathfrak{l}_m}\xi_m^{-1})\,z_m\prod_{1\leqslant l<m}\eta^{\mathfrak{l}_l}\xi_l^{-1} =$$
$$= (1-\eta)\sum_{a=1}^{\ell} D_a\big[t_1 w_{\mathfrak{l}}(t_2,\dots,t_\ell)\big]_{(1,a)},$$

where $(1,a)\in \mathbf{S}^\ell$ *are transpositions. Moreover, if* $\mathcal{R}'(z)$ *is the subspace in* $\mathcal{F}(z)$ *generated by the elements in the left hand side of the relations, then*

$$\dim \mathcal{F}(z)/\mathcal{R}'(z) = \binom{n+\ell-2}{n-2}$$

provided that $\xi_l\xi_m z_m/z_l \neq \eta^r$, $1\leqslant l\leqslant m\leqslant n$, *for any* $r=0,\dots,\ell-1$.

Lemmas 2.23 and 2.24 are proved in Chapter 7.

The subspaces $\mathcal{R}(z),\mathcal{R}'(z)\subset\mathcal{F}(z)$ are called the *coboundary subspaces*.

For $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\xi_m$ relations (2.23) induce the relations

$$\sum_{m=1}^{n}\lfloor w_{\mathfrak{l}+\mathfrak{e}(m)}Dt_1\wedge\ldots\wedge Dt_\ell\rfloor\,(1-\eta^{\mathfrak{l}_m+1})\,(\xi_m-\eta^{\mathfrak{l}_m}\xi_m^{-1})\prod_{1\leqslant l\leqslant m}\eta^{-\mathfrak{l}_l}\xi_l = 0,$$
$$\mathfrak{l}\in\mathcal{Z}_{\ell-1}^n,$$

in the cohomology group $H^\ell(z)$, where $\lfloor\alpha\rfloor$ denotes the cohomological class of a form α. Under assumptions of Theorem 2.18 we have

$$\mathcal{H}(z)\simeq\mathcal{F}(z)/\mathcal{R}(z). \tag{2.25}$$

Similarly, for $\kappa=p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\xi_m^{-1}$ relations (2.24) induce the relations

$$\sum_{m=1}^{n}\lfloor w_{\mathfrak{l}+\mathfrak{e}(m)}Dt_1\wedge\ldots\wedge Dt_\ell\rfloor\,(1-\eta^{\mathfrak{l}_m+1})\,(\xi_m-\eta^{\mathfrak{l}_m}\xi_m^{-1})\,z_m\prod_{1\leqslant l<m}\eta^{\mathfrak{l}_l}\xi_l^{-1} = 0,$$
$$\mathfrak{l}\in\mathcal{Z}_{\ell-1}^n,$$

in the cohomology group $H^\ell(z)$, and under assumptions of Theorem 2.18 we have

$$\mathcal{H}(z) \simeq \mathcal{F}(z)/\mathcal{R}'(z). \tag{2.26}$$

For any permutation $\tau \in \mathbf{S}^n$ we define a basis $\{w_{\mathfrak{l}}^\tau\}_{\mathfrak{l}\in\mathcal{Z}_\ell^n}$ in the trigonometric hypergeometric space of a fiber by formulae similar to (2.20). Namely, set

$$\begin{aligned} w_{\mathfrak{l}}^\tau(t_1,\dots,t_\ell,z_1,\dots,z_n;\xi_1,\dots,\xi_n) &= \\ &= w_{{}^\tau\mathfrak{l}}(t_1,\dots,t_\ell,z_{\tau_1},\dots,z_{\tau_n};\xi_{\tau_1},\dots,\xi_{\tau_n}) \end{aligned} \tag{2.27}$$

where ${}^\tau\mathfrak{l} = (\mathfrak{l}_{\tau_1},\dots,\mathfrak{l}_{\tau_n})$.

Example. For $\ell = 1$ and a permutation $\tau = (n, n-1, \dots, 1)$ the trigonometric weight functions have the form

$$w_{\mathfrak{e}(m)}^\tau(t_1,z_1,\dots,z_n) = \frac{t_1}{t_1 - \xi_m z_m} \prod_{m<l\leqslant n} \frac{\xi_l t_1 - z_l}{t_1 - \xi_l z_l}.$$

The elliptic hypergeometric space

In our study of the Gauss-Manin connection for the discrete local system (2.3) an important role is played by the following *elliptic hypergeometric space.* The elliptic hypergeometric space is an elliptic counterpart of the trigonometric hypergeometric space introduced above.

All over this section we assume that $0 < |p| < 1$. Let

$$\theta(u) = (u;p)_\infty (pu^{-1};p)_\infty (p;p)_\infty$$

be the Jacobi theta-function.

The elliptic hypergeometric space $\mathcal{F}_{ell}$ is the space of functions of variables $t_1,\dots,t_\ell,z_1,\dots,z_n$ spanned over $\mathbb{C}$ by functions which have the form

$$Y(z_1,\dots,z_n)\,\Theta(t_1,\dots,t_\ell,z_1,\dots,z_n) \prod_{m=1}^{n}\prod_{a=1}^{\ell} \frac{1}{\theta(\xi_m^{-1}t_a/z_m)} \prod_{1\leqslant a<b\leqslant\ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)}.$$

Here Y is a meromorphic function on $\mathbb{C}^{\times n}$ and Θ is a holomorphic function on $\mathbb{C}^{\times(\ell+n)}$, symmetric in the variables $t_1,\dots,t_\ell$; we assume that the functions Y and Θ have the properties

$$\Theta(t_1,\dots,pt_a,\dots,t_\ell,z_1,\dots,z_n) = (-t_a)^{-n}\,\kappa\prod_{m=1}^{n} z_m\,\Theta(t_1,\dots,t_\ell,z_1,\dots,z_n)\,,$$

$$Y(z_1,\dots,pz_m,\dots,z_n)\,\Theta(t_1,\dots,t_\ell,z_1,\dots,pz_m,\dots,z_n) =$$

$$= (-p/z_m)^n\prod_{a=1}^{\ell} t_a\,Y(z_1,\dots,z_n)\,\Theta(t_1,\dots,t_\ell,z_1,\dots,z_n)\,.$$

The restriction of the elliptic hypergeometric space to a fiber defines the *elliptic hypergeometric space* $\mathcal{F}_{ell}(z)$ of the fiber. The elliptic hypergeometric space $\mathcal{F}_{ell}(z)$ is a complex finite-dimensional vector space of the same dimension as the trigonometric hypergeometric space of the fiber, see Appendix B.

All elements $f(t,z)$ of the elliptic hypergeometric space satisfy the periodicity conditions

$$f(t_1,\dots,pt_a,\dots,t_\ell,z_1,\dots,z_n) = \tag{2.28}$$

$$= \kappa\eta^{\ell-2a+1}\prod_{m=1}^{n}\xi_m^{-1}\,f(t_1,\dots,t_\ell,z_1,\dots,z_n)\,,$$

$$f(t_1,\dots,t_\ell,z_1,\dots,pz_m,\dots,z_n) = \xi_m^{\ell}\,f(t_1,\dots,t_\ell,z_1,\dots,z_n)\,.$$

(cf. (2.6)). Therefore, if $\Phi(t,z)$ is a short phase function given by (2.5) and $f(t,z)$ is any element of the elliptic hypergeometric space, then $\Phi(t,z)f(t,z)$ is a phase function of our discrete local system.

Transformation properties (2.28) also mean that the elliptic hypergeometric spaces of fibers over z and z' are naturally identified if the points z and z' lie in the same orbit of the $\mathbb{Z}^n$-action on the base space.

We give basic facts about the elliptic hypergeometric space in Appendix B.

If required, we will write down explicitly the dependence of the elliptic hypergeometric spaces on κ, $z_1,\dots,z_n$, $\xi_1,\dots,\xi_n$ and ℓ, that is

$$\mathcal{F}_{ell} = \mathcal{F}_{ell}[\kappa;z_1,\dots,z_n;\xi_1,\dots,\xi_n;\ell]$$

and

$$\mathcal{F}_{ell}(z) = \mathcal{F}_{ell}[\kappa;z_1,\dots,z_n;\xi_1,\dots,\xi_n;\ell](z)\,.$$

Introduce a new action of the symmetric group $\mathbf{S}^\ell$ on functions,

$$f \mapsto [\![f]\!]_\sigma\,, \qquad \sigma\in\mathbf{S}^\ell, \tag{2.29}$$

$$[\![f]\!]_\sigma(t_1,\dots,t_\ell,z_1,\dots,z_n) = f(t_{\sigma_1},\dots,t_{\sigma_\ell},z_1,\dots,z_n)\prod_{\substack{1\leqslant a<b\leqslant\ell\\ \sigma_a>\sigma_b}}\frac{\eta\,\theta(\eta^{-1}t_{\sigma_b}/t_{\sigma_a})}{\theta(\eta t_{\sigma_b}/t_{\sigma_a})}\,.$$

The elliptic hypergeometric space is invariant with respect to this action. The action commutes with the restriction of functions to a fiber.

The elliptic hypergeometric space of a fiber has $n!$ remarkable bases. The bases are labelled by elements of the symmetric group $\mathbf{S}^n$. First we define the basis corresponding to the unit element of the symmetric group. For any $\mathfrak{l} \in \mathcal{Z}^n_\ell$ define a function $W_\mathfrak{l}(t,z)$ as follows:

$$(2.30)\quad W_\mathfrak{l}(t_1,\dots,t_\ell,z_1,\dots,z_n) = \prod_{k=1}^{n}\prod_{s=1}^{\mathfrak{l}_k} \frac{\theta(\eta)}{\theta(\eta^s)} \times$$

$$\times \sum_{\sigma\in\mathbf{S}^\ell} \Big[\!\Big[\prod_{m=1}^{n}\prod_{a\in\Gamma_m}\Big(\frac{\theta(\eta^{2a-\ell-1}\kappa_m^{-1}t_a/z_m)}{\theta(\xi_m^{-1}t_a/z_m)} \prod_{1\leqslant l<m} \frac{\theta(\xi_l t_a/z_l)}{\theta(\xi_l^{-1}t_a/z_l)}\Big)\Big]\!\Big]_\sigma$$

where $\Gamma_m = \{1+\mathfrak{l}^{m-1},\dots,\mathfrak{l}^m\}$ and $\kappa_m = \kappa \prod_{1\leqslant l<m}\xi_l \prod_{m<l\leqslant n}\xi_l^{-1}$, $m = 1,\dots,n$. The functions $W_\mathfrak{l}$ are called the *elliptic weight functions*.

(2.31) Lemma. *Let* $\mathfrak{l}\in\mathcal{Z}^n_\ell$. *Then*

$$W_\mathfrak{l}(t_1,\dots,t_\ell,z_1,\dots,z_n) = \prod_{1\leqslant a<b\leqslant\ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)} \times$$

$$\times \sum_{\Gamma_1,\dots,\Gamma_n} \Big\{ \prod_{m=1}^{n}\prod_{a\in\Gamma_m}\Big(\frac{\theta(\kappa_{\mathfrak{l},m}^{-1}t_a/z_m)}{\theta(\xi_m^{-1}t_a/z_m)} \prod_{1\leqslant l<m}\frac{\theta(\xi_l t_a/z_l)}{\theta(\xi_l^{-1}t_a/z_l)}\Big) \prod_{\substack{1\leqslant l<m\leqslant n\\ a\in\Gamma_l,\,b\in\Gamma_m}} \frac{\theta(\eta t_a/t_b)}{\theta(t_a/t_b)}\Big\}$$

where
$$\kappa_{\mathfrak{l},m} = \kappa \prod_{1\leqslant i<m}\eta^{-\mathfrak{l}_i}\xi_i \prod_{m<i\leqslant n}\eta^{\mathfrak{l}_i}\xi_i^{-1}$$

and the summation is over all n-tuples $\Gamma_1,\dots,\Gamma_n$ of disjoint subsets of $\{1,\dots,\ell\}$ such that Γ_m has $\mathfrak{l}_m$ elements.

The lemma is proved in Appendix B.

Let $Y_\mathfrak{l}(z)$ be any meromorphic function such that

$$(2.32)\qquad Y_\mathfrak{l}(z_1,\dots,pz_m,\dots,z_n) = \alpha_{\mathfrak{l},m}\,Y_\mathfrak{l}(z_1,\dots,z_n)$$

where
$$\alpha_{\mathfrak{l},m} = \kappa^{\mathfrak{l}_m}\prod_{1\leqslant l<m}\eta^{-\mathfrak{l}_l\mathfrak{l}_m}\xi_l^{\mathfrak{l}_m}\xi_m^{\mathfrak{l}_l}\prod_{m<l\leqslant n}\eta^{\mathfrak{l}_l\mathfrak{l}_m}\xi_l^{-\mathfrak{l}_m}\xi_m^{-\mathfrak{l}_l},$$

$m = 1, \dots, n$. Then the product $Y_{\mathfrak{l}}(z) W_{\mathfrak{l}}(t, z)$ is an element of the elliptic hypergeometric space. The function $Y_{\mathfrak{l}}$ will be called an *adjusting factor* for the weight function $W_{\mathfrak{l}}$. The adjusting factors can be chosen to be meromorphic functions in parameters η, $\xi_1, \dots, \xi_n$ and κ.

Example. Let $c_1, \dots, c_n$ be arbitrary nonzero complex numbers. Let $\alpha_{\mathfrak{l},1}$, $\dots, \alpha_{\mathfrak{l},n}$ be the same as in (2.32). Then the function

$$Y_{\mathfrak{l}}(z_1, \dots, z_n) = \prod_{m=1}^{n} \frac{\theta(c_m z_m / \alpha_{\mathfrak{l},m})}{\theta(c_m z_m)}$$

is an adjusting factor for the weight function $W_{\mathfrak{l}}$.

Notice that an adjusting factor is not unique. In what follows we never need to know the adjusting factors explicitly.

Example. For $\ell = 1$ the elliptic weight functions have the form

$$W_{\mathfrak{e}(m)}(t_1, z_1, \dots, z_n) = \frac{\theta(\kappa_m^{-1} t_1 / z_m)}{\theta(\xi_m^{-1} t_1 / z_m)} \prod_{1 \leqslant l < m} \frac{\theta(\xi_l t_1 / z_l)}{\theta(\xi_l^{-1} t_1 / z_l)}$$

where $\kappa_m = \kappa \prod_{1 \leqslant l < m} \xi_l \prod_{m < l \leqslant n} \xi_l^{-1}$, $\quad m = 1, \dots, n$.

Example. For $n = 1$ the function $W_{(\ell)}$ has the form

$$W_{(\ell)}(t_1, \dots, t_\ell, z_1) = \prod_{a=1}^{\ell} \frac{\theta(\kappa^{-1} t_a / z_1)}{\theta(\xi_1^{-1} t_a / z_1)} \prod_{1 \leqslant a < b \leqslant \ell} \frac{\theta(t_a / t_b)}{\theta(\eta t_a / t_b)} .$$

Example. For $\ell = 2$ and $n = 2$ the functions $W_{\mathfrak{l}}$ have the form

$$W_{(2,0)}(t_1, t_2, z_1, z_2) = \frac{\theta(\kappa^{-1} \xi_2 t_1 / z_1) \theta(\kappa^{-1} \xi_2 t_2 / z_1)}{\theta(\xi_1^{-1} t_1 / z_1) \theta(\xi_1^{-1} t_2 / z_1)} \frac{\theta(t_1 / t_2)}{\theta(\eta t_1 / t_2)} ,$$

$$W_{(1,1)}(t_1, t_2, z_1, z_2) = \frac{\theta(\eta^{-1} \kappa^{-1} \xi_2 t_1 / z_1) \theta(\eta \kappa^{-1} \xi_1^{-1} t_2 / z_2) \theta(\xi_1 t_2 / z_1)}{\theta(\xi_1^{-1} t_1 / z_1) \theta(\xi_2^{-1} t_2 / z_2) \theta(\xi_1^{-1} t_2 / z_1)} +$$

$$+ \frac{\theta(\eta \kappa^{-1} \xi_1^{-1} t_1 / z_2) \theta(\eta^{-1} \kappa^{-1} \xi_2 t_2 / z_1) \theta(\xi_1 t_1 / z_1)}{\theta(\xi_1^{-1} t_1 / z_1) \theta(\xi_2^{-1} t_1 / z_2) \theta(\xi_1^{-1} t_2 / z_1)} \frac{\eta \theta(\eta^{-1} t_1 / t_2)}{\theta(\eta t_1 / t_2)} ,$$

$$W_{(0,2)}(t_1, t_2, z_1, z_2) =$$

$$= \frac{\theta(\kappa^{-1} \xi_1^{-1} t_1 / z_2) \theta(\kappa^{-1} \xi_1^{-1} t_2 / z_2) \theta(\xi_1 t_1 / z_1) \theta(\xi_1 t_2 / z_1)}{\theta(\xi_2^{-1} t_1 / z_2) \theta(\xi_2^{-1} t_2 / z_2) \theta(\xi_1^{-1} t_1 / z_1) \theta(\xi_1^{-1} t_2 / z_1)} \frac{\theta(t_1 / t_2)}{\theta(\eta t_1 / t_2)} .$$

(2.33) Lemma. *The functions* $W_{\mathfrak{l}}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, *restricted to a fiber over* z *form a basis in the elliptic hypergeometric space* $\mathcal{F}_{ell}(z)$ *of the fiber, provided that* $\xi_l \xi_m z_m / z_l \neq p^s \eta^r$, $1 \leqslant l < m \leqslant n$, *for any* $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}$, *and* $\kappa \prod_{1 \leqslant l \leqslant m} \xi_l \prod_{m < l \leqslant n} \xi_l^{-1} \neq p^s \eta^r$ *for any* $m = 1, \dots, n-1$ *and* $r = 1 - \ell, \dots, \ell - 1$, $s \in \mathbb{Z}$.

Lemma 2.33 is proved in Chapter 7.

Let $\mathcal{Q}(z)$ be the space of functions of the form

$$\sum_{\sigma \in \mathbf{S}^\ell} [\![W(t_2, \dots, t_\ell)]\!]_\sigma ,$$

where $W \in \mathcal{F}_{ell}[\eta^{-1}\kappa; z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell - 1](z)$. Let $\mathcal{Q}'(z)$ be the space of functions of the form

$$\sum_{\sigma \in \mathbf{S}^\ell} \Big[\!\!\Big[W(t_1, \dots, t_{\ell-1})\, t_\ell^{-1} \prod_{m=1}^n \frac{\theta(\xi_m t_\ell / z_m)}{\theta(\xi_m^{-1} t_\ell / z_m)} \Big]\!\!\Big]_\sigma ,$$

where $W \in \mathcal{F}_{ell}[\eta\kappa; z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell - 1](z)$. In general, the spaces $\mathcal{Q}(z)$ and $\mathcal{Q}'(z)$ are not subspaces of the elliptic hypergeometric space of the fiber $\mathcal{F}_{ell}(z)$, because their elements do not have the required quasiperiodicity properties. However, if $\kappa = \eta^{1-\ell} \prod_{m=1}^n \xi_m$, then the space $\mathcal{Q}(z)$ is a subspace of $\mathcal{F}_{ell}(z)$, and if $\kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^n \xi_m^{-1}$, then the space $\mathcal{Q}'(z)$ is a subspace of $\mathcal{F}_{ell}(z)$. The spaces $\mathcal{Q}(z), \mathcal{Q}'(z)$ are called the *boundary subspaces*.

For any function $f(t_1, \dots, t_\ell)$ and a point $t^\star = (t_1^\star, \dots, t_\ell^\star)$ we define the multiple residue $\operatorname{Res} f(t)\big|_{t = t^\star}$ by the formula

$$\operatorname{Res} f(t)\big|_{t=t^\star} = \operatorname{Res} \big(\dots \operatorname{Res} f(t_1, \dots, t_\ell)\big|_{t_\ell = t_\ell^\star} \dots \big)\big|_{t_1 = t_1^\star} . \tag{2.34}$$

For any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ define the points $x \triangleright \mathfrak{l}$, $y \triangleleft \mathfrak{l} \in \mathbb{C}^{\times \ell}$ as follows:

$$x \triangleright \mathfrak{l} = (\eta^{1-\mathfrak{l}_1}\xi_1 z_1, \eta^{2-\mathfrak{l}_1}\xi_1 z_1, \dots, \xi_1 z_1, \eta^{1-\mathfrak{l}_2}\xi_2 z_2, \dots, \xi_2 z_2, \dots, \eta^{1-\mathfrak{l}_n}\xi_n z_n, \dots, \xi_n z_n), \tag{2.35}$$

$$y \triangleleft \mathfrak{l} = (\eta^{\mathfrak{l}_1 - 1}\xi_1^{-1} z_1, \eta^{\mathfrak{l}_1 - 2}\xi_1^{-1} z_1, \dots, \xi_1^{-1} z_1, \eta^{\mathfrak{l}_2 - 1}\xi_2^{-1} z_2, \dots, \xi_2^{-1} z_2, \dots, \eta^{\mathfrak{l}_n - 1}\xi_n^{-1} z_n, \dots, \xi_n^{-1} z_n).$$

Example. Let $\ell = 1$ and $\mathfrak{l} = (0, \dots, \underset{m\text{-th}}{1}, \dots, 0)$. Then $x \triangleright \mathfrak{l} = \xi_m z_m$ and $y \triangleleft \mathfrak{l} = \xi_m^{-1} z_m$.

(2.36) Lemma. *Let $\eta^r \neq p^s$ for any $r = 1, \dots, \ell - 1$, $s \in \mathbb{Z}$. Assume that $\xi_l^{-1} \xi_m z_l^{-1} z_m \neq p^s \eta^r$, $l, m = 1, \dots, n$, $l \neq m$, for any $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}$. Then* $\operatorname{Res} f(t)\big|_{t = x \triangleright \mathfrak{m}} = 0$ *for any $f \in \mathcal{Q}(z)$ and $\mathfrak{m} \in \mathcal{Z}_\ell^n$. Moreover, if $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$ and $\eta^\ell \neq p^s$, $s \in \mathbb{Z}$, then*

$$\mathcal{Q}(z) = \{ f \in \mathcal{F}_{ell}(z) \mid \operatorname{Res} f(t)\big|_{t = x \triangleright \mathfrak{m}} = 0 \quad \text{for any } \mathfrak{m} \in \mathcal{Z}_\ell^n \}.$$

(2.37) Lemma. *Let $\eta^r \neq p^s$ for any $r = 1, \dots, \ell - 1$, $s \in \mathbb{Z}$. Assume that $\xi_l {\xi_m}^{-1} z_l^{-1} z_m \neq p^s \eta^r$, $l, m = 1, \dots, n$, $l \neq m$, for any $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}$. Then $f(y \triangleleft \mathfrak{m}) = 0$ for any $f \in \mathcal{Q}'(z)$ and $\mathfrak{m} \in \mathcal{Z}_\ell^n$. Moreover, if $\kappa = p^{-1} \eta^{\ell - 1} \prod_{m=1}^{n} \xi_m^{-1}$ and $\eta^\ell \neq p^s$, $s \in \mathbb{Z}$, then*

$$\mathcal{Q}'(z) = \{ f \in \mathcal{F}_{ell}(z) \mid f(y \triangleleft \mathfrak{m}) = 0 \quad \text{for any } \mathfrak{m} \in \mathcal{Z}_\ell^n \}.$$

Lemmas 2.36, 2.37 are proved in Appendix B.

Example. Let $\ell = 1$. Then the spaces $\mathcal{Q}(z)$ and $\mathcal{Q}'(z)$ are one-dimensional. The space $\mathcal{Q}(z)$ is the space of constant functions in one variable, and the space $\mathcal{Q}'(z)$ is spanned by the function $W(t_1) = t_1^{-1} \prod_{m=1}^{n} \dfrac{\theta(\xi_m t_1 / z_m)}{\theta(\xi_m^{-1} t_1 / z_m)}$.

Let $\kappa = \prod_{m=1}^{n} \xi_m$. Then functions of the elliptic hypergeometric space of a fiber $\mathcal{F}_{ell}(z)$ are p-periodic. The space $\mathcal{F}_{ell}(z)$ has a one-dimensional subspace of constant functions which is the boundary subspace $\mathcal{Q}(z)$. The constant functions are the only functions in $\mathcal{F}_{ell}(z)$ which are regular in $\mathbb{C}^\times$. The regular elliptic weight function is $W_{\mathfrak{e}(1)} = 1$, $\mathfrak{e}(1) = (1, 0, \dots, 0)$.

Let $\kappa = p^{-1} \prod_{m=1}^{n} \xi_m^{-1}$. Then the elliptic weight function $W_{(0, \dots, 0, 1)}$ equals $-\xi_n^{-1} z_n W$ and generates the boundary subspace $\mathcal{Q}'(z) \subset \mathcal{F}_{ell}(z)$. The function W is the only function in $\mathcal{F}_{ell}(z)$ which vanishes at all the points $\xi_1^{-1} z_1, \dots, \xi_n^{-1} z_n$.

(2.38) Lemma. *Let $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$. Let $\eta^r \neq p^s$ for any $r = 2, \dots, \ell$, $s \in \mathbb{Z}$. Then*

$$\dim \mathcal{F}_{ell}(z) / \mathcal{Q}(z) = \binom{n + \ell - 2}{n - 2}.$$

Moreover, the equivalence classes of functions $W_{\mathfrak{l}}$, $\mathfrak{l}_1 = 0$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, *restricted to a fiber over* z *form a basis in the space* $\mathcal{F}_{ell}(z)/\mathcal{Q}(z)$, *provided that* $\xi_l \xi_m z_m / z_l \neq p^s \eta^r$, $1 \leqslant l < m \leqslant n$, *for any* $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}$, *and* $\prod_{1 \leqslant l \leqslant m} \xi_l^2 \neq p^s \eta^r$, $m = 1, \dots, n-1$, *for any* $r = 0, \dots, 2\ell - 2$, $s \in \mathbb{Z}$.

(2.39) Lemma. *Let* $\kappa = p^{-1} \eta^{\ell - 1} \prod_{m=1}^{n} \xi_m^{-1}$. *Let* $\eta^r \neq p^s$ *for any* $r = 2, \dots, \ell$, $s \in \mathbb{Z}$. *Then*

$$\dim \mathcal{F}_{ell}(z)/\mathcal{Q}'(z) = \binom{n + \ell - 2}{n - 2}.$$

Moreover, the equivalence classes of functions $W_{\mathfrak{l}}$, $\mathfrak{l}_n = 0$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, *restricted to a fiber over* z *form a basis in the space* $\mathcal{F}_{ell}(z)/\mathcal{Q}'(z)$, *provided that* $\xi_l \xi_m z_m / z_l \neq p^s \eta^r$, $1 \leqslant l < m \leqslant n$, *for any* $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}$, *and* $\prod_{m < l \leqslant n} \xi_l^2 \neq p^s \eta^r$, $m = 1, \dots, n-1$, *for any* $r = 0, \dots, 2\ell - 2$, $s \in \mathbb{Z}$.

Lemmas 2.38, 2.39 are proved in Appendix B.

For any permutation $\tau \in \mathbf{S}^n$ we define a basis $\{W^\tau_{\mathfrak{l}}\}_{\mathfrak{l} \in \mathcal{Z}^n_\ell}$ in the elliptic hypergeometric space of a fiber by formulae similar to (2.30). Namely, set

$$\begin{aligned} &W^\tau_{\mathfrak{l}}(t_1, \dots, t_\ell, z_1, \dots, z_n; \xi_1, \dots, \xi_n) = \\ &\qquad = W_{{}^\tau\mathfrak{l}}(t_1, \dots, t_\ell, z_{\tau_1}, \dots, z_{\tau_n}; \xi_{\tau_1}, \dots, \xi_{\tau_n}) \end{aligned} \tag{2.40}$$

where ${}^\tau\mathfrak{l} = (\mathfrak{l}_{\tau_1}, \dots, \mathfrak{l}_{\tau_n})$.

Example. For $\ell = 1$ and a permutation $\tau = (n, n-1, \dots, 1)$ the elliptic weight functions have the form

$$W^\tau_{\mathfrak{e}(m)}(t_1, z_1, \dots, z_n) = \frac{\theta(\tilde\kappa_m^{-1} t_1 / z_m)}{\theta(\xi_m^{-1} t_1 / z_m)} \prod_{m < l \leqslant n} \frac{\theta(\xi_l t_1 / z_l)}{\theta(\xi_l^{-1} t_1 / z_l)}$$

where $\tilde\kappa_m = \kappa \prod_{1 \leqslant l < m} \xi_l^{-1} \prod_{m < l \leqslant n} \xi_l$, $m = 1, \dots, n$.

3. *R*-matrices and the *qKZ* connection

Highest weight $U_q(\mathfrak{sl}_2)$*-modules*

Let q be a nonzero complex number which is not a root of unity. Consider the quantum group $U_q(\mathfrak{sl}_2)$ with generators $E, F, q^{\pm H}$ and relations:

$$q^H q^{-H} = q^{-H} q^H = 1\,,$$

$$q^H E = qE q^H\,, \qquad q^H F = q^{-1} F q^H\,,$$

$$[E, F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}\,.$$

Let the coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ be given by

$$\Delta(q^H) = q^H \otimes q^H, \qquad \Delta(q^{-H}) = q^{-H} \otimes q^{-H},$$

$$\Delta(E) = E \otimes q^{-H} + q^H \otimes E\,, \qquad \Delta(F) = F \otimes q^{-H} + q^H \otimes F\,.$$

The coproduct defines a $U_q(\mathfrak{sl}_2)$-module structure on a tensor product of $U_q(\mathfrak{sl}_2)$-modules.

For a $U_q(\mathfrak{sl}_2)$-module V let $V = \bigoplus_\lambda V_\lambda$ be its weight decomposition. Let $V^* = \bigoplus_\lambda V_\lambda^*$ be its restricted dual. Define a structure of a $U_q(\mathfrak{sl}_2)$-module on V^* by

$$\langle E\varphi, x\rangle = \langle \varphi, Fx\rangle\,, \qquad \langle F\varphi, x\rangle = \langle \varphi, Ex\rangle\,, \qquad \langle q^{\pm H}\varphi, x\rangle = \langle \varphi, q^{\pm H}x\rangle\,.$$

This $U_q(\mathfrak{sl}_2)$-module structure on V^* will be called the *dual* module structure. For any $U_q(\mathfrak{sl}_2)$-modules V_1, V_2, the tautological map $V_1^* \otimes V_2^* \to (V_1 \otimes V_2)^*$ is an isomorphism of $U_q(\mathfrak{sl}_2)$-modules.

Let V be the $U_q(\mathfrak{sl}_2)$-module with highest weight q^Λ. Let $V = \bigoplus_{l=0}^{\infty} V_{\Lambda - l}$ be its weight decomposition. For any nonzero complex number u define an operator $u^{\Lambda - H} \in \mathrm{End}(V)$ by $u^{\Lambda - H} v = u^l v$ for any $v \in V_{\Lambda - l}$.

Let $V_1, \dots, V_n$ be $U_q(\mathfrak{sl}_2)$-modules with highest weights $q^{\Lambda_1}, \dots, q^{\Lambda_n}$, respectively. We have the weight decompositions

$$V_1 \otimes \ldots \otimes V_n = \bigoplus_{\ell=0}^{\infty} (V_1 \otimes \ldots \otimes V_n)_\ell$$

and
$$(V_1 \otimes \ldots \otimes V_n)^* = \bigoplus_{\ell=0}^{\infty} (V_1 \otimes \ldots \otimes V_n)^*_\ell$$
where $()_\ell$ denotes the eigenspace of q^H with eigenvalue $q^{\sum_{m=1}^{n} \Lambda_m - \ell}$.

Let $F(V_1 \otimes \ldots \otimes V_n)^*_{\ell-1} \subset (V_1 \otimes \ldots \otimes V_n)^*_\ell$ be the image of the operator F. Let $(V_1 \otimes \ldots \otimes V_n)^{sing}_\ell \subset V_1 \otimes \ldots \otimes V_n$ be the kernel of the operator E. There is a natural pairing

$$(V_1 \otimes \ldots \otimes V_n)^{sing}_\ell \otimes (V_1 \otimes \ldots \otimes V_n)^*_\ell / F(V_1 \otimes \ldots \otimes V_n)^*_{\ell-1} \to \mathbb{C}. \tag{3.1}$$

Let $z_1, \ldots, z_n$ be nonzero complex numbers. Set
$$E_z = \sum_{m=1}^{n} q^{-H} \otimes \ldots \otimes \underset{m\text{-th}}{z_m E} \otimes \ldots \otimes q^H$$
and
$$F_z = \sum_{m=1}^{n} q^{-H} \otimes \ldots \otimes \underset{m\text{-th}}{z_m F} \otimes \ldots \otimes q^H.$$

Let $F_z(V_1 \otimes \ldots \otimes V_n)^*_{\ell-1} \subset (V_1 \otimes \ldots \otimes V_n)^*_\ell$ be the image of the operator F_z. Let $(V_1 \otimes \ldots \otimes V_n)^{sing}_{\ell,z} \subset V_1 \otimes \ldots \otimes V_n$ be the kernel of the operator E_z. There is a natural pairing

$$(V_1 \otimes \ldots \otimes V_n)^{sing}_{\ell,z} \otimes (V_1 \otimes \ldots \otimes V_n)^*_\ell / F_z(V_1 \otimes \ldots \otimes V_n)^*_{\ell-1} \to \mathbb{C}. \tag{3.2}$$

Let $V_1, \ldots, V_n$ be Verma modules, then pairings (3.1) and (3.2) are nondegenerate provided
$$\prod_{m=1}^{n} \prod_{s=0}^{\ell-1} (1 - q^{4\Lambda_m - 2s}) \neq 0.$$

The trigonometric R-matrix

Let V_1, V_2 be Verma modules for $U_q(\mathfrak{sl}_2)$ with highest weights q^{Λ_1}, q^{Λ_2} and generating vectors v_1, v_2, respectively. Consider an $\mathrm{End}(V_1 \otimes V_2)$-valued meromorphic function $R_{V_1 V_2}(x)$ with the following properties:

$$R_{V_1V_2}(x)\,(F \otimes q^{-H} + q^H \otimes F) = (F \otimes q^H + q^{-H} \otimes F)\,R_{V_1V_2}(x), \tag{3.3}$$
$$R_{V_1V_2}(x)\,(F \otimes q^H + xq^{-H} \otimes F) = (F \otimes q^{-H} + xq^H \otimes F)\,R_{V_1V_2}(x)$$

in $\operatorname{End}(V_1 \otimes V_2)$ and

$$R_{V_1V_2}(x)\, v_1 \otimes v_2 = v_1 \otimes v_2\,. \tag{3.4}$$

Such a function $R_{V_1V_2}(x)$ exists and is uniquely determined. $R_{V_1V_2}(x)$ is called the $\mathfrak{sl}_2$ *trigonometric R-matrix* for the tensor product $V_1 \otimes V_2$.

The trigonometric R-matrix $R_{V_1V_2}(x)$ also satifies the following relations:

$$\begin{aligned} R_{V_1V_2}(x)\,(E\otimes q^{-H} + q^{H}\otimes E) &= (E\otimes q^{H} + q^{-H}\otimes E)\,R_{V_1V_2}(x)\,,\\ R_{V_1V_2}(x)\,(xE\otimes q^{H} + q^{-H}\otimes E) &= (xE\otimes q^{-H} + q^{H}\otimes E)\,R_{V_1V_2}(x)\,,\\ R_{V_1V_2}(x)\,q^{H}\otimes q^{H} &= q^{H}\otimes q^{H}R_{V_1V_2}(x)\,. \end{aligned} \tag{3.5}$$

In particular, $R_{V_1V_2}(x)$ respects the weight decomposition of $V_1 \otimes V_2$.

$R_{V_1V_2}(x)$ satisfies the inversion relation

$$P_{V_1V_2}R_{V_1V_2}(x) = \big(R_{V_2V_1}(x^{-1})\big)^{-1}P_{V_1V_2} \tag{3.6}$$

where $P_{V_1V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is the permutation map.

Let $V_1 \otimes V_2 = \bigoplus_{l=0}^{\infty} V^{(l)}$ be the decomposition of the $U_q(\mathfrak{sl}_2)$-module $V_1 \otimes V_2$ into the direct sum of irreducibles, where the irreducible module $V^{(l)}$ is generated by a singular vector of weight $q^{\Lambda_1+\Lambda_2-l}$. Let $\Pi^{(l)}$ be the projector onto $V^{(l)}$ along the other summands. Then we have

$$R_{V_1V_2}(x), = R_{V_1V_2}(\infty)\sum_{l=0}^{\infty}\Pi^{(l)}\cdot\prod_{s=0}^{l-1}\frac{x-q^{2s-2\Lambda_1-2\Lambda_2}}{x-q^{2\Lambda_1+2\Lambda_2-2s}} \tag{3.7}$$

where

$$R_{V_1V_2}(\infty) = q^{2\Lambda_1\Lambda_2-2H\otimes H}\sum_{k=0}^{\infty}(q^2-1)^{2k}\prod_{s=1}^{k}(1-q^{2s})^{-1}\,(q^{-H}E\otimes q^{H}F)^{k}\,.$$

Let V_1, V_2, V_3 be Verma modules. The corresponding R-matrices satisfy the Yang-Baxter equation:

$$R_{V_1V_2}(x/y)\,R_{V_1V_3}(x)\,R_{V_2V_3}(y) = R_{V_2V_3}(y)\,R_{V_1V_3}(x)\,R_{V_1V_2}(x/y)\,. \tag{3.8}$$

All of the properties of $R_{V_1V_2}(x)$ given above are well known (cf. [T], [D1], [J], [CP]).

The quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$

The trigonometric R-matrix is connected with an action of the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ in a tensor product of $U_q(\mathfrak{sl}_2)$-modules. The quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra which contains $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra. We give the necessary facts about $U_q'(\widetilde{\mathfrak{gl}_2})$ in this section.

Let q be a complex number, $q \neq \pm 1$. The *quantum loop algebra* $U_q(\widetilde{\mathfrak{gl}_2})$ is a unital associative algebra with generators $L_{ij}^{(+0)}$, $L_{ji}^{(-0)}$, $1 \leqslant j \leqslant i \leqslant 2$, and $L_{ij}^{(s)}$, $i,j = 1,2$, $s = \pm 1, \pm 2, \dots$, subject to relations (3.9) [RS], [DF].

Let e_{ij}, $i,j = 1,2$, be the 2×2 matrix with the only nonzero entry 1 at the intersection of the i-th row and j-th column. Set

$$R(x) = (xq - q^{-1})(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) + (x-1)(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}) + \\ + x(q-q^{-1})\, e_{12}\otimes e_{21} + (q-q^{-1})\, e_{21}\otimes e_{12}\,.$$

Introduce the generating series $L_{ij}^{\pm}(u) = L_{ij}^{(\pm 0)} + \sum_{s=1}^{\infty} L_{ij}^{(\pm s)} u^{\pm s}$. The relations in $U_q(\widetilde{\mathfrak{gl}_2})$ have the form

$$(3.9) \qquad L_{ii}^{(+0)} L_{ii}^{(-0)} = 1\,, \qquad L_{ii}^{(-0)} L_{ii}^{(+0)} = 1\,, \qquad i = 1,2\,,$$

$$R(x/y)\, L_{(1)}^{+}(x)\, L_{(2)}^{+}(y) = L_{(2)}^{+}(y)\, L_{(1)}^{+}(x)\, R(x/y)\,,$$

$$R(x/y)\, L_{(1)}^{+}(x)\, L_{(2)}^{-}(y) = L_{(2)}^{-}(y)\, L_{(1)}^{+}(x)\, R(x/y)\,,$$

$$R(x/y)\, L_{(1)}^{-}(x)\, L_{(2)}^{-}(y) = L_{(2)}^{-}(y)\, L_{(1)}^{-}(x)\, R(x/y)\,,$$

where $L_{(1)}^{\nu}(u) = \sum_{ij} e_{ij}\otimes 1\otimes L_{ij}^{\nu}(u)$ and $L_{(2)}^{\nu}(u) = \sum_{ij} 1\otimes e_{ij}\otimes L_{ij}^{\nu}(u)$, $\quad \nu = \pm$.

Elements $L_{11}^{(+0)}L_{22}^{(+0)}$, $L_{22}^{(+0)}L_{11}^{(+0)}$, $L_{11}^{(-0)}L_{22}^{(-0)}$, $L_{22}^{(-0)}L_{11}^{(-0)}$ are central in $U_q(\widetilde{\mathfrak{gl}_2})$. Impose the following relations:

$$L_{11}^{(+0)}L_{22}^{(+0)} = 1\,, \qquad L_{22}^{(+0)}L_{11}^{(+0)} = 1\,,$$

$$L_{11}^{(-0)}L_{22}^{(-0)} = 1\,, \qquad L_{22}^{(-0)}L_{11}^{(-0)} = 1\,,$$

in addition to relations (3.9). Denote the corresponding quotient algebra by $U_q'(\widetilde{\mathfrak{gl}_2})$.

The quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra with a coproduct $\Delta : U'_q(\widetilde{\mathfrak{gl}_2}) \to U'_q(\widetilde{\mathfrak{gl}_2}) \otimes U'_q(\widetilde{\mathfrak{gl}_2})$:

$$\Delta : L^{\nu}_{ij}(u) \mapsto \sum_k L^{\nu}_{ik}(u) \otimes L^{\nu}_{kj}(u)\,, \qquad \nu = \pm\,.$$

There is an important one-parametric family of automorphisms $\rho_x : U'_q(\widetilde{\mathfrak{gl}_2}) \to U'_q(\widetilde{\mathfrak{gl}_2})$:

$$\rho_x : L^{\nu}_{ij}(u) \mapsto L^{\nu}_{ij}(u/x)\,, \qquad \nu = \pm\,,$$

that is

$$\rho_x : L^{(\pm 0)}_{ij} \mapsto L^{(\pm 0)}_{ij} \qquad \text{and} \qquad \rho_x : L^{(s)}_{ij} \mapsto x^{-s} L^{(s)}_{ij}\,, \qquad s \in \mathbb{Z}_{\neq 0}\,.$$

The quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ contains $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra; the embedding is given by

$$E \mapsto -L^{(+0)}_{21}/(q-q^{-1})\,, \qquad F \mapsto L^{(-0)}_{12}/(q-q^{-1})\,, \qquad q^H \mapsto L^{(-0)}_{11}\,.$$

There is also an *evaluation homomorphism* $\epsilon : U'_q(\widetilde{\mathfrak{gl}_2}) \to U_q(\mathfrak{sl}_2)$:

$$\epsilon : L^{+}_{11}(u) \mapsto q^{-H} - q^{H}u\,, \qquad \epsilon : L^{+}_{12}(u) \mapsto -F(q-q^{-1})u\,,$$

$$\epsilon : L^{+}_{21}(u) \mapsto -E(q-q^{-1})\,, \qquad \epsilon : L^{+}_{22}(u) \mapsto q^{H} - q^{-H}u\,,$$

$$\epsilon : L^{-}_{11}(u) \mapsto q^{H} - q^{-H}u^{-1}\,, \qquad \epsilon : L^{-}_{12}(u) \mapsto F(q-q^{-1}),$$

$$\epsilon : L^{-}_{21}(u) \mapsto E(q-q^{-1})u^{-1}\,, \qquad \epsilon : L^{-}_{22}(u) \mapsto q^{-H} - q^{H}u^{-1},$$

that is

$$\epsilon : L^{(+0)}_{11} \mapsto q^{-H}, \quad \epsilon : L^{(1)}_{11} \mapsto -q^{H}, \quad \epsilon : L^{(1)}_{12} \mapsto -F(q-q^{-1})\,,$$

$$\epsilon : L^{(+0)}_{21} \mapsto -E(q-q^{-1})\,, \quad \epsilon : L^{(+0)}_{22} \mapsto q^{H}, \quad \epsilon : L^{(1)}_{22} \mapsto -q^{-H},$$

$$\epsilon : L^{(-0)}_{11} \mapsto q^{H}, \quad \epsilon : L^{(-1)}_{11} \mapsto -q^{-H}, \quad \epsilon : L^{(-0)}_{12} \mapsto F(q-q^{-1})\,,$$

$$\epsilon : L^{(-1)}_{21} \mapsto E(q-q^{-1})\,, \quad \epsilon : L^{(-0)}_{22} \mapsto q^{-H}, \quad \epsilon : L^{(+1)}_{22} \mapsto -q^{H}$$

and $\epsilon : L^{(s)}_{ij} \mapsto 0$ for all other generators $L^{(s)}_{ij}$.

Both the automorphisms ρ_x and ϵ restricted to the subalgebra $U_q(\mathfrak{sl}_2)$ are the identity maps.

For any $U_q(\mathfrak{sl}_2)$-module V denote by $V(x)$ the $U'_q(\widetilde{\mathfrak{gl}_2})$-module which is obtained from the module V via the homomorphism $\epsilon \circ \rho_x$. The module $V(x)$ is called the *evaluation module*.

Let V_1, V_2 be Verma modules for $U_q(\mathfrak{sl}_2)$ with generating vectors v_1, v_2, respectively. For generic complex numbers x, y the $U'_q(\widetilde{\mathfrak{gl}_2})$-modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$ are isomorphic and the trigonometric R-matrix $P_{V_1V_2}R_{V_1V_2}(x/y)$ intertwines them [T], [CP]. The vectors $v_1 \otimes v_2$ and $v_2 \otimes v_1$ are respective generating vectors of the $U'_q(\widetilde{\mathfrak{gl}_2})$-modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$. The trigonometric R-matrix $R_{V_1V_2}(x/y)$ can be defined as the unique element of $\operatorname{End}(V_1 \otimes V_2)$ with property (3.4) and such that

$$P_{V_1V_2}R_{V_1V_2}(x/y) \,:\, V_1(x) \otimes V_2(y) \to V_2(y) \otimes V_1(x) \tag{3.10}$$

is an isomorphism of the $U'_q(\widetilde{\mathfrak{gl}_2})$-modules.

The trigonometric qKZ connection associated with $\mathfrak{sl}_2$

Let $V_1, \dots, V_n$ be $U_q(\mathfrak{sl}_2)$-modules. The *qKZ* connection is a discrete connection on the trivial bundle over $\mathbb{C}^{\times n}$ with fiber $V_1 \otimes \dots \otimes V_n$. We define it below.

Let $V_1, \dots, V_n$ be Verma modules with highest weights $q^{\Lambda_1}, \dots, q^{\Lambda_n}$, respectively. Let $R_{V_iV_j}(x)$ be the trigonometric R-matrices. Let $R_{ij}(x) \in \operatorname{End}(V_1 \otimes \dots \otimes V_n)$ be defined in a standard way:

$$R_{ij}(x) = \sum \mathrm{id} \otimes \dots \otimes \underset{i\text{-th}}{r(x)} \otimes \dots \otimes \underset{j\text{-th}}{r'(x)} \otimes \dots \otimes \mathrm{id} \tag{3.11}$$

provided that $R_{V_iV_j}(x) = \sum r(x) \otimes r'(x) \in \operatorname{End}(V_i \otimes V_j)$. For any $X \in U_q(\mathfrak{sl}_2)$ set

$$X_m = \mathrm{id} \otimes \dots \otimes \underset{m\text{-th}}{X} \otimes \dots \otimes \mathrm{id}\,.$$

Let p, κ be complex numbers. For any $m = 1, \dots, n$ set

$$\begin{aligned} K_m(z_1, \dots, z_n) = R_{m,m-1}(pz_m/z_{m-1}) \dots R_{m,1}(pz_m/z_1)\, \kappa^{\Lambda_m - H_m} \times \\ \times R_{m,n}(z_m/z_n) \dots R_{m,m+1}(z_m/z_{m+1})\,. \end{aligned} \tag{3.12}$$

(3.13) Theorem. [FR] *The linear maps* $K_m(z)$ *obey the flatness conditions*

$$K_l(z_1,\dots,pz_m,\dots,z_n)\,K_m(z_1,\dots,z_n) = \\ = K_m(z_1,\dots,pz_l,\dots,z_n)\,K_l(z_1,\dots,z_n)\,,$$

$l,m=1,\dots,n$.

The maps $K_1(z),\dots,K_n(z)$ define a flat connection on the trivial bundle over $\mathbb{C}^{\times n}$ with fiber $V_1\otimes\dots\otimes V_n$. This connection is called the *qKZ connection.*

By (3.5) the operators $K_m(z)$ commute with the action of q^H in $V_1\otimes\dots\otimes V_n$:

$$[K_m(z_1,\dots,z_n),q^H]=0\,,\qquad m=1,\dots,n\,,$$

and, therefore, preserve the weight decomposition of $V_1\otimes\dots\otimes V_n$. Hence, the *qKZ* connection induces the dual flat connection on the trivial bundle over $\mathbb{C}^{\times n}$ with fiber $(V_1\otimes\dots\otimes V_n)^*$. This connection will be called the *dual qKZ connection.*

(3.14) Lemma. *For any* $z\in\mathbb{C}^{\times n}$ *such that* $q^{2\Lambda_l+2\Lambda_m-2r}z_l/z_m\neq p^s$ *for all* $r=0,\dots,\ell-1$, *and* $l,m=1,\dots,n$, $l\neq m$, $s\in\mathbb{Z}$, *the linear maps* $K_1^*(z)$, $\dots,K_n^*(z)$ *define isomorphisms of* $(V_1\otimes\dots\otimes V_n)^*_\ell$.

This statement follows from formulae (3.7) and (3.12).

If $\kappa=q^{2\sum_{m=1}^{n}\Lambda_m-2\ell+2}$, then the dual *qKZ* connection admits a trivial discrete subbundle with fiber $F(V_1\otimes\dots\otimes V_n)^*_{\ell-1}$ and, therefore, it induces a flat connection on the trivial bundle with fiber $(V_1\otimes\dots\otimes V_n)^*_\ell/F(V_1\otimes\dots\otimes V_n)^*_{\ell-1}$.

If $\kappa=p^{-1}q^{-2\sum_{m=1}^{n}\Lambda_m+2\ell-2}$, then the dual *qKZ* connection admits a discrete subbundle with fiber $F_z(V_1\otimes\dots\otimes V_n)^*_{\ell-1}$ and, therefore, it induces a flat connection on the discrete vector bundle with fiber $(V_1\otimes\dots\otimes V_n)^*_\ell/F_z(V_1\otimes\dots\otimes V_n)^*_{\ell-1}$.

Let $V_1,\dots,V_n$ be $U_q(\mathfrak{sl}_2)$-modules. The *qKZ equation* for a $V_1\otimes\dots\otimes V_n$ valued function $\Psi(z_1,\dots,z_n)$ is the following system of equations

$$\Psi(z_1,\dots,pz_m,\dots,z_n) = K_m(z_1,\dots,z_n)\,\Psi(z_1,\dots,z_n)\,,\qquad m=1,\dots,n\,.$$

The *qKZ* equation is a remarkable difference equation, see [S], [FR], [JM], [Lu].

Modules over the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ and the elliptic R-matrices

In this section we recall the definitions concerning the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$, $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules and the R-matrices associated with tensor products of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules. For a more detailed exposition on the subject and proofs see [F], [FV].

Fix two complex numbers ρ, γ such that $\operatorname{Im} \rho > 0$. Set $p = e^{2\pi i\rho}$ and $\eta = e^{-4\pi i\gamma}$. Let $\theta(u) = (u;p)_\infty (pu^{-1};p)_\infty (p;p)_\infty$ be the Jacobi theta-function and

$$\alpha(x,\lambda) = \frac{\eta\,\theta(x)\,\theta(\lambda/\eta)}{\theta(\eta x)\,\theta(\lambda)}, \qquad \beta(x,\lambda) = \frac{\theta(\eta)\,\theta(x\lambda)}{\theta(\eta x)\theta(\lambda)}.$$

Let e_{ij}, $i,j = 1,2$, be the 2×2 matrix with the only nonzero entry 1 at the intersection of the i-th row and j-th column. Set

$$\begin{aligned} R(x,\lambda) &= e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + \alpha(x,\lambda)\,e_{11}\otimes e_{22} + \\ &\quad + \alpha(x,\lambda^{-1})\,e_{22}\otimes e_{11} + \beta(x,\lambda)\,e_{12}\otimes e_{21} + \beta(x,\lambda^{-1})\,e_{21}\otimes e_{12}. \end{aligned}$$

Remark. In [FV] the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ is described in terms of the additive theta-function

$$\Theta(u) = -\sum_{m=-\infty}^{\infty} \exp\bigl(\pi i(m+1/2)^2\rho + 2\pi i(m+1/2)(u+1/2)\bigr),$$

which is related to the multiplicative theta-function $\theta(x)$ by the equality

$$\Theta(u) = i\exp(\pi i\rho/4 - \pi i u)\,\theta(e^{2\pi i u}).$$

Let $\mathfrak{h}$ be the one-dimensional Lie algebra with the generator H. Let V be an $\mathfrak{h}$-module. Say that V is *diagonalizable* if V is a direct sum of finite-dimensional eigenspaces of H:

$$V = \bigoplus_{\mu} V_\mu, \qquad Hv = \mu v \quad \text{for } v \in V_\mu.$$

For a function $X(\mu)$ taking values in $\operatorname{End}(V)$ we set $X(H)v = X(\mu)v$ for any $v \in V_\mu$.

Let $V_1, \dots, V_n$ be diagonalizable $\mathfrak{h}$-modules. We have the decomposition

$$V_1 \otimes \dots \otimes V_n = \bigoplus_{\mu_1,\dots,\mu_n} (V_1)_{\mu_1} \otimes \dots \otimes (V_n)_{\mu_n} .$$

Set $H_m = \mathrm{id} \otimes \dots \otimes \underset{m\text{-th}}{H} \otimes \dots \otimes \mathrm{id}$. For a function $X(\mu_1, \dots, \mu_n)$ taking values in $\mathrm{End}(V_1 \otimes \dots \otimes V_n)$ we set $X(H_1, \dots, H_n)v = X(\mu_1, \dots, \mu_n)v$ for any $v \in (V_1)_{\mu_1} \otimes \dots \otimes (V_n)_{\mu_n}$.

By definition [FV], a *module over the elliptic quantum group* $E_{\rho,\gamma}(\mathfrak{sl}_2)$ is a diagonalizable $\mathfrak{h}$-module V together with four $\mathrm{End}(V)$-valued functions $T_{ij}(u, \lambda)$, $i, j = 1, 2$, which are meromorphic in $u, \lambda \in \mathbb{C}^\times$ and obey the following relations

$$[T_{ij}(u, \lambda), H] = (j - i)\, H , \qquad i, j = 1, 2 ,$$

$$R(x/y, \eta^{1\otimes 1\otimes 2H}\lambda)\, T_{(1)}(x, \lambda)\, T_{(2)}(y, \eta^{2H\otimes 1\otimes 1}\lambda) =$$
$$= T_{(2)}(y, \lambda)\, T_{(1)}(x, \eta^{1\otimes 2H\otimes 1}\lambda)\, R(x/y, \lambda) .$$

Here $T_{(1)}(u, \lambda) = \sum_{ij} e_{ij} \otimes \mathrm{id} \otimes T_{ij}(u, \lambda)$, $T_{(2)}(u, \lambda) = \sum_{ij} \mathrm{id} \otimes e_{ij} \otimes T_{ij}(u, \lambda)$, and H acts in $\mathbb{C}^2$ as $(e_{11} - e_{22})/2$.

Example. Fix a complex number Λ. Consider a diagonalizable $\mathfrak{h}$-module $V^\Lambda = \bigoplus_{k\in\mathbb{Z}_{\geqslant 0}} \mathbb{C} v^{[k]}$, such that

$$H v^{[k]} = (\Lambda - k)\, v^{[k]} .$$

Let x be a nonzero complex number. Set

$$T_{11}(u, \lambda)\, v^{[k]} = \frac{\theta(\eta^{\Lambda-k} u/x)\, \theta(\eta^{-k}\lambda)}{\theta(\eta^\Lambda u/x)\, \theta(\lambda)}\, \eta^k v^{[k]} ,$$

$$T_{12}(u, \lambda)\, v^{[k]} = \frac{\theta(\eta^{\Lambda-k-1}\lambda u/x)\, \theta(\eta)}{\theta(\eta^\Lambda u/x)\, \theta(\lambda)}\, v^{[k+1]} ,$$

$$T_{21}(u, \lambda)\, v^{[k]} = \frac{u\, \theta(\eta^{\Lambda-k+1}\lambda x/u)\, \theta(\eta^{2\Lambda-k+1})\, \theta(\eta^k)}{x\, \theta(\eta^\Lambda u/x)\, \theta(\lambda)\, \theta(\eta)}\, \eta^{k-1-\Lambda} v^{[k-1]} ,$$

$$T_{22}(u, \lambda)\, v^{[k]} = \frac{\theta(\eta^{k-\Lambda} u/x)\, \theta(\eta^{2\Lambda-k}\lambda)}{\theta(\eta^\Lambda u/x)\, \theta(\lambda)}\, v^{[k]} .$$

These formulae make V^Λ into an $E_{\rho,\gamma}(\mathfrak{sl}_2)$-module $V^\Lambda(x)$, which is called the *evaluation Verma module* with *highest weight* Λ and *evaluation point* x [FV].

For any complex vector space V denote by $Fun(V)$ the space of V-valued meromorphic functions on $\mathbb{C}^\times$. The space V is naturally embedded in $Fun(V)$ as the subspace of constant functions.

Let V_1, V_2 be complex vector spaces. Any function $\varphi \in Fun(\mathrm{Hom}(V_1, V_2))$ induces a linear map

$$^{Fun}\varphi : Fun(V_1) \to Fun(V_2), \qquad {}^{Fun}\varphi : f(\lambda) \mapsto \varphi(\lambda) f(\lambda).$$

For any $E_{\rho,\gamma}(\mathfrak{sl}_2)$-module V we define the associated *operator algebra* acting on the space $Fun(V)$. The operator algebra is generated by meromorphic functions in λ, η^H acting pointwise,

$$\varphi(\lambda, \eta^H) : f(\lambda) \mapsto \varphi(\lambda, \eta^H) f(\lambda), \tag{3.15}$$

and by values and residues with respect to x of the operator-valued meromorphic functions $\widetilde{T}_{ij}(x)$, $i, j = 1, 2$, defined below:

$$\widetilde{T}_{i1}(x) : f(\lambda) \mapsto T_{i1}(x, \lambda) f(\eta\lambda), \qquad \widetilde{T}_{i2}(x) : f(\lambda) \mapsto T_{i2}(x, \lambda) f(\eta^{-1}\lambda).$$

The relations obeyed by the generators of the operator algebra are described in detail in [FV].

Let V_1, V_2 be $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules. An element $\varphi \in Fun(\mathrm{Hom}(V_1, V_2))$ such that the induced map $^{Fun}\varphi$ intertwines the actions of the respective operator algebras is called a *morphism* of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules V_1, V_2. A morphism φ is called an *isomorphism* if the linear map $\varphi(\lambda)$ is nondegenerate for generic λ.

Example. Evaluation Verma modules $V^\Lambda(x)$ and $V^{\mathrm{M}}(x)$ are isomorphic if $\eta^\Lambda = \eta^{\mathrm{M}}$ with the tautological isomorphism.

The elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ has the coproduct Δ^{ell}:

$$\Delta^{ell} : H \mapsto H \otimes 1 + 1 \otimes H,$$

$$\Delta^{ell} : T_{ij}(u, \lambda) \mapsto \sum_k \bigl(1 \otimes T_{ik}(u, \eta^{2H\otimes 1}\lambda)\bigr) \bigl(T_{kj}(u, \lambda) \otimes 1\bigr).$$

The precise meaning of the coproduct is that it defines an $E_{\rho,\gamma}(\mathfrak{sl}_2)$-module structure on the tensor product $V_1 \otimes V_2$ of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules V_1, V_2. If V_1, V_2, V_3 are $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules, then the modules $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$ are naturally isomorphic [F].

Remark. Notice that we take the coproduct Δ^{ell} which is opposite to the coproduct used in [F], [FV], [FTV1]. The coproduct Δ^{ell} is in a sence opposite to the coproduct Δ taken for the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$.

(3.16) Theorem. [FV], [FTV1] *Let $V^{\Lambda}(x)$, $V^{\mathrm{M}}(y)$ be evaluation Verma modules. Then for any Λ, M and generic x, y there is a unique isomorphism $\widetilde{R}_{(x,y)}$ of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules $V^{\Lambda}(x) \otimes V^{\mathrm{M}}(y)$, $V^{\mathrm{M}}(y) \otimes V^{\Lambda}(x)$ such that $\widetilde{R}_{(x,y)}(\lambda) v^{[0]} \otimes v^{[0]} = v^{[0]} \otimes v^{[0]}$. Moreover, $\widetilde{R}_{(x,y)}$ has the form*

$$\widetilde{R}_{(x,y)}(\lambda) = P_{V^{\Lambda}V^{\mathrm{M}}} R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(x/y, \lambda)$$

where $R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(u, \lambda)$ is a meromorphic function of $u, \lambda \in \mathbb{C}^{\times}$ with values in $\mathrm{End}(V^{\Lambda} \otimes V^{\mathrm{M}})$ *and $P_{V^{\Lambda}V^{\mathrm{M}}} : V^{\Lambda} \otimes V^{\mathrm{M}} \to V^{\mathrm{M}} \otimes V^{\Lambda}$ is the permutation map.*

(3.17) Corollary. *The function $R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(x, \lambda)$ satisfies the inversion relation*

$$P_{V^{\Lambda}V^{\mathrm{M}}} R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(x, \lambda) = \left(R^{ell}_{V^{\mathrm{M}}V^{\Lambda}}(x^{-1}, \lambda)\right)^{-1} P_{V^{\Lambda}V^{\mathrm{M}}}.$$

The function $R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(x, \lambda)$ is called the $\mathfrak{sl}_2$ *dynamical elliptic R-matrix* for the tensor product $V^{\Lambda} \otimes V^{\mathrm{M}}$.

(3.18) Theorem. [FV], [FTV1] *For any complex numbers $\Lambda, \mathrm{M}, \mathrm{N}$ the corresponding elliptic R-matrices satisfy the dynamical Yang-Baxter equation in the space* $\mathrm{End}(V^{\Lambda} \otimes V^{\mathrm{M}} \otimes V^{\mathrm{N}})$:

$$R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(x/y, \eta^{1\otimes 1\otimes 2H}\lambda)\, R^{ell}_{V^{\Lambda}V^{\mathrm{N}}}(x, \lambda)\, R^{ell}_{V^{\mathrm{M}}V^{\mathrm{N}}}(y, \eta^{2H\otimes 1\otimes 1}\lambda) =$$
$$= R^{ell}_{V^{\mathrm{M}}V^{\mathrm{N}}}(y, \lambda)\, R^{ell}_{V^{\Lambda}V^{\mathrm{N}}}(x, \eta^{1\otimes 2H\otimes 1}\lambda)\, R^{ell}_{V^{\Lambda}V^{\mathrm{M}}}(x/y, \lambda).$$

One can associate a discrete flat connection with the dynamical elliptic R-matrix. This connection is studied in [FTV1], [FTV2].

4. Tensor coordinates on the hypergeometric spaces

In this chapter we identify the Gauss-Manin connection and the qKZ connection.

Tensor coordinates on the trigonometric hypergeometric spaces of fibers

Let $V_1, \dots, V_n$ be $U_q(\mathfrak{sl}_2)$ Verma modules with highest weights $q^{\Lambda_1}, \dots, q^{\Lambda_n}$ and generating vectors $v_1, \dots, v_n$, respectively. Consider the weight subspace $(V_1 \otimes \dots \otimes V_n)_\ell$ with a basis given by monomials $F^{\mathfrak{l}_1} v_1 \otimes \dots \otimes F^{\mathfrak{l}_n} v_n$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$. The dual space $(V_1 \otimes \dots \otimes V_n)_\ell^*$ has the dual basis denoted by $(F^{\mathfrak{l}_1} v_1 \otimes \dots \otimes F^{\mathfrak{l}_n} v_n)^*$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$.

Consider the trigonometric $\mathfrak{sl}_2$-type local system with connection coefficients (2.3) where the parameters $\xi_1, \dots, \xi_n$ and η are related to the parameter q and the highest weights $q^{\Lambda_1}, \dots, q^{\Lambda_n}$ as follows:

$$\eta = q^2, \qquad \xi_m = q^{2\Lambda_m}, \qquad m = 1, \dots, n.$$

Let $\mathcal{F} = \mathcal{F}[z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell]$ be the corresponding trigonometric hypergeometric space. For any $z \in \mathbb{C}^{\times n}$ and for any $\tau \in \mathbf{S}^n$ denote by $B_\tau(z)$ the following linear map:

$$B_\tau(z) : (V_{\tau_1} \otimes \dots \otimes V_{\tau_n})_\ell^* \to \mathcal{F}(z), \tag{4.1}$$

$$B_\tau(z) : (F^{\mathfrak{l}_{\tau_1}} v_{\tau_1} \otimes \dots \otimes F^{\mathfrak{l}_{\tau_n}} v_{\tau_n})^* \mapsto b_{\mathfrak{l}} w_{\mathfrak{l}}^\tau(t, z),$$

where $\mathcal{F}(z)$ is the trigonometric hypergeometric space of the fiber and

$$b_{\mathfrak{l}} = \prod_{m=1}^n q^{\mathfrak{l}_m(\mathfrak{l}_m - 1)/2 + \mathfrak{l}_m \Lambda_m},$$

(cf. (2.20), (2.27)). The linear maps $B_\tau(z)$ are called the *tensor coordinates* on the trigonometric hypergeometric space of a fiber. The composition maps

$$B_{\tau,\tau'}(z) : (V_{\tau'_1} \otimes \dots \otimes V_{\tau'_n})_\ell^* \to (V_{\tau_1} \otimes \dots \otimes V_{\tau_n})_\ell^*,$$

$$B_{\tau,\tau'}(z) = B_\tau^{-1}(z) \circ B_{\tau'}(z),$$

are called the *transition functions*, cf. [V3], [TV3].

(4.2) Lemma. *Let $q^{2\Lambda_l + 2\Lambda_m - 2r} z_l / z_m \neq 1$ for any $r = 0, \dots, \ell - 1$, and $l, m = 1, \dots, n$, $l \neq m$. Then for any permutation τ the linear map $B_\tau(z)$: $(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})_\ell^* \to \mathcal{F}(z)$ is nondegenerate.*

The statement follows from Lemma 2.22 since $\xi_l\xi_m z_l/z_m \neq \eta^r$ for any $r = 0, \dots, \ell-1$, and $l, m = 1, \dots, n$, $l \neq m$.

Consider a tensor product $V_{\tau_1}(z_{\tau_1}) \otimes \dots \otimes V_{\tau_n}(z_{\tau_n})$ of evaluation modules over $U'_q(\widetilde{\mathfrak{gl}_2})$ coinciding with $V_{\tau_1} \otimes \dots \otimes V_{\tau_n}$ as a $U_q(\mathfrak{sl}_2)$-module.

(4.3) Lemma. *For any $\varphi \in (V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_\ell$ we have*

$$\langle \varphi, L^+_{12}(t_1) \dots L^+_{12}(t_\ell) v_{\tau_1} \otimes \dots \otimes v_{\tau_n} \rangle = \bigl(B_\tau(z)\varphi\bigr)(t_1, \dots, t_\ell) \times$$

$$\times\ (q - q^{-1})^\ell q^{\ell(1-\ell)/2 - \ell \sum_{m=1}^{n} \Lambda_m} \prod_{a=1}^{\ell} \prod_{m=1}^{n} (\xi_m - t_a/z_m) \prod_{1 \leqslant a < b \leqslant \ell} \frac{\eta t_a - t_b}{t_a - t_b}\,,$$

$$\langle \varphi, L^-_{12}(t_1) \dots L^-_{12}(t_\ell) v_{\tau_1} \otimes \dots \otimes v_{\tau_n} \rangle = \bigl(B_\tau(z)\varphi\bigr)(t_1, \dots, t_\ell) \times$$

$$\times\ (q - q^{-1})^\ell q^{\ell(1-\ell)/2 - \ell \sum_{m=1}^{n} \Lambda_m} \prod_{a=1}^{\ell} \prod_{m=1}^{n} (1 - \xi_m z_m/t_a) \prod_{1 \leqslant a < b \leqslant \ell} \frac{\eta t_a - t_b}{t_a - t_b}\,.$$

It is easy to see that the right hand sides of the formulae above are polynomials in $t_1, \dots, t_\ell$ and $t_1^{-1}, \dots, t_\ell^{-1}$, respectively, cf. (2.10) and (4.1). So both the formulae make sense without additional prescriptions.

Lemma 4.3 is proved in Chapter 7.

(4.4) Theorem. [V3] *For any $\tau \in \mathbf{S}^n$ and any transposition $(m, m+1)$, $m = 1, \dots, n-1$, the transition function*

$$B_{\tau, \tau\cdot(m,m+1)}(z) : (V_{\tau_1} \otimes \dots \otimes V_{\tau_{m+1}} \otimes V_{\tau_m} \otimes \dots \otimes V_{\tau_n})^*_\ell \to$$

$$\to (V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_\ell$$

equals the operator $\bigl(P_{V_{\tau_m} V_{\tau_{m+1}}} R_{V_{\tau_m} V_{\tau_{m+1}}}(z_{\tau_m}/z_{\tau_{m+1}})\bigr)^*$ acting in the m-th and $(m+1)$-th factors.

The theorem follows from Lemma 4.3 and formula (3.10).

Each $B_\tau(z)$ induces a linear map $(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_\ell \to \mathcal{H}(z)$ which also will be denoted by $B_\tau(z)$.

(4.5) Theorem. *Let $\kappa \neq p^s \eta^{-r} \prod_{m=1}^{n} \xi_m$ and $\kappa \neq p^{-s-1} \eta^r \prod_{m=1}^{n} \xi_m^{-1}$ for any $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}_{\geqslant 0}$. Let $0 < |p| < 1$. Let (2.13) – (2.15) hold. Then for any $\tau \in \mathbf{S}^n$ the map $B_\tau(z) : (V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_\ell \to \mathcal{H}(z)$ is an isomorphism.*

This statement follows from Theorem 2.16 and Lemma 4.2. The assumption of the theorem means that $\kappa \neq p^s q^{2\sum_{m=1}^n \Lambda_m - 2r}$ and $\kappa \neq p^{-s-1} q^{-2\sum_{m=1}^n \Lambda_m + 2r}$ for any $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}_{\geqslant 0}$.

It is easy to see that for any $\tau \in \mathbf{S}^n$ the images of $F(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_{\ell-1}$ and $F_z(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_{\ell-1}$ under the map $B_\tau(z)$ coincide respectively with the coboundary subspaces $\mathcal{R}(z)$ and $\mathcal{R}'(z)$ in the trigonometric hypergeometric space of the fiber $\mathcal{F}(z)$.

(4.6) Theorem. *Let* $0 < |p| < 1$. *Let* (2.13) – (2.15) *hold. Let* $\prod_{m=1}^n \xi_m^2 \neq p^s \eta^r$ *for any* $r = \ell - 1, \dots, 2\ell - 2$, $s \in \mathbb{Z}_{<0}$. *If* $\kappa = \eta^{1-\ell} \prod_{m=1}^n \xi_m$, *that is* $\kappa = q^{2\sum_{m=1}^n \Lambda_m - 2\ell + 2}$, *then for any* $\tau \in \mathbf{S}^n$ *the map* $B_\tau(z)$ *induces an isomorphism*

$$(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_\ell / F(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_{\ell-1} \to \mathcal{H}(z).$$

Similarly, if $\kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^n \xi_m^{-1}$, *that is* $\kappa = p^{-1} q^{-2\sum_{m=1}^n \Lambda_m + 2\ell - 2}$, *then for any* $\tau \in \mathbf{S}^n$ *the map* $B_\tau(z)$ *induces an isomorphism*

$$(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_\ell / F_z(V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^*_{\ell-1} \to \mathcal{H}(z).$$

The statements follow from Theorem 2.18 and Lemmas 2.22 – 2.24. The assumption of the theorem means that $q^{4\sum_{m=1}^n \Lambda_m - 2r} \neq p^s$ for any $r = \ell - 1, \dots, 2\ell - 2$, $s \in \mathbb{Z}_{<0}$.

Taking into account formulae (3.1) and (3.2) we get an isomorphism

$$\big((V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^{sing}_\ell\big)^* \to \mathcal{H}(z)$$

for $\kappa = \eta^{1-\ell} \prod_{m=1}^n \xi_m$ and an isomorphism

$$\big((V_{\tau_1} \otimes \dots \otimes V_{\tau_n})^{sing}_{\ell,z}\big)^* \to \mathcal{H}(z)$$

for $\kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^n \xi_m^{-1}$.

(4.7) Theorem. [V3], [TV1] *For any* $m = 1, \dots, n$, *the following diagram is commutative:*

$$\begin{array}{ccc} (V_{\tau_1}\otimes\ldots\otimes V_{\tau_n})^*_\ell & \xrightarrow{K^*_m(z_{\tau_1},\ldots,z_{\tau_n})} & (V_{\tau_1}\otimes\ldots\otimes V_{\tau_n})^*_\ell \\ {\scriptstyle B_\tau(z_1,\ldots,pz_m,\ldots,z_n)}\big\downarrow & & \big\downarrow{\scriptstyle B_\tau(z_1,\ldots,z_n)} \\ \mathcal{H}(z_1,\ldots,pz_m,\ldots,z_n) & \xrightarrow[A_m(z_1,\ldots,z_n)]{} & \mathcal{H}(z_1,\ldots,z_n) \end{array}$$

Here $A_m(z)$ are the operators of the Gauss-Manin connection, $K^*_m(z)$ are the operators dual to $K_m(z)$, and $K_m(z)$ are the operators of the qKZ connection in $(V_{\tau_1}\otimes\ldots\otimes V_{\tau_n})_\ell$ defined by (3.12).

(4.8) Corollary. *The construction above identifies the qKZ connection and the Gauss-Manin connection restricted to the hypergeometric subbundle.*

Tensor coordinates on the elliptic hypergeometric spaces of fibers

Let $V^e_1(z_1),\ldots,V^e_n(z_n)$ be evaluation Verma modules over $E_{\rho,\gamma}(\mathfrak{sl}_2)$ with highest weights $\Lambda_1,\ldots,\Lambda_n$ and evaluation points $z_1,\ldots,z_n$, respectively. Let $V^e_1,\ldots,V^e_n$ be the corresponding $\mathfrak{h}$-modules. The weight subspace $(V^e_1\otimes\ldots\otimes V^e_n)_\ell$ has a basis given by the monomials $v^{[\mathfrak{l}_1]}\otimes\ldots\otimes v^{[\mathfrak{l}_n]}$, $\mathfrak{l}\in\mathcal{Z}^n_\ell$.

Let $\mathcal{F}_{ell}=\mathcal{F}_{ell}[\kappa;z_1,\ldots,z_n;\xi_1,\ldots,\xi_n;\ell]$ be the elliptic hypergeometric space where the parameters $\xi_1,\ldots,\xi_n$ and the highest weights $\Lambda_1,\ldots,\Lambda_n$ are related as follows:

$$\xi_m=\eta^{\Lambda_m},\qquad m=1,\ldots,n.$$

For any $z\in\mathbb{B}$ and for any $\tau\in\mathbf{S}^n$ denote by $C_\tau(z)$ the following linear map:

$$C_\tau(z):(V^e_{\tau_1}\otimes\ldots\otimes V^e_{\tau_n})_\ell\to\mathcal{F}_{ell}(z),$$
$$C_\tau(z):v^{[\mathfrak{l}_{\tau_1}]}\otimes\ldots\otimes v^{[\mathfrak{l}_{\tau_n}]}\mapsto c^\tau_\mathfrak{l}\,W^\tau_\mathfrak{l}(t,z).$$

Here $\mathcal{F}_{ell}(z)$ is the elliptic hypergeometric space of the fiber and $c^\tau_\mathfrak{l}=c_{\tau\mathfrak{l}}(\xi_{\tau_1},\ldots,\xi_{\tau_n})$ where ${}^\tau\mathfrak{l}=(\mathfrak{l}_{\tau_1},\ldots,\mathfrak{l}_{\tau_n})$ and

$$(4.9)\qquad c_\mathfrak{l}(\xi_1,\ldots,\xi_n)=\prod_{m=1}^n\prod_{s=1}^{\mathfrak{l}_m}\frac{\theta(\eta^s)\theta(\eta^{1-s}\xi^2_m)}{\theta(\eta)}\prod_{1\leqslant l<m\leqslant n}\eta^{-\mathfrak{l}_l\mathfrak{l}_m}\xi_m^{2\mathfrak{l}_l}\times$$
$$\times\Big(\prod_{s=1}^{\mathfrak{l}_1}\theta\big(\eta^{s-\ell}\kappa^{-1}\prod_{m=1}^n\xi_m\big)\prod_{s=1}^{\mathfrak{l}_n}\theta\big(\eta^{\ell-s}\kappa^{-1}\prod_{m=1}^n\xi_m^{-1}\big)\times$$
$$\times\prod_{m=1}^{n-1}\prod_{\substack{s=-\mathfrak{l}_m\\ s\neq 0}}^{\mathfrak{l}_{m+1}}\theta\big(\eta^s\kappa^{-1}\prod_{1\leqslant l\leqslant m}\eta^{\mathfrak{l}_l}\xi_l^{-1}\prod_{m<l\leqslant n}\eta^{-\mathfrak{l}_l}\xi_l\big)\Big)^{-1}.$$

The linear maps $C_\tau(z)$ are called the *tensor coordinates* on the elliptic hypergeometric space of a fiber. The composition maps

$$C_{\tau,\tau'}(z) : (V^e_{\tau'_1} \otimes \ldots \otimes V^e_{\tau'_n})_\ell \to (V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n})_\ell ,$$

$$C_{\tau,\tau'}(z) = C^{-1}_\tau(z) \circ C_{\tau'}(z) ,$$

are called the *transition functions*, cf. [V3], [TV3].

Example. For $\ell = 1$ the coefficients $c_\mathfrak{l}$ have the form

$$c_{\mathfrak{e}(m)}(\xi_1, \ldots, \xi_n) =$$
$$= \theta(\xi_m^2) \Big(\theta\big(\kappa^{-1} \prod_{1\leqslant l\leqslant m} \xi_l^{-1} \prod_{m<l\leqslant n} \xi_l\big)\, \theta\big(\kappa^{-1} \prod_{1\leqslant l<m} \xi_l^{-1} \prod_{m\leqslant l\leqslant n} \xi_l\big)\Big)^{-1} \prod_{m<l\leqslant n} \xi_l^2$$

where $\mathfrak{e}(m) = (0, \ldots, \underset{m\text{-th}}{1}, \ldots, 0)$, $m = 1, \ldots, n$.

Example. For $\ell = 2$ and $n = 2$ the coefficients $c_\mathfrak{l}$ have the form

$$c_{(2,0)}(\xi_1, \xi_2) = \xi_2^4 \,\frac{\theta(\eta^2)}{\theta(\eta)}\, \theta(\xi_1^2)\,\theta(\eta^{-1}\xi_1^2) \times$$
$$\times \big(\theta(\kappa^{-1}\xi_1\xi_2)\,\theta(\eta^{-1}\kappa^{-1}\xi_1\xi_2)\,\theta(\kappa^{-1}\xi_1^{-1}\xi_2)\,\theta(\eta\kappa^{-1}\xi_1^{-1}\xi_2)\big)^{-1},$$

$$c_{(1,1)}(\xi_1, \xi_2) = \eta^{-1}\xi_2^2\, \theta(\xi_1^2)\,\theta(\xi_2^2) \times$$
$$\times \big(\theta(\eta^{-1}\kappa^{-1}\xi_1\xi_2)\,\theta(\eta\kappa^{-1}\xi_1^{-1}\xi_2)\,\theta(\eta^{-1}\kappa^{-1}\xi_1^{-1}\xi_2)\,\theta(\eta\kappa^{-1}\xi_1^{-1}\xi_2^{-1})\big)^{-1},$$

$$c_{(0,2)}(\xi_1, \xi_2) = \frac{\theta(\eta^2)}{\theta(\eta)}\, \theta(\xi_2^2)\,\theta(\eta^{-1}\xi_2^2) \times$$
$$\times \big(\theta(\kappa^{-1}\xi_1^{-1}\xi_2)\,\theta(\eta^{-1}\kappa^{-1}\xi_1^{-1}\xi_2)\,\theta(\kappa^{-1}\xi_1^{-1}\xi_2^{-1})\,\theta(\eta\kappa^{-1}\xi_1^{-1}\xi_2^{-1})\big)^{-1}.$$

(4.10) Lemma. *Let $\eta^r \neq p^s$ for any $r = 2, \ldots, \ell$, $s \in \mathbb{Z}$. Let*

$$\eta^{\Lambda_l+\Lambda_m} z_l/z_m \neq p^s\eta^r \qquad \text{and} \qquad \kappa^{\pm1} \prod_{m=1}^{n} \eta^{\Lambda_m} \neq p^s\eta^r$$

for any $l, m = 1, \ldots, n$, and $r = 0, \ldots, \ell - 1$, $s \in \mathbb{Z}$. Assume that for a permutation τ we have $\kappa \prod_{1\leqslant l\leqslant m} \eta^{\Lambda_{\tau_l}} \prod_{m<l\leqslant n} \eta^{-\Lambda_{\tau_l}} \neq p^s\eta^r$ for any $m = 1, \ldots,$

$n-1$, and $r=1-\ell,\dots,\ell-1$, $s\in\mathbb{Z}$. Then the map $C_\tau(z):(V^e_{\tau_1}\otimes\dots\otimes V^e_{\tau_n})_\ell\to\mathcal{F}_{\mathrm{ell}}(z)$ is an isomorphism.

The statement follows from Lemma 2.33.

Remark. The map $C_\tau(z)$ considered as a function of κ has a simple pole at $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\xi_m$ because of the factor $\theta\bigl(\eta^{1-\ell}\kappa^{-1}\prod_{m=1}^{n}\xi_m\bigr)$ in formula (4.9) for the coefficients $c_{\mathfrak{l}}$, and

$$\begin{aligned}\operatorname{Ker}\bigl(\operatorname{Res}^+C_\tau(z)\bigr)&=v^{[0]}\otimes(V^e_{\tau_2}\otimes\dots\otimes V^e_{\tau_n})_\ell\,,\\ \operatorname{Im}\bigl(\operatorname{Res}^+C_\tau(z)\bigr)&=\mathcal{Q}(z)\,,\end{aligned}\tag{4.11}$$

where $\operatorname{Res}^+C_\tau(z)$ is the residue of $C_\tau(z)$ at $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\xi_m$ and $\mathcal{Q}(z)$ is the boundary subspace.

Similarly, the map $C_\tau(z)$ has a simple pole at $\kappa=p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\xi_m^{-1}$ because of the factor $\theta\bigl(\eta^{\ell-1}\kappa^{-1}\prod_{m=1}^{n}\xi_m^{-1}\bigr)$ in formula (4.9) for the coefficients $c_{\mathfrak{l}}$, and

$$\begin{aligned}\operatorname{Ker}\bigl(\operatorname{Res}^-C_\tau(z)\bigr)&=(V^e_{\tau_1}\otimes\dots\otimes V^e_{\tau_{n-1}})_\ell\otimes v^{[0]}\,,\\ \operatorname{Im}\bigl(\operatorname{Res}^-C_\tau(z)\bigr)&=\mathcal{Q}'(z)\,,\end{aligned}\tag{4.12}$$

where $\operatorname{Res}^-C_\tau(z)$ is the residue of $C_\tau(z)$ at $\kappa=p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\xi_m^{-1}$ and $\mathcal{Q}'(z)$ is the boundary subspace.

(4.13) Lemma. *Let $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$. Let $\eta^r\neq p^s$ for any $r=2,\dots,\ell$, $s\in\mathbb{Z}$. Let $\eta^{\Lambda_l+\Lambda_m}z_l/z_m\neq p^s\eta^r$ for any $l,m=2,\dots,n$, and $r=0,\dots,\ell-1$, $s\in\mathbb{Z}$. Let $\eta^{2\Lambda_1}\neq p^s\eta^r$ for any $r=0,\dots,\ell-1$, $s\in\mathbb{Z}$. Let $\prod_{m=1}^{n}\eta^{2\Lambda_m}\neq p^s\eta^r$ for any $r=\ell-1,\dots,2\ell-2$, $s\in\mathbb{Z}$. Assume that for a permutation τ we have $\prod_{1\leqslant l\leqslant m}\eta^{2\Lambda_{\tau_l}}\neq p^s\eta^r$ for any $m=2,\dots,n-1$, and $r=0,\dots,2\ell-2$, $s\in\mathbb{Z}$. Then the linear map $C_\tau(z):v^{[0]}\otimes(V^e_{\tau_2}\otimes\dots\otimes V^e_{\tau_n})_\ell\to\mathcal{F}_{\mathrm{ell}}(z)/\mathcal{Q}(z)$ is an isomorphism.*

The statement follows from Lemma 2.38.

(4.14) Lemma. *Let* $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$. *Let* $\eta^r \neq p^s$ *for any* $r = 2, \dots, \ell$, $s \in \mathbb{Z}$. *Let* $\eta^{\Lambda_l+\Lambda_m} z_l/z_m \neq p^s\eta^r$ *for any* $l, m = 1, \dots, n-1$, *and* $r = 0, \dots, \ell-1$, $s \in \mathbb{Z}$. *Let* $\eta^{2\Lambda_n} \neq p^s\eta^r$ *for any* $r = 0, \dots, \ell-1$, $s \in \mathbb{Z}$. *Let* $\prod_{m=1}^{n}\eta^{2\Lambda_m} \neq p^s\eta^r$ *for any* $r = \ell-1, \dots, 2\ell-2$, $s \in \mathbb{Z}$. *Assume that for a permutation* τ *we have* $\prod_{m<l\leqslant n}\eta^{2\Lambda_{\tau_l}} \neq p^s\eta^r$ *for any* $m = 1, \dots, n-2$, *and* $r = 0, \dots, 2\ell-2$, $s \in \mathbb{Z}$. *Then the linear map* $C_\tau(z) : (V^e_{\tau_1} \otimes \dots \otimes V^e_{\tau_{n-1}})_\ell \otimes v^{[0]} \to \mathcal{F}_{ell}(z)/\mathcal{Q}'(z)$ *is an isomorphism.*

The statement follows from Lemma 2.39.

Consider a tensor product $V^e_{\tau_1}(z_{\tau_1}) \otimes \dots \otimes V^e_{\tau_n}(z_{\tau_n})$ of evaluation Verma modules over $E_{\rho,\gamma}(\mathfrak{sl}_2)$ coinciding with $V^e_{\tau_1} \otimes \dots \otimes V^e_{\tau_n}$ as an $\mathfrak{h}$-module.

(4.15) Lemma. *Let* $\lambda = \kappa \prod_{m=1}^{n}\xi_m^{-1}$. *Then for any* $v \in (V^e_{\tau_1} \otimes \dots \otimes V^e_{\tau_n})_\ell$ *we have*

$$T_{21}(t_1, \lambda) \dots T_{21}(t_\ell, \eta^{\ell-1}\lambda)v = \big(C_\tau(z)v\big)(t_1, \dots, t_\ell) \prod_{s=0}^{\ell-1}\theta(\eta^s\kappa^{-1}\prod_{m=1}^{n}\xi_m^{-1}) \times$$

$$\times \prod_{a=1}^{\ell}\prod_{m=1}^{n}\frac{\theta(\xi_m^{-1}t_a/z_m)}{\theta(\xi_m t_a/z_m)} \prod_{1\leqslant a<b\leqslant\ell}\frac{\theta(\eta t_a/t_b)}{\theta(t_a/t_b)}\, v^{[0]} \otimes \dots \otimes v^{[0]},$$

The lemma is proved in Chapter 7.

(4.16) Theorem. *For any* $\tau \in \mathbf{S}^n$ *and any transposition* $(m, m+1)$, $m = 1, \dots, n-1$, *the transition function*

$$C_{\tau,\tau\cdot(m,m+1)}(z) : V^e_{\tau_1} \otimes \dots \otimes V^e_{\tau_{m+1}} \otimes V^e_{\tau_m} \otimes \dots \otimes V^e_{\tau_n} \to V^e_{\tau_1} \otimes \dots \otimes V^e_{\tau_n}$$

equals the operator

$$P_{V^e_{\tau_{m+1}}V^e_{\tau_m}} R^{ell}_{V^e_{\tau_{m+1}}V^e_{\tau_m}}\big(z_{\tau_{m+1}}/z_{\tau_m}, (\eta^H \otimes \dots \otimes \eta^H \otimes \underset{m\text{-th}}{\eta^{-H}} \otimes \dots \otimes \eta^{-H})\eta^{-1}\kappa\big).$$

The theorem follows from Lemma 4.15 and Theorem 3.16.

Remark. The elliptic R-matrix in Theorem 4.16 has an operator

$$K = (\eta^H \otimes \dots \otimes \eta^H \otimes \underset{m\text{-th}}{\eta^{-H}} \otimes \dots \otimes \eta^{-H})\eta^{-1}\kappa$$

at the place of the second argument, and $R^{ell}_{V^e_{\tau_{m+1}}V^e_{\tau_m}}(x, \lambda)$ commutes with K

for any values of x, λ. The operator $R^{ell}_{V^e_{\tau_{m+1}} V^e_{\tau_m}}(x, K)$ is understood in the standard way:

$$R^{ell}_{V^e_{\tau_{m+1}} V^e_{\tau_m}}(x, K)v = R^{ell}_{V^e_{\tau_{m+1}} V^e_{\tau_m}}(x, \lambda)v$$

for any $v \in V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_{m+1}} \otimes V^e_{\tau_m} \otimes \ldots \otimes V^e_{\tau_n}$ such that $Kv = \lambda v$.

Tensor products of the hypergeometric spaces

Let $\mathcal{F}[z_1, \ldots, z_m; \xi_1, \ldots, \xi_m; l]$ and $\mathcal{F}_{ell}[\alpha; z_1, \ldots, z_m; \xi_1, \ldots, \xi_m; l]$ be respectively the trigonometric and the elliptic hypergeometric spaces defined for the projection $\mathbb{C}^{l+m} \to \mathbb{C}^m$. In particular, in our previous notations we have

$$\mathcal{F} = \mathcal{F}[z_1, \ldots, z_n; \xi_1, \ldots, \xi_n; \ell] \quad \text{and} \quad \mathcal{F}_{ell} = \mathcal{F}_{ell}[\kappa; z_1, \ldots, z_n; \xi_1, \ldots, \xi_n; \ell].$$

There are maps

$$\begin{aligned} \chi : \mathcal{F}[z_1, \ldots, z_k; \xi_1, \ldots, \xi_k; j] \otimes \\ \otimes \mathcal{F}[z_{k+1}, \ldots, z_{k+m}; \xi_{k+1}, \ldots, \xi_{k+m}; l] \to \\ \to \mathcal{F}[z_1, \ldots, z_{k+m}; \xi_1, \ldots, \xi_{k+m}; j+l] \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \chi_{ell} : \mathcal{F}_{ell}[\alpha\eta^l \prod_{i=1}^m \xi_{i+k}^{-1}; z_1, \ldots, z_k; \xi_1, \ldots, \xi_k; j]((z_1, \ldots, z_k)) \otimes \\ \otimes \mathcal{F}_{ell}[\alpha\eta^{-j} \prod_{i=1}^k \xi_i; z_{k+1}, \ldots, z_{k+m}; \xi_{k+1}, \ldots, \xi_{k+m}; l]((z_{k+1}, \ldots, z_{k+m})) \to \\ \to \mathcal{F}_{ell}[\alpha; z_1, \ldots, z_{k+m}; \xi_1, \ldots, \xi_{k+m}; j+l]((z_1, \ldots, z_{k+m})) \end{aligned}$$

which are respectively defined by $\chi : f \otimes g \mapsto f \star g$ and $\chi_{ell} : f \otimes g \mapsto f * g$ where

$$(f \star g)(t_1, \ldots, t_{j+l}) = \\ = \frac{1}{j!\,l!} \sum_{\sigma \in \mathbf{S}^{j+l}} \Big[f(t_1, \ldots, t_j)\, g(t_{j+1}, \ldots, t_{j+l}) \prod_{i=1}^k \prod_{a=1}^l \frac{\xi_i t_{a+j} - z_i}{t_{a+j} - \xi_i z_i} \Big]_\sigma$$

and

$$(f * g)(t_1, \ldots, t_{j+l}) = \\ = \frac{1}{j!\,l!} \sum_{\sigma \in \mathbf{S}^{j+l}} \Big[\!\Big[f(t_1, \ldots, t_j)\, g(t_{j+1}, \ldots, t_{j+l}) \prod_{i=1}^k \prod_{a=1}^l \frac{\theta(\xi_i t_{a+j}/z_i)}{\theta(\xi_i^{-1} t_{a+j}/z_i)} \Big]\!\Big]_\sigma .$$

We have the next lemmas.

(4.18) Lemma. *Assume that* $\xi_i\xi_{j+k}z_{j+k}/z_i \neq \eta^r$ *for any* $i=1,\dots,k$, $j=1,\dots,m$, $r=0,\dots,l-1$. *Then the map*

$$\chi : \bigoplus_{i+j=l} \mathcal{F}[z_1,\dots,z_k;\xi_1,\dots,\xi_k;i]\big((z_1,\dots,z_k)\big) \otimes$$
$$\otimes\, \mathcal{F}[z_{k+1},\dots,z_{k+m};\xi_{k+1},\dots,\xi_{k+m};j]\big((z_{k+1},\dots,z_{k+m})\big) \to$$
$$\to \mathcal{F}[z_1,\dots,z_{k+m};\xi_1,\dots,\xi_{k+m};l]\big((z_1,\dots,z_{k+m})\big)$$

defined by linearity is bijective.

(4.19) Lemma. *Assume that* $\xi_i\xi_{j+k}z_{j+k}/z_i \neq p^s\eta^r$ *and* $\alpha\prod_{a=1}^{k}\xi_a\prod_{a=1}^{m}\xi_{a+k}^{-1} \neq p^s\eta^{\pm r}$ *for any* $i=1,\dots,k$, $j=1,\dots,m$, $r=0,\dots,l-1$, $s\in\mathbb{Z}$. *Then the map*

$$\chi_{ell} : \bigoplus_{i+j=l} \mathcal{F}_{ell}\Big[\alpha\eta^j\prod_{a=1}^{m}\xi_{a+k}^{-1};z_1,\dots,z_k;\xi_1,\dots,\xi_k;i\Big]\big((z_1,\dots,z_k)\big) \otimes$$
$$\otimes\, \mathcal{F}_{ell}\Big[\alpha\eta^{-i}\prod_{a=1}^{k}\xi_a;z_{k+1},\dots,z_{k+m};\xi_{k+1},\dots,\xi_{k+m};j\Big]\big((z_{k+1},\dots,z_{k+m})\big) \to$$
$$\to \mathcal{F}_{ell}[\alpha;z_1,\dots,z_{k+m};\xi_1,\dots,\xi_{k+m};l]\big((z_1,\dots,z_{k+m})\big)$$

defined by linearity is bijective

Lemmas 4.18 and 4.19 are proved in Chapter 7.

It is clear that for any functions f,g,h we have $(f\star g)\star h = f\star(g\star h)$ and for any functions f,g,h we have $(f*g)*h = f*(g*h)$. Lemmas 4.18, 4.19 can be extended naturally to an arbitrary number of factors.

The map χ_{ell} admits the following generalization. Fix a nonnegative integer k. Let $n_0,\dots,n_k$ be integers such that

$$0 = n_0 < n_1 < \dots < n_k\,.$$

Fix nonnegative integers $l_1,\dots,l_k$. Let

$$\mathcal{F}^i_{ell} = \mathcal{F}_{ell}[\alpha_i;z_{n_{i-1}+1},\dots,z_{n_i};\xi_{n_{i-1}+1},\dots,\xi_{n_i};l_i]$$

where $\alpha_i = \alpha\eta^{\sum_{j>i} l_j - \sum_{j<i} l_j} \prod_{j \leqslant n_{i-1}} \xi_j \prod_{j>n_i} \xi_j^{-1}$. Let $h(z_1, \ldots, z_{n_k})$ be a meromorphic function on $\mathbb{C}^{\times n_k}$ such that

$$h(z_1, \ldots, pz_i, \ldots, z_{n_k}) = \xi_i^{\sum_{j<i} l_j - \sum_{j>i} l_j} h(z_1, \ldots, z_{n_k})$$

for any $i = 1, \ldots, n_k$. Then we have a well defined map

$$\mathcal{F}_{ell}^1 \otimes \ldots \otimes \mathcal{F}_{ell}^k \to \mathcal{F}_{ell}[\alpha; z_1, \ldots, z_{n_k}; \xi_1, \ldots, \xi_{n_k}; l_1 + \ldots + l_k], \tag{4.20}$$

$$f_1 \otimes \ldots \otimes f_k \mapsto (f_1 * \ldots * f_k) h$$

of the elliptic hypergeometric spaces. We call the function $h(z_1, \ldots, z_{n_k})$ an *adjusting factor* for the tensor product of the elliptic hypergeometric spaces $\mathcal{F}_{ell}^1 \otimes \ldots \otimes \mathcal{F}_{ell}^k$.

5. The hypergeometric pairing and the hypergeometric solutions of the qKZ equation

In this chapter we define the main object of this paper, the hypergeometric pairing. We define a pairing between the trigonometric and the elliptic hypergeometric spaces of a fiber.

The hypergeometric integral

For any functions $w \in \mathcal{F}(z)$ and $W \in \mathcal{F}_{ell}(z)$ we define the *hypergeometric integral* by

$$(5.1) \qquad I(W, w) = \int_{\widetilde{\mathbb{T}}^\ell} \Phi(t)\, w(t)\, W(t)\ (dt/t)^\ell$$

where $(dt/t)^\ell = \prod_{a=1}^{\ell} dt_a/t_a$, $\Phi(t)$ is the short phase function defined in (5.2) and $\widetilde{\mathbb{T}}^\ell$ is a suitable deformation of the torus

$$\mathbb{T}^\ell = \{ t \in \mathbb{C}^\ell \mid |t_1| = 1, \ \ldots\ , |t_\ell| = 1 \}.$$

Recall that we always have $0 < |p| < 1$. Let $(u)_\infty = \prod_{k=0}^{\infty} (1 - p^k u)$. We take $\varPhi(x; \alpha) = \dfrac{(x\alpha^{-1})_\infty}{(x\alpha)_\infty}$ in (2.5) so that the short phase function has the form

$$(5.2) \quad \Phi(t_1, \ldots, t_\ell, z_1, \ldots, z_n) = \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{(\xi_m^{-1} t_a/z_m)_\infty}{(\xi_m t_a/z_m)_\infty} \prod_{1 \leqslant a < b \leqslant \ell} \frac{(\eta t_a/t_b)_\infty}{(\eta^{-1} t_a/t_b)_\infty}.$$

We define the hypergeometric integral as follows. Assume that $|\eta| > 1$ and $|z_m| = 1$, $|\xi_m| < 1$, $m = 1, \ldots, n$. Set

$$(5.3) \qquad I(W, w) = \int_{\mathbb{T}^\ell} \Phi(t)\, w(t)\, W(t)\ (dt/t)^\ell .$$

Notice that the integrand has simple poles at the hyperplanes

$$(5.4) \qquad \begin{aligned} t_a/z_m &= (p^s \xi_m)^{\pm 1}, & a &= 1, \ldots, \ell, & m &= 1, \ldots, n, \\ t_a/t_b &= (p^s \eta^{-1})^{\pm 1}, & 1 &\leqslant a < b \leqslant \ell, \end{aligned}$$

for $s \in \mathbb{Z}_{\geqslant 0}$ and essential singularities at the coordinate hyperplanes. The set of hyperplanes (5.4) could be decomposed into subsets corresponding to couples $\{a, m\}$ or $\{a, b\}$. Under the above assumptions for each subset of the hyperplanes the torus $\mathbb{T}^\ell$ separates the hyperplanes corresponding to different choices of the sign.

The hypergeometric integral for generic $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ and arbitrary η is defined by analytic continuation with respect to $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ and η. This analytic continuation makes sence since the integrand is analytic in $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ and η, cf. (2.5), (2.20), (2.30). More precisely, first we define the hypergeometric integral for basis functions $w_{\mathfrak{l}}$, $W_{\mathfrak{m}}$ and then extend the definition by linearity to arbitrary functions $w \in \mathcal{F}(z)$, $W \in \mathcal{F}_{ell}(z)$. The result of the analytic continuation can be represented as an integral of the integrand over a suitably deformed torus. Namely, the poles of the integrand of the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ are located at the hyperplanes

$$(5.5) \qquad t_a = p^s \xi_m z_m\,, \qquad t_a = p^{-s} \xi_m^{-1} z_m\,, \qquad t_a = p^s \eta^{-1} t_b\,, \qquad t_a = p^{-s} \eta t_b\,,$$

$1 \leqslant b < a \leqslant \ell$, $m = 1, \dots, n$, $s \in \mathbb{Z}_{\geqslant 0}$. We deform $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ and η in such a way that the topology of the complement in $\mathbb{C}^{\times \ell}$ to the union of hyperplanes (5.5) does not change. We deform accordingly the torus $\mathbb{T}^\ell$ so that it does not intersect the hyperplanes (5.5) at every moment of the deformation. The deformed torus is denoted by $\widetilde{\mathbb{T}}^\ell$. Then the analytic continuation of the integral (5.3) is given by formula (5.1).

(5.6) Theorem. *For any* $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n$ *the hypergeometric integral* $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ *can be analytically continued as a holomorphic univalued function of complex variables* η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ *to the region:*

$$\begin{gathered}
\eta \neq 0\,, \qquad \xi_m \neq 0\,, \qquad z_m \neq 0\,, \qquad m = 1, \dots, n\,, \\
\eta^{r+1} \neq p^s\,, \qquad \xi_m^2 \neq p^s \eta^r\,, \qquad m = 1, \dots, n\,, \\
\xi_l^{\pm 1} \xi_m^{\pm 1} z_l / z_m \neq p^s \eta^r\,, \qquad l, m = 1, \dots, n\,, \qquad l \neq m\,,
\end{gathered}$$

where $r = 0, \dots, \ell - 1$, $s \in \mathbb{Z}$, *and the combination of signs* ± 1 *can be arbitrary (cf.* (2.13) – (2.15)).

The proof of the theorem is the same as the proof of Theorem 5.7 in [TV3].

Let $\mathcal{R}(z)$, $\mathcal{R}'(z) \subset \mathcal{F}(z)$ be the coboundary subspaces and let $\mathcal{Q}(z)$, $\mathcal{Q}'(z) \subset \mathcal{F}_{ell}(z)$ be the boundary subspaces.

(5.7) Lemma. *Let* $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$. *Let* (2.13) – (2.15) *hold. Then*

a) *For any* $w \in \mathcal{R}(z)$ *and* $W \in \mathcal{F}_{ell}(z)$ *the hypergeometric integral* $I(W,w)$ *equals zero.*

b) *For any* $w \in \mathcal{F}(z)$ *and* $W \in \mathcal{Q}(z)$ *the hypergeometric integral* $I(W,w)$ *equals zero.*

Example. Let $\ell = 1$ and $\kappa = \prod_{m=1}^{n} \xi_m$. Then the space $\mathcal{Q}(z)$ is one-dimensional and is spanned by the function $W(t_1) = 1$. Assume for simplicity that $|z_m| = 1$, $|\xi_m| < 1$ for any $m = 1, \dots, n$. Then the hypergeometric integral $I(W,w)$ is given by

$$I(W,w) = \int_{|t_1|=1} w(t_1) \prod_{m=1}^{n} \frac{(\xi_m^{-1} t_1/z_m)_\infty}{(\xi_m t_1/z_m)_\infty} \frac{dt_1}{t_1} .$$

Since $w(0) = 0$ for any $w \in \mathcal{F}(z)$, the integrand is regular in the disk $|t_1| \leqslant 1$. Hence, $I(W,w) = 0$ for any $w \in \mathcal{F}(z)$.

(5.8) Lemma. *Let* $\kappa = p^{-1}\eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$. *Let* (2.13) – (2.15) *hold. Then*

a) *For any* $w \in \mathcal{R}'(z)$ *and* $W \in \mathcal{F}_{ell}(z)$ *the hypergeometric integral* $I(W,w)$ *equals zero.*

b) *For any* $w \in \mathcal{F}(z)$ *and* $W \in \mathcal{Q}'(z)$ *the hypergeometric integral* $I(W,w)$ *equals zero.*

Example. Let $\ell = 1$ and $\kappa = \prod_{m=1}^{n} \xi_m^{-1}$. Then the space $\mathcal{Q}'(z)$ is one-dimensional and is spanned by the function $W(t_1) = t_1^{-1} \prod_{m=1}^{n} \dfrac{\theta(\xi_m t_1/z_m)}{\theta(\xi_m^{-1} t_1/z_m)}$. Assume for simplicity that $|z_m| = 1$, $|\xi_m| < 1$ for any $m = 1, \dots, n$. Then the hypergeometric integral $I(W,w)$ is given by

$$I(W,w) = \int_{|t_1|=1} w(t_1) \prod_{m=1}^{n} \frac{(p\xi_m^{-1} z_m/t_1)_\infty}{(p\xi_m z_m/t_1)_\infty} \frac{dt_1}{t_1^2} .$$

Since $w(t_1) = O(1)$ as $t_1 \to \infty$ for any $w \in \mathcal{F}(z)$, the integrand is regular in the domain $|t_1| \geqslant 1$ and behaves as $O(t_1^{-2})$ as $t_1 \to \infty$. Hence, $I(W,w) = 0$ for any $w \in \mathcal{F}(z)$.

Lemmas 5.7 and 5.8 are proved in Chapter 7.

Determinant formulae for the hypergeometric pairing

The hypergeometric integral defines the *hypergeometric pairing*

$$I : \mathcal{F}_{ell}(z) \otimes \mathcal{F}(z) \to \mathbb{C}$$

which induces the hypergeometric pairings

$$I^\circ : \mathcal{F}_{ell}(z)/\mathcal{Q}(z) \otimes \mathcal{F}(z)/\mathcal{R}(z) \to \mathbb{C}$$

for $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$ and

$$I' : \mathcal{F}_{ell}(z)/\mathcal{Q}'(z) \otimes \mathcal{F}(z)/\mathcal{R}'(z) \to \mathbb{C}$$

for $\kappa = p^{-1}\eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$. According to (2.17), (2.25) and Lemmas 2.38, 2.39 this can be respectively written as

$$I : \mathcal{F}_{ell}(z) \otimes \mathcal{H}(z) \to \mathbb{C}\,,$$

$$I^\circ : \mathcal{F}_{ell}(z)/\mathcal{Q}(z) \otimes \mathcal{H}(z) \to \mathbb{C} \qquad \text{and} \qquad I' : \mathcal{F}_{ell}(z)/\mathcal{Q}'(z) \otimes \mathcal{H}(z) \to \mathbb{C}\,.$$

Set $d(n,m,\ell,s) = \sum_{\substack{i,j\geqslant 0 \\ i+j<\ell \\ i-j=s}} \binom{m-1+i}{m-1}\binom{n-m-1+j}{n-m-1}$.

(5.9) Theorem. *Let* $\kappa \neq p^s\eta^{-r} \prod_{m=1}^{n} \xi_m$ *and* $\kappa \neq p^{-s-1}\eta^{r} \prod_{m=1}^{n} \xi_m^{-1}$, $r = 0, \ldots, \ell-1$, $s \in \mathbb{Z}_{\geqslant 0}$. *Let* (2.13) – (2.15) *hold. Then the hypergeometric pairing* $I : \mathcal{F}_{ell}(z) \otimes \mathcal{F}(z) \to \mathbb{C}$ *is nondegenerate. Moreover,*

$$\det\big[I(W_{\mathfrak{l}}, w_{\mathfrak{m}})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} = (2\pi i)^{\ell\binom{n+\ell-1}{n-1}}\, \ell!^{\binom{n+\ell-1}{n-1}}\, \eta^{-n\binom{n+\ell-1}{n+1}} \times$$

$$\times \prod_{m=1}^{n} \xi_m^{(n-m)\binom{n+\ell-1}{n}} \prod_{s=1-\ell}^{\ell-1}\prod_{m=1}^{n-1} \theta\Big(\eta^s\kappa^{-1}\prod_{1\leqslant l\leqslant m}\xi_l^{-1}\prod_{m<l\leqslant n}\xi_l\Big)^{d(n,m,\ell,s)} \times$$

$$\times \prod_{s=0}^{\ell-1}\bigg[\frac{(\eta^{-1})_\infty^n\,(\eta^{s+1-\ell}\kappa^{-1}\prod\xi_m)_\infty\,(p\eta^{s+1-\ell}\kappa\prod\xi_m)_\infty}{(\eta^{-s-1})_\infty^n\,(p)_\infty^{2n-1}\prod(\eta^{-s}\xi_m^2)_\infty} \times$$

$$\times \prod_{1\leqslant l<m\leqslant n} \frac{(\eta^s\xi_l^{-1}\xi_m^{-1}z_l/z_m)_\infty}{(\eta^{-s}\xi_l\xi_m z_l/z_m)_\infty}\bigg]^{\binom{n+\ell-s-2}{n-1}}.$$

(5.10) Theorem. *Let* $\kappa = \eta^{1-\ell}\prod_{m=1}^{n}\xi_m$. *Let* (2.13) – (2.15) *hold. If* $\prod_{m=1}^{n}\xi_m^2 \neq p^s\eta^r$ *for all* $r = \ell-1,\dots,2\ell-2$ *and* $s\in\mathbb{Z}_{<0}$, *then the hypergeometric pairing* $I^\circ : \mathcal{F}_{ell}(z)/\mathcal{Q}(z)\otimes\mathcal{F}(z)/\mathcal{R}(z)\to\mathbb{C}$ *is nondegenerate. Moreover,*

$$\det\bigl[I(W_{\mathfrak{l}},w_{\mathfrak{m}})\bigr]_{\substack{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n\\ \mathfrak{l}_1=\mathfrak{m}_1=0}} = (2\pi i)^{\ell\binom{n+\ell-2}{n-2}}\,\ell!^{\binom{n+\ell-2}{n-2}}\,\eta^{(1-n)\binom{n+\ell-2}{n}}\times$$
$$\times\prod_{m=1}^{n}\xi_m^{(n-m)\binom{n+\ell-2}{n-1}}\prod_{s=1-\ell}^{\ell-1}\prod_{m=2}^{n-1}\theta\Bigl(\eta^{s+\ell-1}\prod_{1\leqslant l\leqslant m}\xi_l^{-2}\Bigr)^{d(n-1,m-1,\ell,s)}\times$$
$$\times\prod_{s=0}^{\ell-1}\Biggl[\frac{(\eta^{-1})_\infty^{n-1}\,(p\eta^{s+2-2\ell}\prod\xi_m^2)_\infty\,(\eta^s\xi_1^{-2})_\infty}{(\eta^{-s-1})_\infty^{n-1}\,(p)_\infty^{2n-3}\prod_{1<m\leqslant n}(\eta^{-s}\xi_m^2)_\infty}\times$$
$$\prod_{1\leqslant l<m\leqslant n}\frac{(\eta^s\xi_l^{-1}\xi_m^{-1}z_l/z_m)_\infty}{(\eta^{-s}\xi_l\xi_m z_l/z_m)_\infty}\Biggr]^{\binom{n+\ell-s-3}{n-2}}.$$

(5.11) Theorem. *Let* $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\xi_m^{-1}$. *Let* (2.13) – (2.15) *hold. If* $\prod_{m=1}^{n}\xi_m^2 \neq p^s\eta^r$ *for all* $r = \ell-1,\dots,2\ell-2$ *and* $s\in\mathbb{Z}_{<0}$, *then the hypergeometric pairing* $I' : \mathcal{F}_{ell}(z)/\mathcal{Q}'(z)\otimes\mathcal{F}(z)/\mathcal{R}'(z)\to\mathbb{C}$ *is nondegenerate. Moreover,*

$$\det\bigl[I(W_{\mathfrak{l}},w_{\mathfrak{m}})\bigr]_{\substack{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n\\ \mathfrak{l}_n=\mathfrak{m}_n=0}} = (2\pi i)^{\ell\binom{n+\ell-2}{n-2}}\,\ell!^{\binom{n+\ell-2}{n-2}}\,\eta^{(1-n)\binom{n+\ell-2}{n}}\times$$
$$\times\prod_{m=1}^{n-1}\xi_m^{(n-m-1)\binom{n+\ell-2}{n-1}}\prod_{s=1-\ell}^{\ell-1}\prod_{m=1}^{n-2}\theta\Bigl(p\eta^{s+1-\ell}\prod_{m<l\leqslant n}\xi_l^{2}\Bigr)^{d(n-1,m,\ell,s)}\times$$
$$\times\prod_{s=0}^{\ell-1}\Biggl[\frac{(\eta^{-1})_\infty^{n-1}\,(p\eta^{s+2-2\ell}\prod\xi_m^2)_\infty\,(\eta^s\xi_n^{-2})_\infty}{(\eta^{-s-1})_\infty^{n-1}\,(p)_\infty^{2n-3}\prod_{1\leqslant m<n}(\eta^{-s}\xi_m^2)_\infty}\times$$
$$\prod_{1\leqslant l<m\leqslant n}\frac{(\eta^s\xi_l^{-1}\xi_m^{-1}z_l/z_m)_\infty}{(\eta^{-s}\xi_l\xi_m z_l/z_m)_\infty}\Biggr]^{\binom{n+\ell-s-3}{n-2}}.$$

In Theorems 5.9 – 5.11, the product $\prod$ without limits stands for $\prod_{m=1}^{n}$.
Theorems 5.9 – 5.11 are proved in Chapter 7.

Example. Theorem 5.9 for $n=1$, $\ell=1$ gives

$$\int_C \frac{\theta(ct)}{(at)_\infty (b/t)_\infty}\,\frac{dt}{t} = 2\pi i\,\frac{(pa/c)_\infty\,(bc)_\infty}{(ab)_\infty}\,.$$

Here C is an anticlockwise oriented contour around the origin $t=0$ separating the sets $\{p^s/a \mid s\in\mathbb{Z}_{\leqslant 0}\}$ and $\{p^s b \mid s\in\mathbb{Z}_{\geqslant 0}\}$.

Example. Theorems 5.10 and 5.11 for $n=2$, $\ell=1$ give particular cases of the Askey-Roy formula [GR, (4.11.2)]

$$(5.12)\qquad \int_C \frac{\theta(pt/c)\,\theta(abct)}{(at)_\infty (bt)_\infty (\alpha/t)_\infty (\beta/t)_\infty}\,\frac{dt}{t} = 2\pi i\,\frac{(ab\alpha\beta)_\infty\,\theta(ac)\,\theta(bc)}{(p)_\infty (a\alpha)_\infty (a\beta)_\infty (b\alpha)_\infty (b\beta)_\infty}\,.$$

Here C is an anticlockwise oriented contour around the origin $t=0$ separating the sets $\{p^s/a,\ p^s/b \mid s\in\mathbb{Z}_{\leqslant 0}\}$ and $\{p^s\alpha,\ p^s\beta \mid s\in\mathbb{Z}_{\geqslant 0}\}$.

Example. There are p-analogues of the gamma-function and the power function:

$$\Gamma_p(x) = (1-p)^{1-x}(p)_\infty/(p^x)_\infty\,, \qquad (1-u)_p^{2x} = (p^{-x}u)_\infty/(p^x u)_\infty\,.$$

Introduce new functions $\{-u\}_p^{2x} = \theta(p^{-x}u)/\theta(p^x u)$ and

$$\sin_p(\pi x) = \frac{\pi}{\Gamma_p(x)\,\Gamma_p(1-x)} = \frac{\pi\,\theta(p^x)}{(1-p)\,(p)_\infty^3}\,.$$

We have $(1-u)_p^{2x} = \{-u\}_p^{2x}(1-pu^{-1})_p^{2x}$.

Theorems 5.10 and 5.11 for $\ell=1$ give respectively the following formulae:

$$\det\Biggl[\,\int_C \frac{p^{2\Delta_l}-1}{t-p^{\Delta_l}z_l}\prod_{m=1}^n (1-t/z_m)_p^{2\Delta_m}\,\{-p^{-\sum_{1\leqslant j\leqslant k}\Delta_j}\,t/z_k\}_p^{2\sum_{1\leqslant j<k}\Delta_j}\times$$
$$\times\prod_{1\leqslant j<k}\{-t/z_j\}_p^{-2\Delta_j}\prod_{1\leqslant j<l}\frac{t-p^{-\Delta_j}z_j}{t-p^{\Delta_j}z_j}\,\frac{dt}{p-1}\Biggr]_{k,l=2}^n =$$
$$= \Gamma_p\Bigl(1+2\sum_{m=1}^n\Delta_m\Bigr)^{-1}\prod_{m=1}^n\Gamma_p(1+2\Delta_m)\prod_{1\leqslant l<m\leqslant n}(1-z_l/z_m)_p^{2(\Delta_l+\Delta_m)}\times$$
$$\times\prod_{m=1}^{n-1}2i\,\sin_p\Bigl(-2\pi\sum_{1\leqslant j\leqslant m}\Delta_j\Bigr)$$

and

$$\det\Big[\int_C \frac{p^{\Delta_l}-p^{-\Delta_l}}{t/z_l-p^{\Delta_l}}\prod_{m=1}^{n}(1-t/z_m)_p^{2\Delta_m}\big\{-p^{\sum_{k<j\leqslant n}\Delta_j}t/z_k\big\}_p^{-2\sum_{k\leqslant j\leqslant n}\Delta_j}\times$$

$$\times\prod_{1\leqslant j<k}\{-t/z_j\}_p^{-2\Delta_j}\prod_{1\leqslant j<l}\frac{t-p^{-\Delta_j}z_j}{t-p^{\Delta_j}z_j}\,\frac{dt}{t(p-1)}\Big]_{k,l=1}^{n-1}=$$

$$=\Gamma_p(1+2\sum_{m=1}^{n}\Delta_m)^{-1}\prod_{m=1}^{n}\Gamma_p(1+2\Delta_m)\prod_{1\leqslant l<m\leqslant n}(1-z_l/z_m)_p^{2(\Delta_l+\Delta_m)}\times$$

$$\times\prod_{m=1}^{n-1}2i\sin_p\big(2\pi\sum_{m<j\leqslant n}\Delta_j\big)$$

where C is an anticlockwise oriented contour around the origin $t=0$ separating the sets $\{p^{s-\Delta_m}z_m\mid m=1,\dots,n,\ s\in\mathbb{Z}_{\leqslant 0}\}$ and $\{p^{s+\Delta_m}z_m\mid m=1,\dots,n,\ s\in\mathbb{Z}_{\geqslant 0}\}$. These formulae are analogues of the next formula [V1]:

$$\det\Big[\int_{z_k}^{z_{k+1}}\frac{\lambda_l}{t-z_l}\prod_{m=1}^{n}(t-z_m)^{\lambda_m}\,dt\Big]_{k,l=1}^{n-1}=$$

$$=\Gamma(1+\sum_{m=1}^{n}\lambda_m)^{-1}\prod_{m=1}^{n}\Gamma(1+\lambda_m)\prod_{l\neq m}(z_l-z_m)^{\lambda_m}.$$

Example. Theorem 5.9 for $n=1$ and arbitrary ℓ gives the following q-beta integral

$$\int_{\mathbb{T}^\ell}\prod_{k=1}^{\ell}\frac{\theta(ct_k)}{t_k\,(at_k)_\infty\,(b/t_k)_\infty}\prod_{j=1}^{\ell}\prod_{\substack{k=1\\k\neq j}}^{\ell}\frac{(t_j/t_k)_\infty}{(xt_j/t_k)_\infty}\,d^\ell t= \tag{5.13}$$

$$=(2\pi i)^\ell\,\ell!\prod_{s=0}^{\ell-1}\frac{(x)_\infty\,(x^sbc)_\infty\,(px^sa/c)_\infty}{(x^{s+1})_\infty\,(x^sab)_\infty}$$

where $|a|<1$, $|b|<1$, $|x|<1$. In Chapter 7 we use this formula to prove Theorems 5.9 – 5.11. We give a proof of the formula in Appendix D and show there that the calculation of the integral by residues inside the torus $\mathbb{T}^\ell$ implies the formula for the q-Selberg integral proved by Aomoto [AK, Theorem 3.2], see formula (D.9).

Example. The formulae of Theorems 5.10 and 5.11 for $n = 2$ are particular cases of the following formula

$$\int_{\mathbb{T}^\ell} \prod_{k=1}^{\ell} \frac{\theta(pt_k/c)\,\theta(x^{\ell-1}abct_k)}{t_k\,(at_k)_\infty\,(bt_k)\,(\alpha/t_k)_\infty\,(\beta/t_k)_\infty} \prod_{j=1}^{\ell}\prod_{\substack{k=1\\k\neq j}}^{\ell} \frac{(t_j/t_k)_\infty}{(xt_j/t_k)_\infty}\,d^\ell t = \tag{5.14}$$

$$= (2\pi i)^\ell\,\ell!\prod_{s=0}^{\ell-1}\frac{(x)_\infty\,(x^{\ell+s-1}ab\alpha\beta)_\infty\,\theta(x^sac)\,\theta(x^sbc)}{(x^{s+1})_\infty\,(x^sa\alpha)_\infty\,(x^sa\beta)_\infty\,(x^sb\alpha)_\infty\,(x^sb\beta)_\infty\,(p)_\infty}$$

where $|a| < 1$, $|b| < 1$, $|\alpha| < 1$, $|\beta| < 1$, $|x| < 1$. This formula is a multidimensional generalization of the Askey-Roy formula (5.12). In Appendix E we give a proof of this formula and show that the calculation of the integral by residues outside the torus $\mathbb{T}^\ell$ implies the formula for the most general multidimensional q-beta integral conjectured by Askey [As, Conjecture 8], see formula (E.8).

Remark. It is plausible that the assumptions on p, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ of Theorems 5.9 – 5.11 as well as of Theorems and Lemmas 2.16, 2.18, 4.5, 4.6, 5.7, 5.8, 6.2, 6.5, 6.6 could be replaced by the following weaker assumptions: the step p and the parameter η are such that (2.13) holds, and the parameters $\xi_1, \dots, \xi_n$ and $z_1, \dots, z_n$ are such that

$$\xi_l\xi_m z_l/z_m \neq p^s\eta^r\,, \qquad l, m = 1, \dots, n\,, \qquad s \in \mathbb{Z}\,,$$

for any $r = 0, \dots, \ell - 1$ and $s \in \mathbb{Z}$.

The hypergeometric solutions of the qKZ equation

Let W be any element of the elliptic hypergeometric space $\mathcal{F}_{ell}$. The restriction of the function W to a fiber defines an element $W|_z \in \mathcal{F}_{ell}(z)$ of the elliptic hypergeometric space of the fiber. The hypergeometric pairing allows us to consider the element $W|_z \in \mathcal{F}_{ell}(z)$ as an element $s_W(z)$ of the space $\mathcal{H}^*(z)$ dual to the hypergeometric cohomology group $\mathcal{H}(z)$. This construction defines a section of the bundle over $\mathbb{C}^{\times n}$ with fiber $\mathcal{H}^*(z)$.

There is a simple but important statement.

(5.15) Theorem. *Let* $\xi_1, \dots, \xi_n$ *obey* (2.14). *Then the section* s_W *is a periodic section with respect to the Gauss-Manin connection.*

The theorem is proved in Chapter 7.

Consider the hypergeometric pairing as a map $\bar{I}(z) : \mathcal{F}_{ell}(z) \to \big(\mathcal{F}(z)\big)^*$ so that for any $W \in \mathcal{F}_{ell}$ we have $s_W = \bar{I}(z)W|_z$. Let $V_1, \dots, V_n$ be Verma modules over $U_q(\mathfrak{sl}_2)$ with highest weights $q^{\Lambda_1}, \ldots, q^{\Lambda_n}$. Recall that

$$q^2 = \eta\,, \qquad \eta^{\Lambda_m} = \xi_m\,, \qquad m = 1, \ldots, n\,.$$

The map $\bar I(z)$ and the tensor coordinates $B_\tau(z)$ induce a map

$$\bigl(B_\tau(z)\bigr)^* \circ \bar I(z) \,:\, \mathcal{F}_{ell}(z) \,\to\, (V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell\,.$$

(5.16) Lemma. *Let* $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$. *Let* (2.13) – (2.15) *hold. Then for any* $W \in \mathcal{F}_{ell}(z)$ *we have that* $\bigl(B_\tau(z)\bigr)^* \circ \bar I(z) \cdot W \in (V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell^{sing}$.

(5.17) Lemma. *Let* $\kappa = p^{-1}\eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$. *Let* (2.13) – (2.15) *hold. Then for any* $W \in \mathcal{F}_{ell}(z)$ *we have that* $\bigl(B_\tau(z)\bigr)^* \circ \bar I(z) \cdot W \in (V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_{\ell,z}^{sing}$.

Lemmas 5.16, 5.17 follow from Lemmas 5.7, 5.8, respectively.

A section s_W and the tensor coordinates B_τ induce a section

$$\Psi_W^\tau \,:\, z \,\mapsto\, B_\tau^*(z^{\tau^{-1}})\, s_W(z^{\tau^{-1}}) \in (V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell \tag{5.18}$$

of the trivial bundle with fiber $(V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell$. Here $z^\sigma = (z_{\sigma_1}, \ldots, z_{\sigma_n})$. If τ is the identity permutation, then we write Ψ_W instead of Ψ_W^{id}.

Theorems 5.15, 4.7 and Lemmas 5.16, 5.17 imply the following statement.

(5.19) Corollary. *The section* Ψ_W^τ *is a solution of the qKZ equation with values in* $(V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell$. *Moreover,*

i) *if* $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$, *then* Ψ_W^τ *takes values in* $(V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell^{sing}$;

ii) *if* $\kappa = p^{-1}\eta^{\ell-1} \prod_{m=1}^{n} \xi_m^{-1}$, *then* Ψ_W^τ *takes values in* $(V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_{\ell,z}^{sing}$.

We call solutions Ψ_W^τ the *hypergeometric solutions* of the *qKZ* equation.

Let $S_{m,\tau}(z_1, \ldots, z_n) : V_{\tau_1} \otimes \ldots \otimes V_{\tau_n} \to V_{\tau_1} \otimes \ldots \otimes V_{\tau_{m+1}} \otimes V_{\tau_m} \otimes \ldots \otimes V_{\tau_n}$ equal the operator $P_{V_{\tau_m} V_{\tau_{m+1}}} R_{V_{\tau_m} V_{\tau_{m+1}}}(z_m/z_{m+1})$ acting in the m-th and $(m+1)$-th factors. Define operators $\hat S_{\tau,\tau'}$ acting on functions of $z_1, \ldots, z_n$

by the following formulae:

$$(5.20)\qquad \big(\hat S_{\tau\cdot(m,m+1),\tau} f\big)(z_1,\dots,z_n) =$$
$$= S_{m,\tau}(z_1,\dots,z_{m+1},z_m,\dots,z_n)\, f(z_1,\dots,z_{m+1},z_m,\dots,z_n)\,,$$
$$\hat S_{\tau,\tau'}\,\hat S_{\tau',\tau''} = \hat S_{\tau,\tau''}$$

where $(m,m+1)$ is a transposition, $m=1,\dots,n-1$, and $\tau,\tau',\tau''\in \mathbf{S}^n$ are arbitrary permutations. The operator $\hat S_{\tau,\tau'}$ acts on a function taking values in $V_{\tau'_1}\otimes\dots\otimes V_{\tau'_n}$ and the result is a function taking values in $V_{\tau_1}\otimes\dots\otimes V_{\tau_n}$.

(5.21) Lemma. *Formulae* (5.20) *define operators* $\hat S_{\tau,\tau'}$ *selfconsistently.*

The statement follows from the inversion relation (3.6) and the Yang-Baxter equation (3.8).

The *qKZ* equation has the following important property.

(5.22) Theorem. *The qKZ equation is functorial. Namely, for any permutations* $\tau,\tau'\in\mathbf{S}^n$ *and any solution* Ψ *of the qKZ equation with values in* $V_{\tau'_1}\otimes\dots\otimes V_{\tau'_n}$, *the function* $\hat S_{\tau,\tau'}\Psi$ *is a solution of the qKZ equation with values in* $V_{\tau_1}\otimes\dots\otimes V_{\tau_n}$.

(5.23) Theorem. *The hypergeometric solutions* Ψ^{τ}_W *of the qKZ equation are functorial. Namely, for any permutations* $\tau,\tau'\in\mathbf{S}^n$ *and any function* $W\in\mathcal{F}_{ell}$ *we have that* $\hat S_{\tau,\tau'}\Psi^{\tau'}_W=\Psi^{\tau}_W$.

The statement follows from Theorem 4.4.

The hypergeometric map

Let $V^e_1(z_1),\dots,V^e_n(z_n)$ be evaluation Verma modules over $E_{\rho,\gamma}(\mathfrak{sl}_2)$ with highest weights $\Lambda_1,\dots,\Lambda_n$ and evaluation points $z_1,\dots,z_n$. Let $V^e_1,\dots,V^e_n$ be the corresponding $\mathfrak{h}$-modules. The tensor coordinates $B_\tau(z)$, $C_{\tau'}(z)$ induce the *hypergeometric map*

$$(5.24)\qquad I_{\tau,\tau'}(z) : (V^e_{\tau'_1}\otimes\dots\otimes V^e_{\tau'_n})_\ell \to (V_{\tau_1}\otimes\dots\otimes V_{\tau_n})_\ell\,,$$
$$I_{\tau,\tau'}(z) = \big(B_\tau(z)\big)^*\circ \bar I(z)\circ C_{\tau'}(z)\,.$$

(5.25) Theorem. *Let* (2.13) – (2.15) *hold. Let* $\kappa^{\pm1}\prod_{m=1}^{n}\eta^{\Lambda_m}\neq p^s\eta^r$ *for any* $r=0,\dots,\ell-1$, $s\in\mathbb{Z}$. *Assume that for a permutation* τ' *we have* $\kappa\prod_{1\leqslant l\leqslant m}\eta^{\Lambda_{\tau'_l}}\prod_{m<l\leqslant n}\eta^{-\Lambda_{\tau'_l}}\neq p^s\eta^r$ *for any* $m=1,\dots,n-1$, *and* $r=1-\ell,\dots,$

$\ell-1$, $s\in\mathbb{Z}$. Then the hypergeometric map $I_{\tau,\tau'}(z)$ is well defined and nondegenerate.

The statement follows from Lemmas 4.2, 4.10 and Theorem 5.9.

(5.26) Theorem. *Let (2.13) – (2.15) hold. Let $\prod_{m=1}^{n}\eta^{2\Lambda_m}\neq p^s\eta^r$ for any $r=\ell-1,\dots,2\ell-2$, $s\in\mathbb{Z}$. Assume that for a permutation τ' we have $\prod_{1\leqslant l\leqslant m}\eta^{2\Lambda_{\tau'_l}}\neq p^s\eta^r$ for any $m=1,\dots,n-1$, and $r=0,\dots,2\ell-2$, $s\in\mathbb{Z}$. Then the hypergeometric map $I_{\tau,\tau'}(z)$ is well defined and nondegenerate for any κ in the punctured neighbourhood of $\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$. Moreover, $I_{\tau,\tau'}(z)$ considered as a function of κ has a finite limit as $\kappa\to\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ and the limit $\lim I_{\tau,\tau'}(z)$ is nondegenerate.*

Define the hypergeometric map $I_{\tau,\tau'}(z)$ at $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ by the analytic continuation with respect to κ. Notice that the restriction of the map $I_{\tau,\tau'}(z)$ to the subspace $v^{[0]}\otimes(V^e_{\tau'_2}\otimes\dots\otimes V^e_{\tau'_n})_\ell$ is regular at $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$, since in this case all the maps involved in definition (5.24) are well defined at $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$.

(5.27) Corollary. *Let $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ and the assumptions of Theorem 5.26 hold. Then*

$$I_{\tau,\tau'}(z)\big(v^{[0]}\otimes(V^e_{\tau'_2}\otimes\dots\otimes V^e_{\tau'_n})_\ell\big)\;=\;(V_{\tau_1}\otimes\dots\otimes V_{\tau_n})^{sing}_\ell .$$

The statement follows from Lemma 5.16.

(5.28) Theorem. *Let (2.13) – (2.15) hold. Let $\prod_{m=1}^{n}\eta^{2\Lambda_m}\neq p^s\eta^r$ for any $r=\ell-1,\dots,2\ell-2$, $s\in\mathbb{Z}$. Assume that for a permutation τ' we have $\prod_{m<l\leqslant n}\eta^{2\Lambda_{\tau'_l}}\neq p^s\eta^r$ for any $m=1,\dots,n-1$, and $r=0,\dots,2\ell-2$, $s\in\mathbb{Z}$. Then the hypergeometric map $I_{\tau,\tau'}(z)$ is well defined and nondegenerate for any κ in the punctured neighbourhood of $p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$. Moreover, $I_{\tau,\tau'}(z)$ considered as a function of κ has a finite limit as $\kappa\to p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$ and the limit $\lim I_{\tau,\tau'}(z)$ is nondegenerate.*

Define the hypergeometric map $I_{\tau,\tau'}(z)$ at $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$ by the analytic continuation with respect to κ. Notice that the restriction of the map $I_{\tau,\tau'}(z)$ to the subspace $(V^e_{\tau'_1}\otimes\ldots\otimes V^e_{\tau'_{n-1}})_\ell\otimes v^{[0]}$ is regular at $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$, since in this case all the maps involved in definition (5.24) are well defined at $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$.

(5.29) Corollary. *Let $\kappa = p^{-1}\eta^{\ell-1}\prod_{m=1}^{n}\eta^{-\Lambda_m}$ and the assumptions of Theorem 5.28 hold. Then*

$$I_{\tau,\tau'}(z)\big((V^e_{\tau_1}\otimes\ldots\otimes V^e_{\tau_{n-1}})_\ell\otimes v^{[0]}\big) = (V_{\tau_1}\otimes\ldots\otimes V_{\tau_n})^{sing}_{\ell,z}.$$

The statement follows from Lemma 5.17.

Theorems 5.26 and 5.28 are proved in Chapter 7.

Therefore, we constructed the hypergeometric maps

$$I_{\tau,\tau'}(z) : V^e_{\tau'_1}(z_{\tau'_1})\otimes\ldots\otimes V^e_{\tau'_n}(z_{\tau'_n}) \to V_{\tau_1}(z_{\tau_1})\otimes\ldots\otimes V_{\tau_n}(z_{\tau_n})$$

from modules over the elliptic quantum group to modules over the quantum loop algebra. The maps have the following properties:

$$I_{\tau\cdot(m,m+1),\tau'}(z) = P_{V_{\tau_m}V_{\tau_{m+1}}}R_{V_{\tau_m}V_{\tau_{m+1}}}(z_{\tau_m}/z_{\tau_{m+1}})\,I_{\tau,\tau'}(z)\,, \tag{5.30}$$

$$I_{\tau,\tau'\cdot(m,m+1)}(z) = I_{\tau,\tau'}(z)\,\times$$
$$\times\,P_{V^e_{\tau'_{m+1}}V^e_{\tau'_m}}R^{ell}_{V^e_{\tau'_{m+1}}V^e_{\tau'_m}}\big(z_{\tau'_{m+1}}/z_{\tau'_m},(\eta^H\otimes\ldots\otimes\eta^H\otimes\underset{m\text{-th}}{\eta^{-H}}\otimes\ldots\otimes\eta^{-H})\eta^{-1}\kappa\big)$$

where $(m,m+1)$ is a transposition.

For any elliptic weight function $W^\tau_\mathfrak{l}(t,z)$ let $Y^\tau_\mathfrak{l}(z)$ be the corresponding adjusting factor. Recall that this means that the product $Y^\tau_\mathfrak{l}(z)W^\tau_\mathfrak{l}(t,z)$ is an element of the elliptic hypergeometric space. Define a map $Y^\tau(z)\in\mathrm{End}\,(V^e_{\tau_1}\otimes\ldots\otimes V^e_{\tau_n})$ by the rule:

$$Y^\tau(z) : v^{[\mathfrak{l}_{\tau_1}]}\otimes\ldots\otimes v^{[\mathfrak{l}_{\tau_n}]} \to Y^\tau_\mathfrak{l}(z)\,v^{[\mathfrak{l}_{\tau_1}]}\otimes\ldots\otimes v^{[\mathfrak{l}_{\tau_n}]}.$$

The map $Y^\tau(z)$ is called an *adjusting map* for the tensor product $V^e_{\tau_1}\otimes\ldots\otimes V^e_{\tau_n}$.

If $v \in V^e_{\tau'_1} \otimes \ldots \otimes V^e_{\tau'_n}$, then the hypergeometric map $I_{\tau,\tau'}(z)$ and the adjusting map $Y^{\tau'}(z)$ define a section

$$\Psi^{\tau}_{v,Y^{\tau'}} : z \mapsto I_{\tau,\tau'}(z^{\tau^{-1}})Y^{\tau'}(z^{\tau^{-1}}) \cdot v \in V_{\tau_1} \otimes \ldots \otimes V_{\tau_n}$$

where $z^{\sigma} = (z_{\sigma_1}, \ldots, z_{\sigma_n})$.

(5.31) Theorem. *For any adjusting map $Y^{\tau'}(z)$ and any $v \in (V^e_{\tau'_1} \otimes \ldots \otimes V^e_{\tau'_n})_{\ell}$ the section $\Psi^{\tau}_{v,Y^{\tau'}}$ is a solution of the qKZ equation with values in $(V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_{\ell}$. Under assumptions of each of the Theorems 5.9–5.11 all solutions are constructed in this way.*

The theorem is proved in Chapter 7.

Remark. Theorem 5.31 can be reformulated as follows. For a given adjusting map $Y^{\tau}(z)$ the assignment $v \mapsto \Psi^{\mathrm{id}}_{v,Y^{\tau}}$ defines an isomorphism of the space $\mathbb{S}$ of solutions of the *qKZ* equation with values in $V_1 \otimes \ldots \otimes V_n$ and the space $V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes \mathbb{F}$, where $\mathbb{F}$ is the space of functions of $z_1, \ldots, z_n$ which are p-periodic with respect to each of the variables,

$$\mathbf{C}_{\tau} : V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes \mathbb{F} \to \mathbb{S}. \tag{5.32}$$

The compositions of the isomorphisms: $\mathbf{C}_{\tau,\tau'} = \mathbf{C}^{-1}_{\tau} \mathbf{C}_{\tau'}$, define linear maps

$$\mathbf{C}_{\tau,\tau'}(z) : V^e_{\tau'_1} \otimes \ldots \otimes V^e_{\tau'_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \tag{5.33}$$

depending on $z_1, \ldots, z_n$ and p-periodic with respect to all the variables. We call these compositions the transition functions. Theorem 6.2 in the next chapter shows that $\mathbf{C}_{\tau,\tau'}(z)$ is a transition function from the asymptotic solution of the *qKZ* equation in the asymptotic zone $\mathbb{A}_{\tau}$ to the asymptotic solution in the asymptotic zone $\mathbb{A}_{\tau'}$, cf. (6.1).

Notice that $\mathbf{C}_{\tau,\tau'}(z)$ differs from the transition function $C_{\tau,\tau'}(z)$ defined in Chapter 4, namely

$$\mathbf{C}_{\tau,\tau'}(z) = \big(Y^{\tau}(z)\big)^{-1} C_{\tau,\tau'}(z) Y^{\tau'}(z). \tag{5.34}$$

Let $\hat{S}_{\tau,\tau'}$ be operators defined by formulae (5.20). We extend their action to matrix-valued functions in a natural way.

The maps $\bar{I}_{\tau}(z) = I_{\tau,\tau}(z^{\tau^{-1}})Y^{\tau}(z^{\tau^{-1}})$ satisfy the *qKZ* equations with values in $V_{\tau_1} \otimes \ldots \otimes V_{\tau_n}$, respectively. The following theorem describes their "monodromy" properties with respect to permutations of the variables $z_1, \ldots, z_n$ in terms of the elliptic R-matrices.

(5.35) Theorem. *For any permutation* $\tau \in \mathbf{S}^n$ *and any transposition* $(m, m+1)$, $m = 1, \ldots, n-1$, *we have that*

$$\left(\hat{S}_{\tau,\tau\cdot(m,m+1)} \bar{I}_{\tau\cdot(m,m+1)}\right)(z) = \bar{I}_\tau(z) \left(Y^\tau(z^{\tau^{-1}})\right)^{-1} \times$$

$$\times P_{V^e_{\tau_{m+1}} V^e_{\tau_m}} R^{ell}_{V^e_{\tau_{m+1}} V^e_{\tau_m}} \big(z_{m+1}/z_m, (\eta^H \otimes \ldots \otimes \eta^H \otimes \underset{m\text{-th}}{\eta^{-H}} \otimes \ldots \otimes \eta^{-H})\eta^{-1}\kappa\big) \times$$

$$\times\, Y^{\tau\cdot(m,m+1)}\big(z^{(m,m+1)\cdot\tau^{-1}}\big).$$

The statement follows from formulae (5.30).

6. Asymptotic solutions of the qKZ equation

One of the most important characteristics of a differential equation is the monodromy group of its solutions. For the differential KZ equation with values in a tensor product of representations of a simple Lie algebra its monodromy group is described in terms of the corresponding quantum group. This fact establishes a remarkable connection between representation theories of simple Lie algebras and their quantum groups, see [K], [D2], [KL], [SV2], [V2], [V4].

The analogue of the monodromy group for difference equations is the set of transition functions between asymptotic solutions. For a difference equation one defines suitable asymptotic zones in the domain of the definition of the equation and then an asymptotic solution for every zone. Thus, for every pair of asymptotic zones one gets a transition function between the corresponding asymptotic solutions, cf. [TV3].

In this chapter we describe asymptotic zones, asymptotic solutions, and their transition functions for the qKZ equation with values in a tensor product of $U_q(\mathfrak{sl}_2)$-modules. A remarkable fact is that the transition functions are described in terms of the elliptic R-matrices acting in the tensor product of the corresponding $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules. This fact establishes a correspondence between representation theories of quantum loop algebras and elliptic quantum groups, since the qKZ equation is defined in terms of the trigonometric R-matrix action in the tensor product of $U_q(\mathfrak{sl}_2)$-modules (and, therefore, in terms of the quantum loop algebra action), and the elliptic R-matrix action in the tensor product of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules is defined in terms of the action of the elliptic quantum group.

Consider the qKZ equation with values in $(V_1 \otimes \ldots \otimes V_n)_\ell$. For every permutation $\tau \in \mathbf{S}^n$ we consider an asymptotic zone $\mathbb{A}_\tau$ in $\mathbb{C}^{\times n}$ given by

$$\mathbb{A}_\tau = \{ z \in \mathbb{C}^{\times n} \mid |z_{\tau_m}/z_{\tau_{m+1}}| \ll 1, \quad m = 1, \ldots, n-1 \}. \tag{6.1}$$

Say that z tends to limit in the asymptotic zone, $z \rightrightarrows \mathbb{A}_\tau$, if $z_{\tau_m}/z_{\tau_{m+1}} \to 0$ for all $m = 1, \ldots, n-1$.

Say that a basis $\Psi_1, \ldots, \Psi_N$ of solutions of qKZ equation form an *asymptotic solution* in the asymptotic zone $\mathbb{A}_\tau$ if

$$\Psi_j(z) = h_j(z)\bigl(v_j + o(1)\bigr),$$

where $h_1(z), \ldots, h_N(z)$ are meromorphic functions such that

$$h_j(z_1, \ldots, pz_m, \ldots, z_n) = a_{jm}\, h_j(z_1, \ldots, z_n)$$

for suitable numbers a_{jm}, $v_1, \dots, v_N$ are constant vectors which form a basis in $(V_1 \otimes \dots \otimes V_n)_\ell$, and $o(1)$ tends to 0 as $z \rightrightarrows \mathbb{A}_\tau$.

For every permutation $\tau \in \mathbf{S}^n$ we constructed the functions $W^\tau_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, whose restriction to a fiber gives a basis in the elliptic hypergeometric space of the fiber. Let $Y^\tau_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, be the corresponding adjusting factors so that the functions $Y^\tau_\mathfrak{l} W^\tau_\mathfrak{l}$ are in the elliptic hypergeometric space. These functions define a basis $\Psi_{Y^\tau_\mathfrak{l} W^\tau_\mathfrak{l}}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, of solutions of the qKZ equation, cf. (5.18).

Recall that ${}^\tau\mathfrak{l} = (\mathfrak{l}_{\tau_1}, \dots, \mathfrak{l}_{\tau_n})$ for any $\mathfrak{l} \in \mathcal{Z}^n_\ell$. For $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}^n_\ell$ say that $\mathfrak{l} \ll \mathfrak{m}$ if $\mathfrak{l} \neq \mathfrak{m}$ and $\sum_{i=1}^m \mathfrak{l}_i \leqslant \sum_{i=1}^m \mathfrak{m}_i$ for any $m = 1, \dots, n-1$.

(6.2) Theorem. *Let the parameters $\xi_1, \dots, \xi_n$ obey condition* (2.14)*. Then for any permutation $\tau \in \mathbf{S}^n$ the basis $\Psi_{Y^\tau_\mathfrak{l} W^\tau_\mathfrak{l}}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, is an asymptotic solution in the asymptotic zone $\mathbb{A}_\tau$. Namely,*

$$\Psi_{Y^\tau_\mathfrak{l} W^\tau_\mathfrak{l}}(z) = Y^\tau_\mathfrak{l}(z)\bigl(\Omega^\tau_\mathfrak{l} + o(1)\bigr)$$

as z tends to limit in the asymptotic zone, $z \rightrightarrows \mathbb{A}_\tau$, so that at any moment assumption (2.15) *holds. Here*

$$\Omega^\tau_\mathfrak{l} = \Xi^\tau_\mathfrak{l} F^{\mathfrak{l}_1} v_1 \otimes \dots \otimes F^{\mathfrak{l}_n} v_n + \sum_{{}^\tau\mathfrak{m} \gg {}^\tau\mathfrak{l}} N^\tau_{\mathfrak{l}\mathfrak{m}} F^{\mathfrak{m}_1} v_1 \otimes \dots \otimes F^{\mathfrak{m}_n} v_n$$

for suitable constant coefficients $N^\tau_{\mathfrak{l}\mathfrak{m}}$ and $\Xi^\tau_\mathfrak{l}$, and the constant $\Xi^\tau_\mathfrak{l}$ is given by

$$\Xi^\tau_\mathfrak{l} = (2\pi i)^\ell \ell! \prod_{m=1}^n \Biggl(q^{\mathfrak{l}_m(1-\mathfrak{l}_m)/2 + \mathfrak{l}_m \Lambda_m} \prod_{\substack{1 \leqslant l < m \\ \sigma_l < \sigma_m}} \xi_l^{\mathfrak{l}_m} \prod_{\substack{1 \leqslant l < m \\ \sigma_l > \sigma_m}} \eta^{\mathfrak{l}_l \mathfrak{l}_m} \xi_l^{-\mathfrak{l}_m} \times$$

$$\times \prod_{s=0}^{\mathfrak{l}_m - 1} \frac{(\eta^{-1})_\infty (\eta^{-s} (\kappa^\tau_{\mathfrak{l},m})^{-1} \xi_m)_\infty (p \eta^{-s} \kappa^\tau_{\mathfrak{l},m} \xi_m)_\infty}{(\eta^{-s-1})_\infty (\eta^{-s} \xi_m^2)_\infty (p)_\infty} \Biggr)$$

where $\sigma = \tau^{-1}$ and $\kappa^\tau_{\mathfrak{l},m} = \kappa \prod_{1 \leqslant \sigma_l < \sigma_m} \eta^{-\mathfrak{l}_l} \xi_l \prod_{\sigma_m < \sigma_l \leqslant n} \eta^{\mathfrak{l}_l} \xi_l^{-1}$.

The theorem is proved in Chapter 7.

Remark. The trigonometric R-matrix $R(x)$ has finite limits as x tends to zero or infinity, cf. (3.7). Thus, the qKZ operators $K_1(z), \dots, K_n(z)$ have finite limits as z tends to limit in an asymptotic zone:

$$K_m(z) = K^\tau_m \bigl(1 + o(1)\bigr), \qquad z \rightrightarrows \mathbb{A}_\tau, \qquad m = 1, \dots, n.$$

where K^τ_m are some operators independent of z. The vectors $\Omega^\tau_{\mathfrak l}$ form an eigenbasis of the operators K^τ_m with eigenvalues $a^\tau_{\mathfrak l,m}$, cf. (6.3).

Remark. Recall that the adjusting factors $Y^\tau_{\mathfrak l}(z)$ have the following properties:

$$Y^\tau_{\mathfrak l}(z_1,\dots,pz_m,\dots,z_n) = a^\tau_{\mathfrak l,m}Y^\tau_{\mathfrak l}(z_1,\dots,z_n), \tag{6.3}$$

$$a^\tau_{\mathfrak l,m} = \kappa^{\mathfrak l_m}\prod_{1\leqslant\sigma_l<\sigma_m}\eta^{-\mathfrak l_l\mathfrak l_m}\xi_l^{\mathfrak l_m}\xi_m^{\mathfrak l_l}\prod_{\sigma_m<\sigma_l\leqslant n}\eta^{\mathfrak l_l\mathfrak l_m}\xi_l^{-\mathfrak l_m}\xi_m^{-\mathfrak l_l},$$

for any $\tau\in\mathbf S^\ell$, $\mathfrak l\in\mathcal Z^n_\ell$, $m=1,\dots,n$. Here $\sigma=\tau^{-1}$.

Remark. If the absolute value of κ is sufficiently small, then the relation $Y^\tau_{\mathfrak m}(z)=o\bigl(Y^\tau_{\mathfrak l}(z)\bigr)$ as z tends to limit in the asymptotic zone, $z\rightrightarrows\mathbb A_\tau$, implies that ${}^\tau\mathfrak m\gg{}^\tau\mathfrak l$, cf. (6.3). Similarly, if the absolute value of κ is sufficiently large, then the relation $Y^\tau_{\mathfrak m}(z)=o\bigl(Y^\tau_{\mathfrak l}(z)\bigr)$ as z tends to limit in the asymptotic zone, $z\rightrightarrows\mathbb A_\tau$, implies that ${}^\tau\mathfrak m\ll{}^\tau\mathfrak l$.

For example, assume that $|\eta|=1$, $|\xi_m|=1$, $m=1,\dots,n$, $|\kappa|<1$, and all the adjusting factors $Y^\tau_{\mathfrak m}(z)$ are regular at point $(1,\dots,1)\in\mathbb C^{\times n}$. Let $z=(p^{s_1},\dots,p^{s_n})$ where $s_1,\dots,s_n$ are integers. Then the relation $Y^\tau_{\mathfrak m}(z)=o\bigl(Y^\tau_{\mathfrak l}(z)\bigr)$ as $z\rightrightarrows\mathbb A_\tau$ implies that the sum $\sum\limits_{i=1}^n s_i(\mathfrak m_i-\mathfrak l_i)$ is large positive if all the differences $s_{\tau_m}-s_{\tau_{m+1}}$, $m=1,\dots,n-1$ are large positive. Since $\sum\limits_{i=1}^n\mathfrak l_i=\sum\limits_{i=1}^n\mathfrak m_i$ we have that

$$\sum_{i=1}^n s_i(\mathfrak m_i-\mathfrak l_i) = \sum_{i=1}^{n-1}(s_{\tau_i}-s_{\tau_{i+1}})\sum_{j=1}^i(\mathfrak m_{\tau_j}-\mathfrak l_{\tau_j}).$$

Therefore, $\sum\limits_{j=1}^i\mathfrak m_{\tau_j}\geqslant\sum\limits_{j=1}^i\mathfrak l_{\tau_j}$ for any $i=1,\dots,n-1$, and $\mathfrak m\neq\mathfrak l$, that is ${}^\tau\mathfrak m\gg{}^\tau\mathfrak l$.

Remark. The *qKZ* equation depends meromorphically on parameters κ, ξ_1, $\dots,\xi_n$. Let the adjusting factors $Y^\tau_{\mathfrak l}$ depend meromorphically on κ, $\xi_1,\dots,\xi_n$. Then the basis of solutions $\Psi_{Y^\tau_{\mathfrak l}W^\tau_{\mathfrak l}}$, $\mathfrak l\in\mathcal Z^n_\ell$ also depends meromorphically on κ, $\xi_1,\dots,\xi_n$. The asymptotics of the basis $\Psi_{Y^\tau_{\mathfrak l}W^\tau_{\mathfrak l}}$, $\mathfrak l\in\mathcal Z^n_\ell$, described in Theorem 6.2 determine the basis uniquely. Namely, if a basis of solutions meromorphically depends on the parameters κ, $\xi_1,\dots,\xi_n$ and has asymptotics in $\mathbb A_\tau$ described in Theorem 6.2, then such a basis coincides with the basis $\Psi_{Y^\tau_{\mathfrak l}W^\tau_{\mathfrak l}}$. In fact, elements of any such a basis are linear combinations of the functions $\Psi_{Y^\tau_{\mathfrak l}W^\tau_{\mathfrak l}}$ with coefficients meromorphically depending on κ, $\xi_1,\dots,\xi_n$ and p-periodic in $z_1,\dots,z_n$. To preserve the asymptotics one can add to an element $\Psi_{Y^\tau_{\mathfrak{l}} W^\tau_{\mathfrak{l}}}$ any other functions $\Psi_{Y^\tau_{\mathfrak{m}} W^\tau_{\mathfrak{m}}}$ having smaller asymptotics. If the absolute value of κ is sufficiently small, then one can add only the functions $\Psi_{Y^\tau_{\mathfrak{m}} W^\tau_{\mathfrak{m}}}$ with ${}^\tau\mathfrak{m} \gg {}^\tau\mathfrak{l}$, and if the absolute value of κ is sufficiently large, then one can add only the functions $\Psi_{Y^\tau_{\mathfrak{m}} W^\tau_{\mathfrak{m}}}$ with ${}^\tau\mathfrak{m} \ll {}^\tau\mathfrak{l}$, see the previous Remark. Since the coefficients of added terms are meromorphic they have to be zero.

Remark. The asymptotic solution $\Psi_{Y^\tau_{\mathfrak{l}} W^\tau_{\mathfrak{l}}}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, in the asymptotic zone $\mathbb{A}_\tau$ of the *qKZ* equation with values in $(V_1 \otimes \ldots \otimes V_n)_\ell$, cf. Theorem 6.2, is an image of the monomial basis $v^{[\mathfrak{l}_{\tau_1}]} \otimes \ldots \otimes v^{[\mathfrak{l}_{\tau_n}]}$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$, of $(V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n})_\ell$ under the composition $I_{\mathrm{id},\tau}(z) Y^\tau(z)$ of the hypergeometric map and the adjusting map, cf. Theorem 5.31. The transition functions between the asymptotic solutions are linear maps $\mathbf{C}_{\tau,\tau'}$, see (5.33). Formula (5.34) and Theorem 4.16 show that the transition functions between asymptotic solutions corresponding to neighbouring asymptotic zones are given by the dynamical elliptic R-matrices twisted by the corresponding adjusting maps.

Example. Theorem 6.2, formula (5.34) and Theorem 4.16 allow us to write an elliptic R-matrix as an infinite product of trigonometric R-matrices. Namely, consider the *qKZ* equation with values in the tensor product of two $U_q(\mathfrak{sl}_2)$ Verma modules $V_1 \otimes V_2$ with highest weights $q^{\Lambda_1}, q^{\Lambda_2}$, respectively. Then there are two asymptotic zones $|z_1/z_2| \gg 1$ and $|z_1/z_2| \ll 1$. Our result on the transition function from the first asymptotic zone to the second one is the following statement.

Let V^e_1, V^e_2 be the evaluation Verma module over $E_{\rho,\gamma}(\mathfrak{sl}_2)$ with highest weights Λ_1, Λ_2, respectively. Then we have

$$(6.4) \qquad \left(R^{ell}_{V^e_1 V^e_2}(x)\right)^{-1} = \prod_{s=-\infty}^{\infty} \kappa^{-H\otimes \mathrm{id}} R_{V_1 V_2}(p^s x),$$

provided the infinite product in the right hand side is suitably regularized and the factors of the product are ordered in such a way that s grows from the right to the left, see the example below. The restriction of formula (6.4) to the weight subspace $(V_1 \otimes V_2)_1$ of weight $q^{\Lambda_1+\Lambda_2-1}$ can be transformed into the infinite product formula for 2×2 matrices, which looks as follows.

Let $a, b, c, d, \alpha, \delta, p$ be nonzero complex numbers such that $|p| < 1$ and

$$\alpha/\delta \neq p^s, \qquad a/d \neq p^s, \qquad bc \neq (1-p^s)(p^{-s}\alpha d - \delta a)$$

for any $s \in \mathbb{Z}$. Set $A(u) = \begin{pmatrix} a - \alpha u & bu \\ c & d - \delta u \end{pmatrix}$. Let λ, μ be two solutions of the quadratic equation $\det A(u)=0$. Then

$$\lim_{s\to\infty}\begin{pmatrix} a & 0\\ c & d\end{pmatrix}^{-s}\Big(\prod_{r=-s}^{s}A(p^r u)\Big)\begin{pmatrix}-\alpha & b\\ 0 & -\delta\end{pmatrix}^{-s}u^{-s}p^{s(s+1)/2}=$$

$$=\begin{pmatrix}1 & 0\\ \dfrac{c}{a-d} & 1\end{pmatrix}\times$$

$$\times\begin{pmatrix}\dfrac{a\theta(\alpha u/a)\,(p\lambda\delta/a)_\infty\,(p\mu\delta/a)_\infty}{(pd/a)_\infty\,(p\delta/\alpha)_\infty\,(p)_\infty} & \dfrac{bu\theta(u^{-1}a/\delta)\,(p\lambda\alpha/a)_\infty\,(p\mu\alpha/a)_\infty}{(pd/a)_\infty\,(\alpha/\delta)_\infty\,(p)_\infty}\\ \dfrac{c\theta(\alpha u/d)\,(p\lambda\delta/d)_\infty\,(p\mu\delta/d)_\infty}{(a/d)_\infty\,(p\delta/\alpha)_\infty\,(p)_\infty} & \dfrac{d\theta(\delta u/d)\,(\lambda\alpha/d)_\infty\,(\mu\alpha/d)_\infty}{(a/d)_\infty\,(\alpha/\delta)_\infty\,(p)_\infty}\end{pmatrix}\times$$

$$\times\begin{pmatrix}1 & \dfrac{b}{\delta-\alpha}\\ 0 & 1\end{pmatrix}$$

where the factors of the product are ordered in such a way that r grows from the right to the left.

Theorem 6.2 admits the following generalization. Fix a nonnegative integer k not greater than n. Fix nonnegative integers $n_0,\dots,n_k$ such that

$$0=n_0<n_1<\ldots<n_k=n$$

and consider an asymptotic zone in $\mathbb{C}^{\times n}$ given by

$$\mathbb{A}\;=\;\{z\in\mathbb{C}^{\times n}\mid |z_{m_i}/z_{m_{i+1}}|\ll 1\quad\text{for all } m_1,\dots,m_k \\ \text{such that } n_{i-1}<m_i\leqslant n_i\,,\ i=1,\dots,k\}\,.$$

We say that z tends to limit in the asymptotic zone, $z\rightrightarrows\mathbb{A}$, if $z_l/z_m\to 0$ for all l,m such that $n_{i-1}<l\leqslant n_i<m\leqslant n_{i+1}$ for some $i=1,\dots,k-1$, and $|z_l/z_m|$ remains bounded for all l,m such that $n_{i-1}<l,m\leqslant n_i$ for some $i=1,\dots,k$.

Fix nonnegative integers $\ell_1,\dots,\ell_k$ such that $\sum\limits_{i=1}^{k}\ell_i=\ell$. Let

$$\mathcal{F}^i[\ell_i]\;=\;\mathcal{F}[z_{n_{i-1}+1},\dots,z_{n_i};\xi_{n_{i-1}+1},\dots,\xi_{n_i};\ell_i]$$

and

$$\mathcal{F}_{ell}^{i}[\ell_i] = \mathcal{F}_{ell}[\kappa_i; z_{n_{i-1}+1}, \dots, z_{n_i}; \xi_{n_{i-1}+1}, \dots, \xi_{n_i}; \ell_i]$$

where $\kappa_i = \kappa\eta^{\sum_{j>i}\ell_j - \sum_{j<i}\ell_j} \prod_{l\leqslant n_{i-1}} \xi_l \prod_{l>n_i} \xi_l^{-1}$. Let $Y(z_1, \dots, z_n)$ be an adjusting factor for the tensor product $\mathcal{F}_{ell}^1[\ell_1] \otimes \dots \otimes \mathcal{F}_{ell}^k[\ell_k]$, cf. (4.20). Then we have linear maps

$$\begin{aligned} \mathcal{F}^1[\ell_1] \otimes \dots \otimes \mathcal{F}^k[\ell_k] &\hookrightarrow \mathcal{F} \\ f_1 \otimes \dots \otimes f_k &\mapsto f_1 \star \dots \star f_k \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{F}_{ell}^1[\ell_1] \otimes \dots \otimes \mathcal{F}_{ell}^k[\ell_k] &\hookrightarrow \mathcal{F}_{ell} \\ f_1 \otimes \dots \otimes f_k &\mapsto (f_1 * \dots * f_k)Y \end{aligned}$$

with respect to the tensor products introduced in Chapter 4.

Let $\mathfrak{m} \in \mathcal{Z}_\ell^n$. Say that $\mathfrak{m} \gg (\ell_1, \dots, \ell_k)$ if $\sum_{l=1}^{n_i} \mathfrak{m}_l \geqslant \sum_{j=1}^{i} \ell_j$ for any $i = 1, \dots, k-1$, and at least one of the inequalities is strict.

For any $W \in \mathcal{F}_{ell}^i[\ell_i]$ let $\Psi_W(z_{n_{i-1}+1}, \dots, z_{n_i})$ be the solution of the qKZ equation with values in $(V_{n_{i-1}+1} \otimes \dots \otimes V_{n_i})_{\ell_i}$ corresponding to W (cf. (5.18)).

(6.5) Theorem. *Let the parameters $\xi_1, \dots, \xi_n$ obey condition* (2.14). *Let $W_i \in \mathcal{F}_{ell}^i[\ell_i]$, $i = 1, \dots, k$. Let $W = W_1 * \dots * W_k$ and let Y be an adjusting factor for the tensor product $\mathcal{F}_{ell}^1[\ell_1] \otimes \dots \otimes \mathcal{F}_{ell}^k[\ell_k]$. Then the solution $\Psi_{YW}(z_1, \dots, z_n)$ of the qKZ equation with values in $(V_1 \otimes \dots \otimes V_n)_\ell$ has the following asymptotics as z tends to limit in the asymptotic zone, $z \rightrightarrows \mathbb{A}$, so that at any moment assumption* (2.15) *holds:*

$$\Psi_{YW}(z) = \frac{\ell!}{\ell_1! \dots \ell_k!} \prod_{i=1}^{k} \prod_{1\leqslant j\leqslant n_{i-1}} \xi_j^{\ell_i}\, Y(z)\,\bigl(\Omega_W(z) + o(1)\bigr)$$

where

$$\begin{aligned} \Omega_W(z_1, \dots, z_n) &= \Psi_{W_1}(z_1, \dots, z_{n_1}) \otimes \dots \otimes \Psi_{W_k}(z_{n_{k-1}+1}, \dots, z_n) + \\ &+ \sum_{\mathfrak{m}\gg(\ell_1,\dots,\ell_k)} \prod_{i=1}^{k} N_{W,\mathfrak{m}}^{(i)}(z_{n_{i-1}+1}, \dots, z_{n_i})\, F^{\mathfrak{m}_1} v_1 \otimes \dots \otimes F^{\mathfrak{m}_n} v_n \end{aligned}$$

for suitable coefficients $N_{W,\mathfrak{m}}^{(i)}(z_{n_{i-1}+1}, \dots, z_{n_i})$.

Theorem 6.5 follows from Theorem 6.6 below which describes asymptotics of the hypergeometric pairing.

Let $\ell'_1,\dots,\ell'_k$ be nonnegative integers such that $\sum\limits_{i=1}^{k}\ell'_i = \ell$. Say that $(\ell'_1,\dots,\ell'_k)\gg(\ell_1,\dots,\ell_k)$ if $\sum\limits_{j=1}^{i}\ell'_j \geqslant \sum\limits_{j=1}^{i}\ell_j$ for any $i=1,\dots,k-1$, and $(\ell'_1,\dots,\ell'_k)\neq(\ell_1,\dots,\ell_k)$.

(6.6) Theorem. *Let the parameters $\xi_1,\dots,\xi_n$ obey condition* (2.14). *Let $w_i\in\mathcal{F}^i[\ell'_i]$ and $W_i\in\mathcal{F}^i_{ell}[\ell_i]$, $i=1,\dots,k$. Let $w=w_1\star\dots\star w_k$ and $W=W_1*\dots*W_k$. Then the hypergeometric integral $I(W,w)$ has the following asymptotics as z tends to limit in the asymptotic zone, $z\rightrightarrows\mathbb{A}$, so that at any moment assumption* (2.15) *holds:*

$$I(W,w) = \frac{\ell!}{\ell_1!\dots\ell_k!}\prod_{i=1}^{k}\prod_{1\leqslant j\leqslant n_{i-1}}\xi_j^{\ell_i}\,\Big(\prod_{i=1}^{k}I(W_i,w_i)+o(1)\Big)$$

for $(\ell'_1,\dots,\ell'_k)=(\ell_1,\dots,\ell_k)$ and

$$\begin{aligned} I(W,w) &= O(1) && \text{for } (\ell'_1,\dots,\ell'_k)\gg(\ell_1,\dots,\ell_k),\\ I(W,w) &= o(1), && \text{otherwise}. \end{aligned}$$

The theorem is proved in Chapter 7.

7. Proofs

This chapter contains proofs of the statements formulated in Chapters 2 – 6. Basic facts about the trigonometric and elliptic hypergeometric spaces are given in Appendices A and B, respectively. In particular, we give there proofs of Lemmas 2.21, 2.31, 2.36 – 2.39.

Proof of Lemmas 2.22, 2.33. The statements immediately follows from Theorems A.7, B.8, respectively. □

Proof of Lemmas 2.23, 2.24. The first claims of the lemmas are respectively equivalent to the first and second formulae in (A.5). The second claims of the lemmas are the same as Corollary A.8. □

Lemma 4.3 follows from formula (A.3) in [IK] and the definition of the evaluation modules by induction with respect to the number of factors in the tensor product. Nevertheless, we give here an independent proof of Lemma 4.3 which is similar to the proof of Lemma 4.15 in the elliptic case.

Proof of Lemma 4.3. Without loss of generality we can assume that τ is the identity permutation. We give a proof of the second formula of the lemma. The proof of the first formula is similar.

It suffices to prove the formula for generic values of parameters η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$, since both sides of the formula are analytic functions of the parameters.

Define functions $X_{\mathfrak{l}}(t_1, \dots, t_\ell)$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, by the rule:

$$L_{12}^-(t_1) \dots L_{12}^-(t_\ell) v_1 \otimes \dots \otimes v_n =$$

$$= \sum_{\mathfrak{l} \in \mathcal{Z}_\ell^n} X_{\mathfrak{l}}(t_1, \dots, t_\ell) \prod_{1 \leqslant l < m \leqslant n} q^{\mathfrak{l}_m \Lambda_l - \mathfrak{l}_l \Lambda_m - \mathfrak{l}_l \mathfrak{l}_m} \prod_{a=1}^{\ell} t_a^{\ell(1-n)} F^{\mathfrak{l}_1} v_1 \otimes \dots \otimes F^{\mathfrak{l}_n} v_n .$$

The claim of the lemma means that for any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ the function $X_{\mathfrak{l}}$ coincides with the polynomial $P_{\mathfrak{l}}$ defined by formula (A.9).

For $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n$ say that $\mathfrak{l} \leqq \mathfrak{m}$ if $\sum_{i=1}^{m} \mathfrak{l}_i \leqslant \sum_{i=1}^{m} \mathfrak{m}_i$ for any $m = 1, \dots, n-1$. Say that $\mathfrak{l} \ll \mathfrak{m}$ if $\mathfrak{l} \neq \mathfrak{m}$ and $\mathfrak{l} \leqq \mathfrak{m}$.

By the definition of functions $X_{\mathfrak{l}}$, they are symmetric polynomials in ℓ variables of degree less than n in each of the variable. Hence, they are linear combinations of polynomials $P_{\mathfrak{l}}$:

$$X_{\mathfrak{l}}(t) = (q - q^{-1})^\ell \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} U_{\mathfrak{l}\mathfrak{m}} P_{\mathfrak{m}}(t) , \qquad \mathfrak{l} \in \mathcal{Z}_\ell^n.$$

By Lemmas 7.1, A.12 the matrix U is upper triangular: $U_{\mathfrak{l}\mathfrak{m}} = 0$ unless $\mathfrak{l} \lll \mathfrak{m}$, as a ratio of upper triangular matrices, and simultaneously lower triangular: $U_{\mathfrak{l}\mathfrak{m}} = 0$ unless $\mathfrak{l} \ggg \mathfrak{m}$, as a ratio of lower triangular matrices. Therefore, the matrix U is diagonal. Moreover, by the same lemmas $U_{\mathfrak{l}\mathfrak{l}} = 1$ for any $\mathfrak{l} \in \mathcal{Z}_\ell^n$. Hence, U is the unit matrix. Lemma 4.3 is proved. □

For any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ let $x \triangleright \mathfrak{l}, y \triangleleft \mathfrak{l} \in \mathbb{C}^{\times \ell}$ be the following points:

$$x \triangleright \mathfrak{l} = (\eta^{1-\mathfrak{l}_1}\xi_1 z_1, \eta^{2-\mathfrak{l}_1}\xi_1 z_1, \dots, \xi_1 z_1, \eta^{1-\mathfrak{l}_2}\xi_2 z_2, \dots, \xi_2 z_2, \dots, \\ \eta^{1-\mathfrak{l}_n}\xi_n z_n, \dots, \xi_n z_n),$$

$$y \triangleleft \mathfrak{l} = (\eta^{\mathfrak{l}_1-1}\xi_1^{-1} z_1, \eta^{\mathfrak{l}_1-2}\xi_1^{-1} z_1, \dots, \xi_1^{-1} z_1, \eta^{\mathfrak{l}_2-1}\xi_2^{-1} z_2, \dots, \xi_2^{-1} z_2, \dots, \\ \eta^{\mathfrak{l}_n-1}\xi_n^{-1} z_n, \dots, \xi_n^{-1} z_n),$$

cf. (2.35), (A.11).

(7.1) Lemma. *$X_\mathfrak{l}(x \triangleright \mathfrak{m}) = 0$ unless $\mathfrak{l} \lll \mathfrak{m}$. $X_\mathfrak{l}(y \triangleleft \mathfrak{m}) = 0$ unless $\mathfrak{l} \ggg \mathfrak{m}$. Moreover,*

$$X_\mathfrak{l}(x \triangleright \mathfrak{l}) = (q - q^{-1})^\ell \times \\ \times \prod_{m=1}^{n} \prod_{s=0}^{\mathfrak{l}_m - 1} \Bigl(\prod_{1 \leqslant l < m} (\eta^{-s}\xi_m z_m - \xi_l^{-1} z_l) \prod_{m < l \leqslant n} (\eta^{\mathfrak{l}_l - s}\xi_m z_m - \xi_l z_l) \Bigr).$$

Proof. The proof is given by the straightforward calculation based on the definition of the coproduct in the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$. We illustrate the calculation by the following example.

Let $n = 3$. Let $\Delta^{(3)} = (\Delta \otimes \mathrm{id}) \circ \Delta : U_q'(\widetilde{\mathfrak{gl}_2}) \to U_q'(\widetilde{\mathfrak{gl}_2})^{\otimes 3}$ be the iterated coproduct. We have that

$$\begin{aligned}(7.2) \qquad \Delta^{(3)}(L_{12}^-(t)) = {} & L_{11}^-(t) \otimes L_{11}^-(t) \otimes L_{12}^-(t) + L_{11}^-(t) \otimes L_{12}^-(t) \otimes L_{22}^-(t) + \\ & + L_{12}^-(t) \otimes L_{22}^-(t) \otimes L_{22}^-(t) + L_{12}^-(t) \otimes L_{21}^-(t) \otimes L_{12}^-(t).\end{aligned}$$

Let $\ell = 4$, $\mathfrak{m} = (1, 1, 2)$. Recall that $\eta = q^2$ and $\xi_m = q^{2\Lambda_m}$.

We have to calculate the following expressions

$$(7.3) \qquad L_{12}^-(\xi_1 z_1) L_{12}^-(\xi_2 z_2) L_{12}^-(\eta^{-1}\xi_3 z_3) L_{12}^-(\xi_3 z_3) v_1 \otimes v_2 \otimes v_3,$$

$$(7.4) \qquad L_{12}^-(\xi_1^{-1} z_1) L_{12}^-(\xi_2^{-1} z_2) L_{12}^-(\eta \xi_3^{-1} z_3) L_{12}^-(\xi_3^{-1} z_3) v_1 \otimes v_2 \otimes v_3.$$

To compute $L^-_{12}(\eta^{-1}\xi_3 z_3)\,L^-_{12}(\xi_3 z_3)\,v_1\otimes v_2\otimes v_3$ we need only the first term in the right hand side of formula (7.2), since all other terms vanish. Then to compute $L^-_{12}(\xi_2 z_2)\,L^-_{12}(\eta^{-1}\xi_3 z_3)\,L^-_{12}(\xi_3 z_3)\,v_1\otimes v_2\otimes v_3$ we need only the first and second terms in the right hand side of formula (7.2) for the same reason. Finally, at the last step we have to use all four terms in (7.2) and we find that expression (7.3) is a linear combination of vectors

$$Fv_1\otimes Fv_2\otimes F^2v_3\,,\qquad Fv_1\otimes v_2\otimes F^3v_3\,,$$

$$v_1\otimes F^2v_2\otimes F^2v_3\,,\qquad v_1\otimes Fv_2\otimes F^3v_3\,,\qquad v_1\otimes v_2\otimes F^4v_3\,.$$

The coefficient of vector $Fv_1\otimes Fv_2\otimes F^2v_3$ can be easily calculated and it has the prescribed form.

To calculate expression (7.4) we first use the commutativity of the factors in the product and transform the expression as follows:

$$L^-_{12}(\eta\xi_3^{-1}z_3)\,L^-_{12}(\xi_3^{-1}z_3)\,L^-_{12}(\xi_2^{-1}z_2)\,L^-_{12}(\xi_1^{-1}z_1)\,v_1\otimes v_2\otimes v_3\,.$$

Then to compute $L^-_{12}(\xi_1^{-1}z_1)\,v_1\otimes v_2\otimes v_3$ we need only the third term in the right hand side of formula (7.2), and to compute $L^-_{12}(\xi_2^{-1}z_2)\,L^-_{12}(\xi_1^{-1}z_1)\,v_1\otimes v_2\otimes v_3$ we need only the second and third terms. The rest of the calculation is clear.

The general case can be considered similarly. □

Proof of Lemma 4.15. The proof is similar to the proof of Lemma 4.3. So we give only the main points of the proof.

Without loss of generality we can assume that parameters η, $\xi_1,\dots,\xi_n$, $z_1,\dots,z_n$ are generic, and τ is the identity permutation. Define functions $\Xi_{\mathfrak l}(t_1,\dots,t_\ell)$, $\mathfrak l\in\mathcal Z^n_\ell$, by the rule:

$$(7.5)\quad T_{21}(t_1,\lambda)\dots T_{21}(t_\ell,\eta^{\ell-1}\lambda)\,v^{[\mathfrak l_1]}\otimes\dots\otimes v^{[\mathfrak l_n]}=$$

$$=\Xi_{\mathfrak l}(t_1,\dots,t_\ell)\prod_{s=0}^{\ell-1}\theta(\eta^s\kappa^{-1}\prod_{m=1}^n\xi_m^{-1})\prod_{a=1}^{\ell}\prod_{m=1}^n\theta(\xi_m^{-1}t_a/z_m)\,v^{[0]}\otimes\dots\otimes v^{[0]}\,.$$

Here $\lambda=\kappa\prod\limits_{m=1}^n\xi_m^{-1}$. We have to show that $\Xi_{\mathfrak l}=c_{\mathfrak l}J_{\mathfrak l}$ for any $\mathfrak l\in\mathcal Z^n_\ell$, where the constant $c_{\mathfrak l}$ and the function $J_{\mathfrak l}$ are given by formulae (4.9) and (B.9), respectively.

By the definition of functions $\Xi_{\mathfrak l}$, they are holomorphic function on $\mathbb C^{\times\ell}$ having the property

$$\Xi_{\mathfrak l}(t_1,\dots,pt_a,\dots,t_\ell)=\kappa\prod_{m=1}^n z_m\,(-t_a)^{-n}\,\Xi_{\mathfrak l}(t_1,\dots,t_\ell)\,.$$

Therefore, they are linear combinations of the functions $J_{\mathfrak{l}}$:

$$\Xi_{\mathfrak{l}}(t) = \sum_{\mathfrak{m}\in\mathcal{Z}_\ell^n} U^{ell}_{\mathfrak{l}\mathfrak{m}} J_{\mathfrak{m}}(t), \qquad \mathfrak{l}\in\mathcal{Z}_\ell^n.$$

Lemmas 7.6 and B.10 imply that the matrix U^{ell} is diagonal and $U^{ell}_{\mathfrak{l}\mathfrak{l}} = c_{\mathfrak{l}}$, $\mathfrak{l}\in\mathcal{Z}_\ell^n$. Lemma 4.15 is proved. □

(7.6) Lemma. $\Xi_{\mathfrak{l}}(x\triangleright\mathfrak{m}) = 0$ *unless* $\mathfrak{l} \lll \mathfrak{m}$. $\Xi_{\mathfrak{l}}(y\triangleleft\mathfrak{m}) = 0$ *unless* $\mathfrak{l} \ggg \mathfrak{m}$. *Moreover,*

$$\Xi_{\mathfrak{l}}(x\triangleright\mathfrak{l}) = \prod_{1\leqslant l<m\leqslant n} \eta^{-\mathfrak{l}_l\mathfrak{l}_m}\xi_m^{2\mathfrak{l}_l} \times$$

$$\times \prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\Bigl(\frac{\theta(\eta^s)\theta(\eta^{1-s}\xi_m^2)}{\theta(\eta)\theta\bigl(\eta^{s+1}\kappa^{-1}\prod_{1\leqslant l\leqslant m}\eta^{\mathfrak{l}_l}\xi_l^{-1}\prod_{m<l\leqslant n}\eta^{-\mathfrak{l}_l}\xi_l\bigr)} \times$$

$$\times \prod_{1\leqslant l<m}\theta(\eta^{-s}\xi_l\xi_m z_m/z_l)\prod_{m<l\leqslant n}\theta(\eta^{\mathfrak{l}_l-s}\xi_l^{-1}\xi_m z_m/z_l)\Bigr).$$

Proof. The proof is similar to the proof of Lemma 7.1 and is given by the straightforward calculation based on the definition of the coproduct in the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$. Two remarks on the calculation is to be done.

The calculation becomes more transparent if it is done in the dual picture, that is if we replace formula (7.5) by the dual one

$$\bigl(T_{21}(t_1,\lambda)\ldots T_{21}(t_\ell,\eta^{\ell-1}\lambda)\bigr)^*\bigl(v^{[0]}\otimes\ldots\otimes v^{[0]}\bigr)^* =$$

$$= \sum_{\mathfrak{l}\in\mathcal{Z}_\ell^n}\Xi_{\mathfrak{l}}(t_1,\ldots,t_\ell)\prod_{s=0}^{\ell-1}\theta(\eta^s\kappa^{-1}\prod_{m=1}^{n}\xi_m^{-1}) \times$$

$$\times \prod_{a=1}^{\ell}\prod_{m=1}^{n}\theta(\xi_m^{-1}t_a/z_m)\bigl(v^{[\mathfrak{l}_1]}\otimes\ldots\otimes v^{[\mathfrak{l}_n]}\bigr)^*.$$

The factors in the product in the left hand side of this formula should be put in the suitable order, which can be done using commutation relations in the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$. For instance, if $\ell = 4$ and $\mathfrak{m} = (1,1,2)$, then for the point $x \triangleright \mathfrak{m}$ the suitable form of the corresponding product is

$$T_{21}(\xi_3 z_3, \lambda) T_{21}(\eta^{-1}\xi_3 z_3, \eta\lambda) T_{21}(\xi_2 z_2, \eta^2\lambda) T_{21}(\xi_1 z_1, \eta^3\lambda) = \\ = T_{21}(\xi_1 z_1, \lambda) T_{21}(\xi_2 z_2, \eta\lambda) T_{21}(\eta^{-1}\xi_3 z_3, \eta^2\lambda) T_{21}(\xi_3 z_3, \eta^3\lambda)$$

and for the point $y \triangleleft \mathfrak{m}$ the suitable form of the corresponding product is

$$T_{21}(\xi_1^{-1} z_1, \lambda) T_{21}(\xi_2^{-1} z_2, \eta\lambda) T_{21}(\xi_3^{-1} z_3, \eta^2\lambda) T_{21}(\eta\xi_3^{-1} z_3, \eta^3\lambda) = \\ = T_{21}(\xi_1^{-1} z_1, \lambda) T_{21}(\xi_2^{-1} z_2, \eta\lambda) T_{21}(\eta\xi_3^{-1} z_3, \eta^2\lambda) T_{21}(\xi_3^{-1} z_3, \eta^3\lambda) .$$

The necessary transformation of the product in the general case is similar. □

Proof of Lemmas 4.18, 4.19. The statements follow from the definition of the weight functions, cf. (2.20), (2.30), and Theorems A.7, B.8, respectively. □

We extend the notion of the hypergeometric integral $I(W, w)$ and consider the hypergeometric integral for any function w in the functional space $\widehat{\mathcal{F}}(z)$ of a fiber. Namely, let $w(t, z) \in \widehat{\mathcal{F}}(z)$ be a function of the form

$$P(t_1, \dots, t_\ell, z_1, \dots, z_n, \xi_1, \dots, \xi_n, \eta) \times \\ \times \prod_{s=0}^{r} \Bigl[\prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{1}{(p^s t_a - \xi_m z_m)(\xi_m t_a - p^{s+1} z_m)} \times \\ \times \prod_{1 \leqslant a < b \leqslant \ell} \frac{1}{(p^s \eta t_a - t_b)(t_a - p^{s+1}\eta t_b)} \Bigr]$$

where P is a Laurent polynomial. If $|z_m| = 1$ for any $m = 1, \dots, n$, the absolute values of the parameters $\xi_1, \dots, \xi_n$ are small and the absolute value of the parameter η is large, then we define the hypergeometric integrals $I(W_{\mathfrak{l}}, w)$ by formula (5.3). For generic η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ we define the hypergeometric integrals $I(W_{\mathfrak{l}}, w)$ by the analytic continuation with respect to η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$. Similar to Theorem 5.6 one can show that this hypergeometric integrals can be analytically continued as holomorphic univalued functions of complex variables η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ to the region described in Theorem 5.6. For arbitrary functions $w \in \widehat{\mathcal{F}}(z)$, $W \in \mathcal{F}_{ell}(z)$ we define the hypergeometric integral $I(W, w)$ by linearity.

Let $D\widehat{\mathcal{F}}(z) = \{ Dw \mid w \in \widehat{\mathcal{F}}(z) \}$.

(7.7) Lemma. *Let* $0 < |p| < 1$. *Let* (2.13) – (2.15) *hold. Then the hypergeometric integral* $I(W, w)$ *equals zero for any function* $w \in D\widehat{\mathcal{F}}(z)$.

Proof. The claim is clear if $|z_m| = 1$ for any $m = 1, \dots, n$, the absolute values of the parameters $\xi_1, \dots, \xi_n$ are small and the absolute value of the parameter η is large. For general η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$ the claim is proved by the analytic continuation. □

Proof of Lemma 5.7. The first claim of the lemma follows from Lemma 2.23 and 7.7.

It suffices to prove the second claim under the assumptions that $|z_m| = 1$, $m = 1, \dots, n$, $|\eta| > 1$ and the absolute values of $\xi_1, \dots, \xi_n$ are small, when the hypergeometric integral $I(W, w)$ is given by formula (5.3).

Let $W \in \mathcal{Q}(z)$, that is

$$W(t_1, \dots, t_\ell) = \sum_{\sigma \in \mathbf{S}^\ell} [\![W'(t_2, \dots, t_\ell)]\!]_\sigma$$

for a suitable function $W' \in \mathcal{F}_{ell}[\eta^{-\ell} \prod_{m=1}^{n} \xi_m; z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell - 1](z)$. Due to formula (2.7), we have that

$$I(W, w) = \ell! \int_{\mathbb{T}^\ell} \Phi(t_1, \dots, t_\ell)\, w(t_1, \dots, t_\ell)\, W'(t_2, \dots, t_\ell)\ (dt/t)^\ell \tag{7.8}$$

because the torus $\mathbb{T}^\ell$ is invariant under permutations of the variables $t_1, \dots,$ t_ℓ.

Since $w(0, t_2, \dots, t_\ell) = 0$ for any $w \in \mathcal{F}(z)$, the integrand $\Phi(t)\, w(t)\, W'(t)$ considered as a function of t_1 is regular in the disk $|t_1| \leqslant 1$. Hence, $I(W, w) = 0$ for any $w \in \mathcal{F}(z)$. □

Proof of Lemma 5.8. The proof is similar to the proof of Lemma 5.7. For the proof of the first claim Lemma 2.23 is to be replaced by Lemma 2.24. In the proof of the second claim the corresponding integrand is regular outside the disk $|t_\ell| \geqslant 1$ decreasing as $O(t_\ell^{-2})$ at infinity. □

The hypergeometric integral defines linear functionals $I(W, \cdot)$ on the functional space of a fiber. Lemma 7.7 means that these linear functionals can be considered as elements of the top homology group $H_\ell(z)$, the dual space to the top cohomology group of the de Rham complex of the discrete local system of the fiber.

Proof of Theorem 5.15. Recall, that in general the definition of the hypergeometric integral depends on z. In this proof we will indicate this dependence explicitly as a subscript: $I(\cdot,\cdot) = I_z(\cdot,\cdot)$.

The section s_W is defined by $s_W(z) = I_z(W|_z, \cdot)$ where $W|_z \in \mathcal{F}_{ell}(z)$ denotes the restriction of the function $W \in \mathcal{F}_{ell}$ to the fiber over z. The theorem is a direct corollary of the quasiperiodicity of the function W with respect to each of the variables $z_1, \ldots, z_n$:

$$W(t, z_1, \ldots, pz_m, \ldots, z_n) = \xi_m^\ell W(t, z_1, \ldots, z_n), \qquad m = 1, \ldots, n.$$

Namely, the periodicity of the section s_W with respect to the translation $z_m \mapsto pz_m$ means that

$$I_{z'}(W|_{z'}, w) = I_z(W|_z, (\varphi_{\ell+m})|_z w) \tag{7.9}$$

for any $w \in \widehat{\mathcal{F}}(z')$. Here $z' = (z_1, \ldots, pz_m, \ldots, z_n)$ and $\varphi_{\ell+m}$ is the corresponding connection coefficient of the $\mathfrak{sl}_2$-type local system, see (2.3).

Without loss of generality we can assume that both w and W are meromorphic function of the parameters $\xi_1, \ldots, \xi_n$ and η. So it suffices to prove (7.9) under the assumption that the absolute values of $\xi_1, \ldots, \xi_n$ are small and the absolute value of η is large. Then, the hypergeometric integral is given by formula (5.3) and we have

$$I_{z'}(W|_{z'}, w) = \int_{\mathbb{T}^\ell} \Phi(t, z') w(t) W(t, z') \, (dt/t)^\ell =$$

$$= \int_{\mathbb{T}^\ell} \Phi(t, z) \varphi_{\ell+m}(t, z) w(t) W(t, z) \, (dt/t)^\ell = I_z(W|_z, (\varphi_{\ell+m})|_z w)$$

The middle equality reflects the fact that the product ΦW is a phase function of system of connection coefficients (2.3). □

Proof of Lemmas 5.16, 5.17. We give here a proof of Lemma 5.16. The proof of Lemma 5.17 is similar.

Without loss of generality we can assume that $\tau = \mathrm{id}$. Let

$$v^{\mathfrak{l}} = F^{\mathfrak{l}_1} v_1 \otimes \ldots \otimes F^{\mathfrak{l}_n} v_n, \qquad b_{\mathfrak{l}} = \prod_{m=1}^{n} q^{\mathfrak{l}_m(\mathfrak{l}_m - 1)/2 + \mathfrak{l}_m \Lambda_m},$$

$$BE(t, z) = (q - q^{-1}) \sum_{\mathfrak{l} \in \mathcal{Z}_\ell^n} b_{\mathfrak{l}} w_{\mathfrak{l}}(t, z) E v^{\mathfrak{l}} \in \mathcal{F}(z) \otimes (V_1 \otimes \ldots \otimes V_n)_{\ell-1}.$$

Here E is a generator of $U_q(\mathfrak{sl}_2)$ acting in $V_1 \otimes \ldots \otimes V_n$.

By the definition of the tensor coordinates $B_{\mathrm{id}}(z)$, cf. (4.1), and the map $\bar{I}(z)$, the claim of Lemma 5.16 is equivalent to the following statement:

$$I(W, BE) = 0.$$

Let $\mathfrak{e}(m) = (0, \dots, \underset{m\text{-th}}{1}, \dots, 0)$, $m = 1, \dots, n$. Since

$$(q - q^{-1}) E v^{\mathfrak{l}} = \sum_{m=1}^{n} (q^{\mathfrak{l}_m} - q^{-\mathfrak{l}_m})(q^{2\Lambda_m - \mathfrak{l}_m + 1} - q^{-2\Lambda_m + \mathfrak{l}_m - 1}) \times \\ \times \prod_{1 \leqslant l < m} q^{\Lambda_l - \mathfrak{l}_l} \prod_{m < l \leqslant n} q^{-\Lambda_l + \mathfrak{l}_l} v^{\mathfrak{l} - \mathfrak{e}(m)},$$

and recalling that $\eta = q^2$, $\xi_m = q^{2\Lambda_m}$, $m = 1, \dots, n$, we obtain

$$BE = -q^{\ell - 1 - \sum_{m=1}^{n} \Lambda_m} \times \\ \times \sum_{\mathfrak{l} \in \mathcal{Z}^n_{\ell-1}} \Bigl(\sum_{m=1}^{n} w_{\mathfrak{l} + \mathfrak{e}(m)} (1 - \eta^{\mathfrak{l}_m + 1})(\xi_m - \eta^{\mathfrak{l}_m} \xi_m^{-1}) \prod_{1 \leqslant l \leqslant m} \eta^{-\mathfrak{l}_l} \xi_l \Bigr) b_{\mathfrak{l}} v^{\mathfrak{l}}.$$

Therefore, $BE \in \mathcal{R}(z) \otimes (V_1 \otimes \dots \otimes V_n)_{\ell-1}$, see Lemma 2.23, and applying Lemma 5.7 we complete the proof. □

Our further strategy is as follows. First we prove Theorem 6.6 which, together with formula (5.13), implies Theorem 6.2. Using Theorem 6.2 we prove that the hypergeometric pairing is nondegenerate, cf. Theorems 5.9 – 5.11. At last, we prove Theorems 5.26, 5.28.

Proof of Theorem 6.6. To simplify notations we give a proof only for the case $k = n$, so that $n_m = m$, $m = 1, \dots, n$. The general case is similar.

Let $w^{(m)}_{(l)} \in \mathcal{F}[z_m; \xi_m; l]$ and $W^{(m)}_{(l)}[\alpha] \in \mathcal{F}_{ell}[\alpha; z_m; \xi_m; l]$ be the following functions:

$$w^{(m)}_{(l)}(t_1, \dots, t_l, z_m) = \prod_{s=1}^{l} \frac{1 - \eta}{1 - \eta^s} \sum_{\sigma \in \mathbf{S}^l} \Bigl[\prod_{a=1}^{l} \frac{t_a}{t_a - \xi_m z_m} \Bigr]_\sigma, \tag{7.10}$$

$$W^{(m)}_{(l)}[\alpha](t_1, \dots, t_l, z_m) = \prod_{s=1}^{l} \frac{\theta(\eta)}{\theta(\eta^s)} \sum_{\sigma \in \mathbf{S}^l} \Bigl[\!\Bigl[\prod_{a=1}^{l} \frac{\theta(\eta^{2a-l-1} \alpha^{-1} t_a / z_m)}{\theta(\xi_m^{-1} t_a / z_m)} \Bigr]\!\Bigr]_\sigma,$$

cf. (2.20), (2.30). We have the equalities

$$w_{\mathfrak{l}} = w^{(1)}_{(\mathfrak{l}_1)} \star \ldots \star w^{(n)}_{(\mathfrak{l}_n)} \qquad \text{and} \qquad W_{\mathfrak{l}} = W^{(1)}_{(\mathfrak{l}_1)}[\kappa_{\mathfrak{l},1}] * \ldots * W^{(n)}_{(\mathfrak{l}_n)}[\kappa_{\mathfrak{l},n}]$$

where $\kappa_{\mathfrak{l},m} = \kappa \prod\limits_{1 \leqslant i < m} \eta^{-\mathfrak{l}_i}\xi_i \prod\limits_{m < i \leqslant n} \eta^{\mathfrak{l}_i}\xi_i^{-1}$. Therefore, we have to study the asymptotics of the hypergeometric integrals $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$.

Consider the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$. Due to property (2.7) all the terms in formula (2.30) for the function $W_{\mathfrak{l}}$ give the same contribution to the integral. So we can replace the integrand $\Phi(t)\, w_{\mathfrak{m}}(t)\, W_{\mathfrak{l}}(t)$ by the following integrand

$$F(t) = \ell!\, w_{\mathfrak{m}}(t) \prod_{1 \leqslant a < b \leqslant \ell} \frac{(\eta t_a/t_b)_\infty}{(\eta^{-1} t_a/t_b)_\infty} \times$$
$$\times \prod_{m=1}^{n} \Bigl(\prod_{s=1}^{\mathfrak{l}_m} \frac{\theta(\eta)}{\theta(\eta^s)} \prod_{a \in \Gamma_m} \frac{\theta(\kappa_{\mathfrak{l},m}^{-1} t_a/z_m)}{(p)_\infty\, (\xi_m t_a/z_m)_\infty\, (p\xi_m z_m/t_a)_\infty} \times$$
$$\times \prod_{1 \leqslant l < m} \frac{(p\xi_l^{-1} z_l/t_a)_\infty}{(p\xi_l z_l/t_a)_\infty} \prod_{m < l \leqslant n} \frac{(\xi_l^{-1} t_a/z_l)_\infty}{(\xi_l t_a/z_l)_\infty} \Bigr)$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \ldots, \mathfrak{l}^m\}$, $m = 1, \ldots, n$.

Assume that $\eta > 1$ and $|\xi_m| < 1$ for any $m = 1, \ldots, n$. If $|z_m| = 1$ for all $m = 1, \ldots, n$, then we have

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \int_{\mathbb{T}^\ell} F(t)\, (dt/t)^\ell .$$

The analytic continuation of $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ to the region $|z_1| < \ldots < |z_n|$ is given by

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \int_{\mathbb{T}_1^{\mathfrak{l}_1} \times \ldots \times \mathbb{T}_n^{\mathfrak{l}_n}} F(t)\, (dt/t)^\ell$$

where

$$\mathbb{T}_m^{\mathfrak{l}_m} = \{ (t_{1+\mathfrak{l}^{m-1}}, \ldots, t_{\mathfrak{l}^m}) \in \mathbb{C}^{\mathfrak{l}_m} \mid |t_a| = |z_m|, \ \mathfrak{l}^{m-1} < a \leqslant \mathfrak{l}^m \}$$

since the integrand has no poles at the hyperplanes $t_a = p^{-s}\xi_l^{-1} z_l$, $s \in \mathbb{Z}$, for $\mathfrak{l}^{m-1} < a \leqslant \mathfrak{l}^m$, $m > l$, has no poles at the hyperplanes $t_a = p^s \xi_l z_l$, $s \in \mathbb{Z}$, for $\mathfrak{l}^{m-1} < a \leqslant \mathfrak{l}^m$, $m < l$, and has no poles at the hyperplanes $t_a = p^s \eta t_b$, $s \in \mathbb{Z}$, for $a > b$.

Let z tends to limit in the asymptotic zone, $z \rightrightarrows \mathbb{A}$, that is

$$|z_m/z_{m+1}| \to 0 \qquad \text{for all} \quad m = 1, \dots, n-1\,.$$

Consider the case $\mathfrak{l} = \mathfrak{m}$. Transform the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{l}})$ as above and replace the integrand by its asymptotics as $z \rightrightarrows \mathbb{A}$. Since

$$(7.11) \qquad w_{\mathfrak{l}}(t_1, \dots, t_\ell) \,=\, \prod_{1 \leqslant l < m \leqslant n} \xi_l^{\mathfrak{l}_m} \prod_{m=1}^{n} w^{(m)}_{(\mathfrak{l}_m)}(t_{\mathfrak{l}^{m-1}+1}, \dots, t_{\mathfrak{l}^m}) \,+\, o(1)$$

as $z \rightrightarrows \mathbb{A}$ and $t \in \mathbb{T}_1^{\mathfrak{l}_1} \times \dots \times \mathbb{T}_n^{\mathfrak{l}_n}$, we obtain that

$$I(W_{\mathfrak{l}}, w_{\mathfrak{l}}) \,=\, \ell! \prod_{m=1}^{n} \Bigl(\prod_{s=1}^{\mathfrak{l}_m} \frac{\theta(\eta)}{\theta(\eta^s)} \int\limits_{\mathbb{T}_m^{\mathfrak{l}_m}} w^{(m)}_{(\mathfrak{l}_m)}(t_{\mathfrak{l}^{m-1}+1}, \dots, t_{\mathfrak{l}_m}) \,\times$$

$$\times \prod_{a \in \Gamma_m} \Bigl(\bigl((\xi_m t_a/z_m)_\infty \, (p\,\xi_m z_m/t_a)_\infty \bigr)^{-1} \prod_{\substack{b<a \\ b \in \Gamma_m}} \frac{(\eta t_b/t_a)_\infty}{(\eta^{-1} t_b/t_a)_\infty} \, \frac{dt_a}{t_a} \Bigr) \Bigr) \bigl(1 + o(1)\bigr)$$

as $z \rightrightarrows \mathbb{A}$. Due to (2.7) the integrals over $\mathbb{T}_m^{\mathfrak{l}_m}$ are the hypergeometric integrals $I(W^{(m)}_{(\mathfrak{l}_m)}[\kappa_{\mathfrak{l},m}], w^{(m)}_{(\mathfrak{l}_m)})$ up to simple factors. Hence, we finally obtain that

$$I(W_{\mathfrak{l}}, w_{\mathfrak{l}}) \,=\, \frac{\ell!}{\mathfrak{l}_1! \dots \mathfrak{l}_n!} \prod_{1 \leqslant l < m \leqslant n} \xi_l^{\mathfrak{l}_m} \,\Bigl(\prod_{m=1}^{n} I(W^{(m)}_{(\mathfrak{l}_m)}[\kappa_{\mathfrak{l},m}], w^{(m)}_{(\mathfrak{l}_m)}) + o(1) \Bigr)\,.$$

The hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ for $\mathfrak{l} \neq \mathfrak{m}$ can be treated similarly to the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{l}})$ considered above. The final answer is

$$\begin{aligned} I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) \,&=\, O(1) \qquad && \text{for } \mathfrak{l} \ll \mathfrak{m}\,, \\ I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) \,&=\, o(1)\,, && \text{otherwise}\,, \end{aligned}$$

which completes the proof if $\eta > 1$ and $|\xi_m| < 1$, $m = 1, \dots, n$.

For general $\xi_1, \dots, \xi_n$ and η the proof is similar. The analytic continuation of $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ to the region $|z_1| \ll \dots \ll |z_n|$ is given by

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) \,=\, \int\limits_{\mathbb{T}_1^{\mathfrak{l}_1} \times \dots \times \mathbb{T}_n^{\mathfrak{l}_n}} F(t)\, d^\ell t$$

where $\widetilde{\mathbb{T}}_m^{\mathfrak{l}_m}$ is the respective deformation of $\mathbb{T}_m^{\mathfrak{l}_m}$. On every contour $\widetilde{\mathbb{T}}_m^{\mathfrak{l}_m}$ the quantities t_a/z_m remain bounded and separated from zero as z tends to limit, $z \rightrightarrows \mathbb{A}$, for all a such that $\mathfrak{l}^{m-1} < a \leqslant \mathfrak{l}^m$, and the rest of the proof remains the same as before.

Theorem 6.6 is proved. □

(7.12) Theorem. *Let the parameters $\xi_1, \dots, \xi_n$ obey condition (2.14). Then for any permutation $\tau \in \mathbf{S}^n$ the hypergeometric integral $I(W_{\mathfrak{l}}^\tau, w_{\mathfrak{m}})$ has the following asymptotics as z tends to limit in the asymptotic zone, $z \rightrightarrows \mathbb{A}_\tau$, so that at any moment assumption (2.15) holds:*

$$I(W_{\mathfrak{l}}^\tau, w_{\mathfrak{m}}) =$$

$$= \frac{\ell!}{\mathfrak{l}_1! \dots \mathfrak{l}_n!} \prod_{\substack{1 \leqslant l < m \\ \sigma_l < \sigma_m}} \xi_l^{\mathfrak{l}_m} \prod_{\substack{1 \leqslant l < m \\ \sigma_l > \sigma_m}} \eta^{\mathfrak{l}_l \mathfrak{l}_m} \xi_l^{-\mathfrak{l}_m} \Bigl(\prod_{m=1}^n I(W_{(\mathfrak{l}_m)}^{(m)}[\kappa_{\mathfrak{l},m}^\tau], w_{(\mathfrak{l}_m)}^{(m)}) + o(1) \Bigr),$$

$$I(W_{\mathfrak{l}}^\tau, w_{\mathfrak{m}}) = O(1) \qquad \textit{for } {}^\tau\mathfrak{l} \ll {}^\tau\mathfrak{m},$$

$$I(W_{\mathfrak{l}}^\tau, w_{\mathfrak{m}}) = o(1), \qquad \textit{otherwise}.$$

Here $\sigma = \tau^{-1}$, the functions $w_{(\mathfrak{l}_m)}^{(m)}$ and $W_{(\mathfrak{l}_m)}^{(m)}[\kappa_{\mathfrak{l},m}^\tau]$ are defined by formulae (7.10), ${}^\tau\mathfrak{l} = (\mathfrak{l}_{\tau_1}, \dots, \mathfrak{l}_{\tau_n})$ *and* $\kappa_{\mathfrak{l},m}^\tau = \kappa \prod_{1 \leqslant \sigma_l < \sigma_m} \eta^{-\mathfrak{l}_l} \xi_l \prod_{\sigma_m < \sigma_l \leqslant n} \eta^{\mathfrak{l}_l} \xi_l^{-1}$.

The proof is similar to the proof of Theorem 6.6.

Proof of Theorem 6.2. The statement follows from Theorem 7.12 and formula (5.13). □

Proof of Theorem 5.9. Since both sides of formula (5.9) are analytic functions of $\xi_1, \dots, \xi_n$ and η, it suffices to prove the formula under the assumption that $|\eta| > 1$ and $|\xi_m| < 1$, $m = 1, \dots, n$.

Denote by $F(z)$ the determinant $\det\bigl[I(W_{\mathfrak{l}}, w_{\mathfrak{m}})\bigr]_{\mathfrak{l},\mathfrak{m} \in \mathcal{Z}_\ell^n}$ and by $G(z)$ the right hand side of formula (5.9). Let $Y_{\mathfrak{l}}$ be an adjusting factor for the elliptic weight function $W_{\mathfrak{l}}$ and $\alpha_{\mathfrak{l},m}$ be the corresponding multipliers defined by formulae (2.32).

Since for every $\mathfrak{l} \in \mathcal{Z}_\ell^n$ the section $\Psi_{Y_{\mathfrak{l}} W_{\mathfrak{l}}}$ is a $(V_1 \otimes \dots \otimes V_n)_\ell$-valued solution of the *qKZ* equation, $F(z)$ solves the following system of difference equations:

$$F(z_1, \dots, pz_m, \dots, z_n) = \overset{(\ell)}{\det} K_m(z_1, \dots, z_n) \prod_{\mathfrak{l} \in \mathcal{Z}_\ell^n} \alpha_{\mathfrak{l},m}^{-1} F(z_1, \dots, z_n).$$

Here $\overset{(\ell)}{\det} K_m(z)$ stands for the determinant of the operator $K_m(z)$ (3.12) acting in the weight subspace $(V_1 \otimes \ldots \otimes V_n)_\ell$. Using either formula (3.7) or Theorem A.7 we see that

$$\overset{(\ell)}{\det} K_m(z_1, \ldots, z_n) \prod_{\mathfrak{l} \in \mathcal{Z}_\ell^n} \alpha_{\mathfrak{l},m}^{-1} =$$

$$= \prod_{s=0}^{\ell-1} \Bigl(\prod_{1 \leqslant l < m} \frac{p z_m - \eta^s \xi_l^{-1} \xi_m^{-1} z_l}{p z_m - \eta^{-s} \xi_l \xi_m z_l} \prod_{m < l \leqslant n} \frac{z_l - \eta^{-s} \xi_l \xi_m z_m}{z_l - \eta^s \xi_l^{-1} \xi_m^{-1} z_m} \Bigr)^{\binom{n+\ell-s-2}{n-1}}.$$

Therefore, the ratio $F(z)/G(z)$ is a p-periodic function of each of the variables $z_1, \ldots, z_n$:

$$\frac{F}{G}(z_1, \ldots, p z_m, \ldots, z_n) = \frac{F}{G}(z_1, \ldots, z_n).$$

Theorem 6.6 implies that the ratio $F(z)/G(z)$ tends to 1 as z tends to limit in the asymptotic zone, $z \rightrightarrows \mathbb{A}$. Hence, this ratio equals 1 identically, which completes the proof of the determinant formula.

Let functions $G_\mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be defined by formula (B.3). They form a basis in the elliptic hypergeometric space of the fiber $\mathcal{F}_{ell}(z)$. Using Theorem B.8 we have that

$$\det\bigl[I(G_\mathfrak{l}, w_\mathfrak{m})\bigr]_{\mathfrak{l},\mathfrak{m} \in \mathcal{Z}_\ell^n} = \Xi^{-1} (2\pi i)^{\ell \binom{n+\ell-1}{n-1}} \ell!^{\binom{n+\ell-1}{n-1}} \eta^{-n \binom{n+\ell-1}{n+1}} \times$$

$$\times \prod_{m=1}^n z_m^{(n-m)\binom{n+\ell-1}{n}} \prod_{s=0}^{\ell-1} \Bigl[\frac{(\eta^{-1})_\infty^n (\eta^{s+1-\ell} \kappa^{-1} \prod \xi_m)_\infty (p \eta^{s+1-\ell} \kappa \prod \xi_m)_\infty}{(\eta^{-s-1})_\infty^n (p)_\infty^{2n-1} \prod (\eta^{-s} \xi_m^2)_\infty} \times$$

$$\times \prod_{1 \leqslant l < m \leqslant n} \frac{1}{(p)_\infty (\eta^{-s} \xi_l \xi_m z_l / z_m)_\infty (p \eta^{-s} \xi_l \xi_m z_m / z_l)_\infty} \Bigr]^{\binom{n+\ell-s-2}{n-1}}$$

where Ξ is the constant given in Theorem B.8. Under the assumptions of Theorem 5.9 we have that $\det\bigl[I(G_\mathfrak{l}, w_\mathfrak{m})\bigr]_{\mathfrak{l},\mathfrak{m} \in \mathcal{Z}_\ell^n} \neq 0$ which means that the hypergeometric pairing $I : \mathcal{F}_{ell}(z) \otimes \mathcal{F}(z) \to \mathbb{C}$ is nondegenerate. Theorem 5.9 is proved. □

Proof of Theorem 5.10. Since both sides of formula (5.9) are analytic functions of $\xi_1, \ldots, \xi_n$ and η, it suffices to prove the formula under the assumption that $|\eta| > 1$ and $|\xi_m| < 1$, $m = 1, \ldots, n$.

Consider the determinant $\det\big[I(W_{\mathfrak l},w_{\mathfrak m})\big]_{\mathfrak l,\mathfrak m\in\mathcal Z_\ell^n}$ as a function of κ and denote it by $Det(\kappa;\ell)$. Set $\varepsilon=\big(1-\kappa^{-1}\eta^{1-\ell}\prod_{m=1}^{n}\xi_m\big)$. We will show that

$$(7.13)\quad Det(\kappa;\ell)=\prod_{s=0}^{\ell-1}\Big(\frac{2\pi i\varepsilon\ell}{(1-\eta^{-s}\xi_1^2)}\,\frac{(1-\eta)}{(1-\eta^{s+1})}\,\frac{\theta(\eta)}{\theta(\eta^{s+1})}\Big)^{\binom{n+\ell-s-3}{n-2}}\times$$
$$\times\ \Big(Det\big(\eta^{-\ell}\prod_{m=1}^{n}\xi_m;\ell-1\big)\det\big[I(W_{\mathfrak l},w_{\mathfrak m})\big]_{\substack{\mathfrak l,\mathfrak m\in\mathcal Z_\ell^n\\ \mathfrak l_1=\mathfrak m_1=0}}+o(1)\Big)$$

as $\varepsilon\to0$, that is $\kappa\to\eta^{1-\ell}\prod_{m=1}^{n}\xi_m$. This equality and the determinant formula (5.9) imply the determinant formula (5.10).

Let $\mathfrak e(m)=(0,\dots,\underset{m\text{-th}}{1},\dots,0)$, $m=1,\dots,n$. Introduce a new basis $w'_{\mathfrak l}$, $\mathfrak l\in\mathcal Z_\ell^n$, of the trigonometric hypergeometric space of a fiber by the rule: $w'_{\mathfrak l}=w_{\mathfrak l}$ for $\mathfrak l_1=0$ and

$$w'_{\mathfrak l}=w_{\mathfrak l}(1-\eta^{\mathfrak l_1})(1-\eta^{1-\mathfrak l_1}\xi_1^2)\ -$$
$$-\sum_{m=2}^{n}w_{\mathfrak l-\mathfrak e(1)+\mathfrak e(m)}(1-\eta^{\mathfrak l_m+1})(\xi_m-\eta^{\mathfrak l_m}\xi_m^{-1})\prod_{1\leqslant l\leqslant m}\eta^{-\mathfrak l_l}\xi_l$$

for $\mathfrak l_1>0$. We have that

$$Det(\kappa;\ell)=\prod_{s=0}^{\ell-1}\big((1-\eta^{s+1})(1-\eta^{-s}\xi_1^2)\big)^{-\binom{n+\ell-s-3}{n-2}}\det\big[I(W_{\mathfrak l},w'_{\mathfrak m})\big]_{\mathfrak l,\mathfrak m\in\mathcal Z_\ell^n}.$$

The main property of the new basis follows from Lemma 5.7. Namely, we have that $I(W_{\mathfrak l},w_{\mathfrak m})=O(\varepsilon)$ as $\varepsilon\to0$, if either $\mathfrak l_1>0$ or $\mathfrak m_1>0$. Therefore,

$$\det\big[I(W_{\mathfrak l},w'_{\mathfrak m})\big]_{\mathfrak l,\mathfrak m\in\mathcal Z_\ell^n}=$$
$$=\det\big[I(W_{\mathfrak l},w_{\mathfrak m})\big]_{\substack{\mathfrak l,\mathfrak m\in\mathcal Z_\ell^n\\ \mathfrak l_1=\mathfrak m_1=0}}\det\big[I(W_{\mathfrak l},w'_{\mathfrak m})\big]_{\substack{\mathfrak l,\mathfrak m\in\mathcal Z_\ell^n\\ \mathfrak l_1>0,\mathfrak m_1>0}}+o\big(\varepsilon^{\binom{n+\ell-2}{n-1}}\big).$$

If $\mathfrak{m}_1 > 0$, then by Lemmas 2.23 and A.5

$$w'_{\mathfrak{m}}(t_1, \dots, t_\ell) =$$
$$= \varepsilon \sum_{a=1}^{\ell} \bigl[w_{\mathfrak{m}-\mathfrak{e}(1)}(t_2, \dots, t_\ell) \bigr]_{(1,a)} + (1-\varepsilon) \sum_{a=1}^{\ell} D_a \bigl[w_{\mathfrak{m}-\mathfrak{e}(1)}(t_2, \dots, t_\ell) \bigr]_{(1,a)}$$

where $(1,a) \in \mathbf{S}^\ell$ are transpositions. Then using Lemma 7.7 we see that $I(W_{\mathfrak{l}}, w'_{\mathfrak{m}}) = \varepsilon\, I(W_{\mathfrak{l}}, w''_{\mathfrak{m}-\mathfrak{e}(1)})$ where

$$w''_{\mathfrak{n}}(t_1, \dots, t_\ell) = \sum_{a=1}^{\ell} \bigl[w_{\mathfrak{n}}(t_2, \dots, t_\ell) \bigr]_{(1,a)}, \qquad \mathfrak{n} \in \mathcal{Z}^n_{\ell-1}\,. \tag{7.14}$$

The next step is to calculate the hypergeometric integral $I(W_{\mathfrak{l}}, w''_{\mathfrak{m}-\mathfrak{e}(1)})$ at $\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \xi_m$. We will indicate explicitly the dependence of the elliptic weight functions on κ. Namely, the elliptic weight function $W_{\mathfrak{l}}[\kappa]$ is an element of the elliptic hypergeometric space of a fiber $\mathcal{F}_{ell}[\kappa, \sum_{m=1}^{n} \mathfrak{l}_m](z)$.

If $\mathfrak{l}_1 > 0$, then

$$W_{\mathfrak{l}}[\eta^{1-\ell} \prod_{m=1}^{n} \xi_m](t_1, \dots, t_\ell) = \frac{\theta(\eta)}{\theta(\eta^{\mathfrak{l}_1})} \sum_{\sigma \in \mathbf{S}^\ell} \bigl[\!\bigl[W_{\mathfrak{l}-\mathfrak{e}(1)}[\eta^{-\ell} \prod_{m=1}^{n} \xi_m](t_2, \dots, t_\ell) \bigr]\!\bigr]_\sigma ,$$

and due to formula (2.7) we have that

$$I(W_{\mathfrak{l}}[\eta^{1-\ell} \prod_{m=1}^{n} \xi_m], w''_{\mathfrak{m}}) = \ell\, \frac{\theta(\eta)}{\theta(\eta^{\mathfrak{l}_1})} \times$$
$$\times \int_{\mathbb{T}^\ell} \Phi(t_1, \dots, t_\ell)\, w''_{\mathfrak{m}}(t_1, \dots, t_\ell)\, W_{\mathfrak{l}-\mathfrak{e}(1)}[\eta^{-\ell} \prod_{m=1}^{n} \xi_m](t_2, \dots, t_\ell)\, (dt/t)^\ell$$

because the torus $\mathbb{T}^\ell$ is invariant under permutations of the variables $t_1, \dots, t_\ell$. Substitute formula (7.14) into the above integral. Similar to the proof of the second claim of Lemma 5.7 we obtain that

$$\int_{\mathbb{T}^\ell} \Phi(t_1, \dots, t_\ell) \bigl[w_{\mathfrak{m}-\mathfrak{e}(1)}(t_2, \dots, t_\ell) \bigr]_{(1,a)} W_{\mathfrak{l}-\mathfrak{e}(1)}[\eta^{-\ell} \prod_{m=1}^{n} \xi_m](t_2, \dots, t_\ell)\, (dt/t)^\ell$$
$$= 2\pi i\, \delta_{1a}\, I(W_{\mathfrak{l}-\mathfrak{e}(1)}, w_{\mathfrak{m}-\mathfrak{e}(1)})\,.$$

Hence,

$$I(W_{\mathfrak l}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m], w''_{\mathfrak m-\mathfrak e(1)}) = 2\pi i\ell\,\frac{\theta(\eta)}{\theta(\eta^{\mathfrak l_1})}\,I(W_{\mathfrak l-\mathfrak e(1)}[\eta^{-\ell}\prod_{m=1}^{n}\xi_m], w_{\mathfrak m-\mathfrak e(1)})$$

and finally

$$\det\big[I(W_{\mathfrak l}, w'_{\mathfrak m})\big]_{\substack{\mathfrak l,\mathfrak m\in\mathcal Z^n_\ell\\ \mathfrak l_1>0,\mathfrak m_1>0}} = \prod_{s=0}^{\ell-1}\Big(\frac{2\pi i\varepsilon\,\ell\,\theta(\eta)}{\theta(\eta^{s+1})}\Big)^{\binom{n+\ell-s-3}{n-2}}\times$$
$$\times\ \Big(Det(\eta^{-\ell}\prod_{m=1}^{n}\xi_m;\ell-1)+o(1)\Big)$$

as $\varepsilon\to 0$. Formula (7.13) is proved.

The rest of the proof is similar to the end of the proof of Theorem 5.9. Consider the space $\mathcal F_{ell}[\kappa\xi_1;z_2,\dots,z_n;\xi_2,\dots,\xi_n;\ell](z)$. It has a basis given by functions $\widetilde G_{\mathfrak l}(t)$, $\mathfrak l\in\mathcal Z^{n-1}_\ell$, defined similarly to formula (B.3). Set

$$G'_{\mathfrak l}(t) = \widetilde G_{\mathfrak l}(t)\prod_{a=1}^{\ell}\frac{\theta(\xi_1 t_a/z_1)}{\theta(\xi_1^{-1}t_a/z_1)}\,,\qquad \mathfrak l\in\mathcal Z^{n-1}_\ell.$$

The functions $W_{\mathfrak l}[\kappa]$ such that $\mathfrak l_1=0$, $\mathfrak l\in\mathcal Z^n_\ell$, are linear combinations of the functions $G'_{\mathfrak m}$, $\mathfrak m\in\mathcal Z^{n-1}_\ell$. Formula 5.10 and Theorem B.8 imply that

$$\det\big[I(G'_{\mathfrak l}, w_{(0,\mathfrak m)})\big]_{\mathfrak l,\mathfrak m\in\mathcal Z^{n-1}_\ell} =$$

$$= K\,(2\pi i\xi_1)^{\ell\binom{n+\ell-2}{n-2}}\,\ell!^{\binom{n+\ell-2}{n-2}}\,\eta^{-n\binom{n+\ell-2}{n}}\prod_{m=2}^{n}z_m^{(n-m)\binom{n+\ell-2}{n-1}}\times$$
$$\times\prod_{s=0}^{\ell-1}\Big[\frac{(\eta^{-1})_\infty^{n-1}\,(p\eta^{s+2-2\ell}\prod\xi_m^2)_\infty\,(\eta^s\xi_1^{-2})_\infty}{(\eta^{-s-1})_\infty^{n-1}\,(p)_\infty^{2n-3}\prod\limits_{1<m\leqslant n}(\eta^{-s}\xi_m^2)_\infty}\prod_{m=2}^{n}\frac{(\eta^s\xi_1^{-1}\xi_m^{-1}z_1/z_m)_\infty}{(\eta^{-s}\xi_1\xi_m z_1/z_m)_\infty}\times$$
$$\times\prod_{2\leqslant l<m\leqslant n}\frac{1}{(p)_\infty\,(\eta^{-s}\xi_l\xi_m z_l/z_m)_\infty\,(p\eta^{-s}\xi_l\xi_m z_m/z_l)_\infty}\Big]^{\binom{n+\ell-s-3}{n-2}}$$

where $K = \Big[(p)_\infty^{n(n-2)}\prod_{m=1}^{n-2}\Big(\frac{e^{2\pi i m/(n-1)}-1}{\theta(e^{2\pi i m/(n-1)})}\Big)^{n-m-1}\Big]^{\binom{n+\ell-2}{n-1}}.$

Under the assumptions of Theorem 5.10 we have that

$$\det\left[I(G'_{\mathfrak{l}}, w_{(0,\mathfrak{m})})\right]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^{n-1}} \neq 0$$

which means that the hypergeometric pairing

$$I : \mathcal{F}_{eu}(z)/\mathcal{Q}(z) \otimes \mathcal{F}(z)/\mathcal{R}(z) \to \mathbb{C}$$

is nondegenerate. Theorem 5.10 is proved. □

Proof of Theorem 5.11. The proof is similar to the proof of Theorem 5.10. □

Proof of Theorem 5.26. Under assumptions of the theorem, for any κ in the puctured neighbourhood of $\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ the assumptions of Theorem 5.25 are valid. Therefore, the hypergeometric map $I_{\tau,\tau'}(z)$ is well defined and nondegenerate for any such κ.

Introduce a matrix X by the rule:

$$I_{\tau,\tau'}(z)\, v^{[\mathfrak{l}_{\tau'_1}]}\otimes\ldots\otimes v^{[\mathfrak{l}_{\tau'_n}]} = \sum_{\mathfrak{m}\in\mathcal{Z}_\ell^n} X_{\mathfrak{l}\mathfrak{m}}\, F^{\mathfrak{m}_{\tau_1}}v_{\tau_1}\otimes\ldots\otimes F^{\mathfrak{m}_{\tau_n}}v_{\tau_n}\,, \qquad \mathfrak{l}\in\mathcal{Z}_\ell^n\,.$$

We have to show that the matrix X has a finite limit as $\kappa\to\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ and $\lim\det X\neq 0$.

Consider the hypergeometric integral $I(W_{\mathfrak{l}}^{\tau'}, w_{\mathfrak{m}}^{\tau})$. It is a holomorphic function of κ since the corresponding integrand is a holomorphic function of κ and the integration is over a compact contour. Hence, Lemmas 5.7 and 2.38 imply that $\left(\kappa-\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}\right)^{-1} I(W_{\mathfrak{l}}^{\tau'}, w_{\mathfrak{m}}^{\tau})$ has a finite limit as $\kappa\to\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ if $\mathfrak{l}_1>0$. Therefore, the entries $X_{\mathfrak{l}\mathfrak{m}}$ with $\mathfrak{l}_1>0$ have finite limits and the entries $X_{\mathfrak{l}\mathfrak{m}}$ with $\mathfrak{l}_1=0$ are regular at $\kappa=\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$. That is the matrix X has a finite limit as $\kappa\to\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$.

The explicit formula for the determinant $\det X$ for general κ can be easily obtained using Theorems 5.9, A.7, B.8, formula (4.9) and the definition of the hypergeometric map $I_{\tau,\tau'}(z)$. It follows from the obtained expression that under the assumptions of Theorem 5.26 the limit of the determinant $\det X$ as $\kappa\to\eta^{1-\ell}\prod_{m=1}^{n}\eta^{\Lambda_m}$ is not equal to zero. The theorem is proved. □

Proof of Theorem 5.28. The proof is similar to the proof of Theorem 5.26. □

Proof of Theorem 5.31. Without loss of generality we can assume that the adjusting map $Y^{\tau'}(z)$ analytically depends on κ. Hence, the same do the section $\Psi^{\tau}_{v,Y^{\tau'}}$. The qKZ operators analytically depend on κ as well.

For general κ the section $\Psi^{\tau}_{v,Y^{\tau'}}$ solves the qKZ equation with values in $(V_{\tau_1} \otimes \ldots \otimes V_{\tau_n})_\ell$ due to Corollary 5.19. For special values of κ:

$$\kappa = \eta^{1-\ell} \prod_{m=1}^{n} \eta^{\Lambda_m} \qquad \text{and} \qquad \kappa = p^{-1} \eta^{\ell-1} \prod_{m=1}^{n} \eta^{-\Lambda_m},$$

the qKZ equation remains valid by the analytic continuation.

Theorems 5.25, 5.26 and 5.28 imply that under the assumptions of each of Theorems 5.9 – 5.11 the sections $\Psi^{\tau}_{v,Y^{\tau'}}$, $v \in (V^{e}_{\tau'_1} \otimes \ldots \otimes V^{e}_{\tau'_n})_\ell$, span the space of solutions of the qKZ equation over the field of p-periodic functions (quasiconstants). Theorem 5.31 is proved. □

A. Basic facts about the trigonometric hypergeometric space

Let $\mathcal{F} = \mathcal{F}[z_1, \ldots, z_n; \xi_1, \ldots, \xi_n; \ell]$ be the trigonometric hypergeometric space.

By construction, the trigonometric hypergeometric space of a fiber has the same dimension as the space of symmetric polynomials in ℓ variables of degree less than n in each of the variables, that is

$$\dim \mathcal{F}(z) = \binom{n+\ell-1}{n-1}.$$

(A.1) Lemma. *For any* $\mathfrak{l} \in \mathcal{Z}_\ell^n$ *the trigonometric weight function* $w_\mathfrak{l}$ *is in the trigonometric hypergeometric space* $\mathcal{F}$.

Proof. It is clear from definiton (2.20) that the function $w_\mathfrak{l}(t,z)$ has the form

$$Q(t_1, \ldots, t_\ell, z_1, \ldots, z_n) \prod_{a=1}^{\ell} t_a \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{1}{t_a - \xi_m z_m} \prod_{1 \leqslant a < b \leqslant \ell} \frac{1}{\eta t_a - t_b}$$

where Q is a polynomial which has degree less than $n + \ell - 1$ in each of the variables $t_1, \ldots, t_\ell$. Furthermore, by construction the function $w_\mathfrak{l}$ as a function of $t_1, \ldots, t_\ell$ is invariant with respect to the action (2.9) of the symmetric group $\mathbf{S}^\ell$, which means that the polynomial Q is skewsymmetric with respect to the variables $t_1, \ldots, t_\ell$. Hence, the polynomial Q is divisible by $\prod_{1 \leqslant a < b \leqslant \ell} (t_a - t_b)$ and the ratio is a polynomial which is symmetric in variables $t_1, \ldots, t_\ell$ and has degree less than n in each of the variables $t_1, \ldots, t_\ell$; that is the function $w_\mathfrak{l}$ is in the trigonometric hypergeometric space. □

(A.2) Corollary. *Let* $n = 1$. *Then*

$$w_{(\ell)}(t_1, \ldots, t_\ell, z_1) = \prod_{a=1}^{\ell} \frac{t_a}{t_a - \xi_1 z_1} \prod_{1 \leqslant a < b \leqslant \ell} \frac{t_a - t_b}{\eta t_a - t_b}.$$

Proof. Denote by $f(t, z_1)$ the ratio

$$w_{(\ell)}(t, z_1) \left[\prod_{a=1}^{\ell} \frac{t_a}{t_a - \xi_1 z_1} \prod_{1 \leqslant a < b \leqslant \ell} \frac{t_a - t_b}{\eta t_a - t_b} \right]^{-1}.$$

Since for $n = 1$ the trigonometric hypergeometric space of a fiber is one-dimensional, $f(t, z_1)$ does not depend on t. Let $t^\star = (t_1, \eta^{-1}t_1, \ldots, \eta^{1-\ell}t_1)$. Only the term corresponding to the identity permutation in the right hand side of (2.20) contributes to $w(t^\star, z_1)$ and by a straightforward calculation we get $f(t^\star, z_1) = 1$. □

(A.3) Lemma. *Let* $n = 1$. *Then*

$$(1-\eta^\ell)(\eta^{1-\ell}\xi_1 - \xi_1^{-1})\, w_{(\ell)}(t_1, \ldots, t_\ell, z_1) =$$

$$= (1-\eta) \sum_{a=1}^{\ell} \Big[\Big(\frac{\xi_1 t_1 - z_1}{\xi_1^{-1} t_1 - z_1} \prod_{b=2}^{\ell} \frac{\eta^{-1}t_1 - t_b}{\eta t_1 - t_b} - 1 \Big)\, w_{(\ell-1)}(t_2, \ldots, t_\ell) \Big]_{(1,a)},$$

$$(1-\eta^\ell)(\xi_1 - \eta^{\ell-1}\xi_1^{-1})\, w_{(\ell)}(t_1, \ldots, t_\ell, z_1) =$$

$$= (1-\eta) \sum_{a=1}^{\ell} \Big[\Big(\frac{t_1 - \xi_1^{-1} z_1}{t_1 - \xi_1 z_1} \prod_{b=2}^{\ell} \frac{t_1 - \eta t_b}{t_1 - \eta^{-1} t_b} - 1 \Big)\, t_1\, w_{(\ell-1)}(t_2, \ldots, t_\ell) \Big]_{(1,a)}.$$

Here $(1, a) \in \mathbf{S}^\ell$ *are transpositions.*

Proof. Similar to the proof of Lemma A.1 one can show that the right hand sides of the both above formulae are elements of the trigonometric hypergeometric space. The rest of the proof is similar to the proof of Corollary A.2. □

(A.4) Lemma. *Let* $n = 1$. *Then*

$$w_{(\ell)}(t_1, \ldots, t_\ell, z_1) = \sum_{\sigma \in \mathbf{S}^\ell} \Big[\prod_{a=1}^{\ell} \frac{t_a}{t_a - \eta t_{a-1}} \Big]_\sigma .$$

Here $t_0 = \eta^{-1}\xi_1 z_1$.

Proof. The right hand side of the above formula is an element of the trigonometric hypergeometric space. Comparing residues of both sides at $t = (\xi_1 z_1, \ldots, \eta^{\ell-1}\xi_1 z_1)$ completes the proof. □

Proof of Lemma 2.21. Let $\mathbf{S}^{\ell_1} \times \ldots \times \mathbf{S}^{\ell_n} \subset \mathbf{S}^\ell$ be the subgroup of permutations preserving the subsets $\Gamma_m = \{1 + \ell^{m-1}, \ldots, \ell^m\}$, $m = 1, \ldots, n$. The coset space $\mathbf{S}^\ell / (\mathbf{S}^{\ell_1} \times \ldots \times \mathbf{S}^{\ell_n})$ is in one-to-one correspondence with the set of all n-tuples $\Gamma_1, \ldots, \Gamma_n$ of disjoint subsets of $\{1, \ldots, \ell\}$ such that Γ_m has ℓ_m elements. Namely, a permutation $\sigma \in \mathbf{S}^\ell$ corresponds to an n-tuple $\sigma(\Gamma_1), \ldots, \sigma(\Gamma_n)$, and the n-tuple depends only on the coset of the permutation σ.

Perform the summation in the right hand side of formulae (2.20) in two steps. First take a sum over the subgroup $\mathbf{S}^{\mathfrak{l}_1}\times\ldots\times\mathbf{S}^{\mathfrak{l}_n}$. This can be done explicitly using Corollary A.2. The remaining sum over the coset space $\mathbf{S}^{\ell}/(\mathbf{S}^{\mathfrak{l}_1}\times\ldots\times\mathbf{S}^{\mathfrak{l}_n})$ is equivalent to the right hand side of formula (2.21). The lemma is proved. □

(A.5) Lemma. *For any $\mathfrak{l}\in\mathcal{Z}^n_{\ell-1}$ the following relations hold:*

$$\sum_{m=1}^{n} w_{\mathfrak{l}+\mathfrak{e}(m)}\,(1-\eta^{\mathfrak{l}_m+1})\,(\xi_m-\eta^{\mathfrak{l}_m}\xi_m^{-1})\prod_{1\leqslant l\leqslant m}\eta^{-\mathfrak{l}_l}\xi_l\;=$$

$$=\;(1-\eta)\sum_{a=1}^{\ell}\Big[\Big(\prod_{m=1}^{n}\frac{\xi_m t_1-z_m}{\xi_m^{-1}t_1-z_m}\prod_{b=2}^{\ell}\frac{\eta^{-1}t_1-t_b}{\eta t_1-t_b}\;-1\Big)\,w_{\mathfrak{l}}(t_2,\ldots,t_\ell)\Big]_{(1,a)},$$

$$\sum_{m=1}^{n} w_{\mathfrak{l}+\mathfrak{e}(m)}\,(1-\eta^{\mathfrak{l}_m+1})\,(\xi_m-\eta^{\mathfrak{l}_m}\xi_m^{-1})\,z_m\prod_{1\leqslant l<m}\eta^{\mathfrak{l}_l}\xi_l^{-1}\;=$$

$$=\;(1-\eta)\sum_{a=1}^{\ell}\Big[\Big(\prod_{m=1}^{n}\frac{t_1-\xi_m^{-1}z_m}{t_1-\xi_m z_m}\prod_{b=2}^{\ell}\frac{t_1-\eta t_b}{t_1-\eta^{-1}t_b}\;-1\Big)\,t_1\,w_{\mathfrak{l}}(t_2,\ldots,t_\ell)\Big]_{(1,a)}$$

where $(1,a)\in\mathbf{S}^{\ell}$ are transpositions.

The proof is similar to the proof of Lemma 2.21 using Lemma A.3 instead of Corollary A.2.

For any $\mathfrak{l}\in\mathcal{Z}^n_\ell$ denote by $Q_{\mathfrak{l}}(t_1,\ldots,t_\ell)$ the following symmetric polynomial:

$$Q_{\mathfrak{l}}(t_1,\ldots,t_\ell)\;=\;\frac{1}{\mathfrak{l}_1!\ldots\mathfrak{l}_n!}\sum_{\sigma\in\mathbf{S}^\ell}\prod_{m=1}^{n}\prod_{a\in\Gamma_m}t_{\sigma_a}^{m-1}\tag{A.6}$$

where $\Gamma_m=\{1+\mathfrak{l}^{m-1},\ldots,\mathfrak{l}^m\}$, $m=1,\ldots,n$. Consider a basis in the space $\mathcal{F}(z)$ given by functions

$$g_{\mathfrak{l}}(t,z)\;=\;Q_{\mathfrak{l}}(t_1,\ldots,t_\ell)\prod_{a=1}^{\ell}t_a\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{t_a-\xi_m z_m}\prod_{1\leqslant a<b\leqslant\ell}\frac{t_a-t_b}{\eta t_a-t_b}\,,\qquad \mathfrak{l}\in\mathcal{Z}^n_\ell\,.$$

Define a matrix $M(z)$ by the rule:

$$w_{\mathfrak{l}}(t,z)\;=\;\sum_{\mathfrak{m}\in\mathcal{Z}^n_\ell}M_{\mathfrak{l}\mathfrak{m}}(z)\,g_{\mathfrak{m}}(t,z)\,,\qquad \mathfrak{l}\in\mathcal{Z}^n_\ell.$$

(A.7) Theorem.

$$\det M \,=\, \prod_{s=0}^{\ell-1}\ \prod_{1\leqslant l<m\leqslant n} (\eta^s z_l - \xi_l\xi_m z_m)^{\binom{n+\ell-s-2}{n-1}}.$$

The theorem is equivalent to Lemma 2.2 in [T]. Nevertheless, we give below another proof of Theorem A.7 which is similar to the proof of Theorem B.8 in the elliptic case.

(A.8) Corollary. *Let $\mathcal{R}(z)$, $\mathcal{R}'(z)$ be the coboundary subspaces. Then*

$$\dim \mathcal{F}(z)/\mathcal{R}(z) \,=\, \dim \mathcal{F}(z)/\mathcal{R}'(z) \,=\, \binom{n+\ell-2}{n-2}$$

provided that $\xi_l\xi_m z_m/z_l \neq \eta^r$, $1\leqslant l\leqslant m\leqslant n$, *for any* $r=0,\dots,\ell-1$.

Proof. Under assumptions of the corollary, both spaces $\mathcal{F}(z)/\mathcal{R}(z)$ and $\mathcal{F}(z)/\mathcal{R}'(z)$ have bases induced by the set $\{w_{\mathfrak{l}}(t,z)\mid \mathfrak{l}\in\mathcal{Z}^n_\ell,\ \mathfrak{l}_n=0\}$. □

Proof of Theorem A.7. For any $\mathfrak{l}\in\mathcal{Z}^n_\ell$ define a symmetric polynomial $P_{\mathfrak{l}}(t_1, \dots,t_\ell)$ by the rule:

$$\text{(A.9)}\quad w_{\mathfrak{l}}(t,z) \,=\, P_{\mathfrak{l}}(t_1,\dots,t_\ell)\,\times$$
$$\times\,\prod_{a=1}^{\ell} t_a \prod_{1\leqslant l<m\leqslant n}\xi_l^{\mathfrak{l}_m}\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{t_a-\xi_m z_m}\prod_{1\leqslant a<b\leqslant \ell}\frac{t_a-t_b}{\eta t_a-t_b}\,.$$

Introduce new variables $x_1,\dots,x_n$, $y_1,\dots,y_n$:

$$x_m=\xi_m z_m\,,\qquad y_m=\xi_m^{-1}z_m\,,\qquad m=1,\dots,n\,.$$

Then the polynomial $P_{\mathfrak{l}}(t)$ has the form:

$$P_{\mathfrak{l}}(t_1,\dots,t_\ell) \,=\, \sum_{\Gamma_1,\dots,\Gamma_n}\Biggl\{\prod_{\substack{1\leqslant m<l\leqslant n\\ a\in\Gamma_m}}(t_a-x_l)\prod_{\substack{1\leqslant l<m\leqslant n\\ a\in\Gamma_m}}(t_a-y_l)\prod_{\substack{1\leqslant l<m\leqslant n\\ a\in\Gamma_l,\,b\in\Gamma_m}}\frac{\eta t_a-t_b}{t_a-t_b}\Biggr\}$$

where the summation is over all n-tuples $\Gamma_1,\dots,\Gamma_n$ of disjoint subsets of $\{1,\dots,\ell\}$ such that Γ_m has $\mathfrak{l}_m$ elements.

Define a matrix N by the rule:

$$P_{\mathfrak{l}}(t) \,=\, \sum_{\mathfrak{m}\in\mathcal{Z}^n_\ell} N_{\mathfrak{l}\mathfrak{m}}\,Q_{\mathfrak{m}}(t)\,,\qquad \mathfrak{l}\in\mathcal{Z}^n_\ell.$$

Then the claim of the theorem takes the form

$$\text{(A.10)}\qquad \det N = \prod_{s=0}^{\ell-1}\prod_{1\leqslant l<m\leqslant n}(\eta^s y_l - x_m)^{\binom{n+\ell-s-2}{n-1}}.$$

For $\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n$ say that $\mathfrak{l}\leqq\mathfrak{m}$ if $\sum_{i=1}^m \mathfrak{l}_i \leqslant \sum_{i=1}^m \mathfrak{m}_i$ for any $m=1,\dots,n-1$. Say that $\mathfrak{l}\ll\mathfrak{m}$ if $\mathfrak{l}\neq\mathfrak{m}$ and $\mathfrak{l}\leqq\mathfrak{m}$.

For any $x,y\in\mathbb{C}^n$ and $\mathfrak{l}\in\mathcal{Z}_\ell^n$ set

$$\text{(A.11)}\qquad \begin{aligned} x\triangleright\mathfrak{l} &= (\eta^{1-\mathfrak{l}_1}x_1,\eta^{2-\mathfrak{l}_1}x_1,\dots,x_1,\eta^{1-\mathfrak{l}_2}x_2,\dots,x_2,\dots,\\ &\qquad\qquad \eta^{1-\mathfrak{l}_n}x_n,\dots,x_n),\\ y\triangleleft\mathfrak{l} &= (\eta^{\mathfrak{l}_1-1}y_1,\eta^{\mathfrak{l}_1-2}y_1,\dots,y_1,\eta^{\mathfrak{l}_2-1}y_2,\dots,y_2,\dots,\\ &\qquad\qquad \eta^{\mathfrak{l}_n-1}y_n,\dots,y_n). \end{aligned}$$

(A.12) Lemma. *$P_\mathfrak{l}(x\triangleright\mathfrak{m})=0$ unless $\mathfrak{l}\leqq\mathfrak{m}$. $P_\mathfrak{l}(y\triangleleft\mathfrak{m})=0$ unless $\mathfrak{l}\geqq\mathfrak{m}$. Moreover,*

$$P_\mathfrak{l}(x\triangleright\mathfrak{l}) = \prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\Bigl(\prod_{1\leqslant l<m}(\eta^{-s}x_m-y_l)\prod_{m<l\leqslant n}(\eta^{\mathfrak{l}_l-s}x_m-x_l)\Bigr),$$

$$P_\mathfrak{l}(y\triangleleft\mathfrak{l}) = \prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\Bigl(\prod_{1\leqslant l<m}(\eta^{s}y_m-\eta^{\mathfrak{l}_l}y_l)\prod_{m<l\leqslant n}(\eta^{s}y_m-x_l)\Bigr).$$

The proof is straightforward.

Set $\displaystyle D(n,\ell,s) = \sum_{\substack{r\in\mathbb{Z}_{\geqslant0}\\ 2r\leqslant\ell-|s|-1}}\binom{n+\ell-|s|-2r-3}{n-2}$.

(A.13) Lemma.

$$\det\bigl[P_\mathfrak{l}(x\triangleright\mathfrak{m})\bigr]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} =$$

$$= \prod_{s=0}^{\ell-1}\prod_{1\leqslant l<m\leqslant n}(y_l-\eta^{-s}x_m)^{\binom{n+\ell-s-2}{n-1}}\prod_{s=1-\ell}^{\ell-1}\prod_{1\leqslant l<m\leqslant n}(\eta^s x_m-x_l)^{D(n,\ell,s)},$$

$$\det\big[P_{\mathfrak{l}}(y \lhd \mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} = \eta^{n(n-1)/2\cdot\binom{n+\ell-1}{n+1}} \times$$

$$\times \prod_{s=0}^{\ell-1} \prod_{1\leqslant l<m\leqslant n} (\eta^s y_l - x_m)^{\binom{n+\ell-s-2}{n-1}} \prod_{s=1-\ell}^{\ell-1} \prod_{1\leqslant l<m\leqslant n} (\eta^s y_m - y_l)^{D(n,\ell,s)},$$

Proof. Lemma A.12 implies that

$$\det\big[P_{\mathfrak{l}}(x \rhd \mathfrak{m})\big] = \prod_{\mathfrak{l}\in\mathcal{Z}_\ell^n} P_{\mathfrak{l}}(x \rhd \mathfrak{l}) \qquad \text{and} \qquad \det\big[P_{\mathfrak{l}}(y \lhd \mathfrak{m})\big] = \prod_{\mathfrak{l}\in\mathcal{Z}_\ell^n} P_{\mathfrak{l}}(y \lhd \mathfrak{l}).$$

The rest of the proof consists of several applications of identity (G.1). □

Consider two more determinants,

$$\det\big[Q_{\mathfrak{l}}(x \rhd \mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} \qquad \text{and} \qquad \det\big[Q_{\mathfrak{l}}(y \lhd \mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n}.$$

Since

$$\det\big[P_{\mathfrak{l}}(x \rhd \mathfrak{m})\big] = \det N \cdot \det\big[Q_{\mathfrak{l}}(x \rhd \mathfrak{m})\big]$$

and

$$\det\big[P_{\mathfrak{l}}(y \lhd \mathfrak{m})\big] = \det N \cdot \det\big[Q_{\mathfrak{l}}(y \lhd \mathfrak{m})\big],$$

we have

$$\frac{\det\big[P_{\mathfrak{l}}(x \rhd \mathfrak{m})\big]}{\det\big[P_{\mathfrak{l}}(y \lhd \mathfrak{m})\big]} = \frac{\det\big[Q_{\mathfrak{l}}(x \rhd \mathfrak{m})\big]}{\det\big[Q_{\mathfrak{l}}(y \lhd \mathfrak{m})\big]},$$

and by standard arguments of the separation of variables we obtain that

(A.14)
$$\det\big[Q_{\mathfrak{l}}(x \rhd \mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} =$$

$$= C\eta^{n(1-n)/2\cdot\binom{n+\ell-1}{n+1}} \prod_{s=1-\ell}^{\ell-1} \prod_{1\leqslant l<m\leqslant n} (\eta^s x_m - x_l)^{D(n,\ell,s)},$$

$$\det\big[Q_{\mathfrak{l}}(y \lhd \mathfrak{m})\big]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} =$$

$$= C\eta^{n(n-1)/2\cdot\binom{n+\ell-1}{n+1}} \prod_{s=1-\ell}^{\ell-1} \prod_{1\leqslant l<m\leqslant n} (\eta^s y_m - y_l)^{D(n,\ell,s)}$$

where C is a nonzero constant which does not depend on $x_1, \ldots, x_n$, $y_1, \ldots,$ y_n. Formulae (A.13), (A.14) imply formula (A.10) up to a factor:

$$\text{(A.15)} \qquad \det N = C^{-1} \prod_{s=0}^{\ell-1} \prod_{1 \leqslant l < m \leqslant n} (\eta^s y_l - x_m)^{\binom{n+\ell-s-2}{n-1}}.$$

To calculate the constant C we consider its dependence on η. The left hand side of formula (A.15) is a polynomial in η of degree

$$n(n-1)/2 \cdot \binom{n+\ell-1}{n+1},$$

the same as the double product in the right hand side. Thus, C is a rational function in η with no pole at infinity. Moreover, since C does not depend on $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, it has no zeros as a function of η. Hence, C does not depend on η at all.

Let $\eta = 1$, $x_m = x_1$, $y_m = 0$, $m = 1, \ldots, n$, and consider the limit $x_1 \to \infty$. In this limit

$$P_{\mathfrak{l}}(t) = (-x_1)^{\sum_{m=1}^{n} (n-m)\mathfrak{l}_m} \big(Q_{\mathfrak{l}}(t) + o(1)\big).$$

Therefore, $C = 1$. The theorem is proved. □

B. Basic facts about the elliptic hypergeometric space

Let $\omega = \exp(2\pi i/n)$ all over the appendix. Fix a complex number α such that $\alpha^n = p$. Let $\theta(u) = (u,p)_\infty (p/u,p)_\infty (p,p)_\infty$ be the Jacobi theta-function.

Let $A = \kappa \prod_{m=1}^{n} z_m$. Fix a complex number ζ such that $\zeta^n = (-1)^{n-1} A^{-1}$. Let $\mathcal{E}[A]$ be the space of holomorphic functions on $\mathbb{C}^\times$ such that $f(pu) = A(-u)^{-n} f(u)$. It is easy to see that $\dim \mathcal{E}[A] = n$, say by Fourier series. Set

$$\vartheta_l(u) = u^{l-1} \prod_{m=1}^{n} \theta(\zeta \alpha^{l-1} \omega^m u), \qquad l = 1, \dots, n.$$

(B.1) Lemma. *The functions* $\vartheta_1, \dots, \vartheta_n$ *form a basis in the space* $\mathcal{E}[A]$.

Proof. Clearly, $\vartheta_l \in \mathcal{E}[A]$ for any $l = 1, \dots, n$. Moreover,

$$\vartheta_l(\omega u) = \omega^{l-1} \vartheta_l(u),$$

that is the functions $\vartheta_1, \dots, \vartheta_n$ are eigenfunctions of the translation operator with distinct eigenvalues. Hence, they are linearly independent. □

Let $\mathcal{E}_\ell[A]$ be the space of symmetric functions in variables $t_1, \dots, t_\ell$ which are holomorphic on $\mathbb{C}^{\times \ell}$ and have the property

$$f(t_1, \dots, pt_a, \dots, t_\ell) = A(-t_a)^{-n} f(t_1, \dots, t_\ell).$$

In particular, $\mathcal{E}_1[A] = \mathcal{E}[A]$. The space $\mathcal{E}_\ell[A]$ has dimension $\binom{n+\ell-1}{n-1}$. A basis in the space $\mathcal{E}_\ell[A]$ is given by functions $\Theta_{\mathfrak{l}}(t_1, \dots, t_\ell)$, $\mathfrak{l} \in \mathcal{Z}^n_\ell$:

$$\Theta_{\mathfrak{l}}(t_1, \dots, t_\ell) = \frac{1}{\mathfrak{l}_1! \dots \mathfrak{l}_n!} \sum_{\sigma \in \mathbf{S}^\ell} \prod_{m=1}^{n} \prod_{a \in \Gamma_m} \vartheta_m(t_{\sigma_a}). \tag{B.2}$$

Here $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \dots, \mathfrak{l}^m\}$, $m = 1, \dots, n$.

Let $\mathcal{F}_{ell} = \mathcal{F}_{ell}[\kappa; z_1, \dots, z_n; \xi_1, \dots, \xi_n; \ell]$ be the elliptic hypergeometric space. The elliptic hypergeometric space $\mathcal{F}_{ell}(z)$ of a fiber is isomorphic to the space $\mathcal{E}_\ell[A]$ by definition. Therefore

$$\dim \mathcal{F}_{ell}(z) = \binom{n+\ell-1}{n-1}.$$

Set

$$G_{\mathfrak{l}}(t,z) = \Theta_{\mathfrak{l}}(t_1,\dots,t_\ell) \prod_{m=1}^{n}\prod_{a=1}^{\ell} \frac{1}{\theta(\xi_m^{-1}t_a/z_m)} \prod_{1\leqslant a<b\leqslant \ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)}\,. \tag{B.3}$$

The functions $G_{\mathfrak{l}}$, $\mathfrak{l}\in\mathcal{Z}_\ell^n$ form a basis in $\mathcal{F}_{ell}(z)$.

(B.4) Lemma. *For any $\mathfrak{l}\in\mathcal{Z}_\ell^n$ and $z\in\mathbb{C}^{\times n}$ the elliptic weight function $W_{\mathfrak{l}}(t,z)$ is in the elliptic hypergeometric space $\mathcal{F}_{ell}(z)$ of the fiber.*

Proof. It is clear from definiton (2.30) that the function $W_{\mathfrak{l}}(t,z)$ has the form

$$Q(t_1,\dots,t_\ell,z_1,\dots,z_n) \prod_{m=1}^{n}\prod_{a=1}^{\ell} \frac{1}{\theta(t_a\xi_m^{-1}/z_m)} \prod_{1\leqslant a<b\leqslant \ell} \frac{1}{\theta(\eta t_a/t_b)}$$

where Q is a holomorphic function on $\mathbb{C}^{\times(\ell+n)}$ with the properties

$$Q(t_1,\dots,pt_a,\dots,t_\ell,z_1,\dots,z_n) =$$
$$= (-t_a)^{-\ell-n}\prod_{b=1}^{\ell} t_b\cdot p^{1-a}\kappa\prod_{m=1}^{n} z_m\, Q(t_1,\dots,t_\ell,z_1,\dots,z_n)\,,$$

$a=1,\dots,\ell$. Furthermore, by construction the function $W_{\mathfrak{l}}$ as a function of $t_1,\dots,t_\ell$ is invariant with respect to the action (2.29) of the symmetric group $\mathbf{S}^\ell$, which means that the holomorphic function Q is skewsymmetric with respect to the variables $t_1,\dots,t_\ell$. Hence, the ratio Θ of the function Q and the product $\prod_{1\leqslant a<b\leqslant \ell}\theta(t_a/t_b)$ is a holomorphic function on $\mathbb{C}^{\times(\ell+n)}$ which is symmetric in the variables $t_1,\dots,t_\ell$ and has the properties

$$\Theta(t_1,\dots,pt_a,\dots,t_\ell,z_1,\dots,z_n) = (-t_a)^{-n}\kappa\prod_{m=1}^{n} z_m\,\Theta(t_1,\dots,t_\ell,z_1,\dots,z_n)\,,$$

$a=1,\dots,\ell$; that is the function $W_{\mathfrak{l}}$ is in the elliptic hypergeometric space of the fiber. □

(B.5) Corollary. *Let $n=1$. Then*

$$W_{(\ell)}(t_1,\dots,t_\ell,z_1) = \prod_{a=1}^{\ell} \frac{\theta(\kappa^{-1}t_a/z_1)}{\theta(\xi_1^{-1}t_a/z_1)} \prod_{1\leqslant a<b\leqslant \ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)}\,.$$

The proof is similar to the proof of Corollary A.2.

(B.6) Lemma. *Let* $n = 1$. *Then*

$$W_{(\ell)}(t_1,\dots,t_\ell,z_1) = \prod_{s=0}^{\ell-1} \frac{\theta(\eta^{-s}\kappa\xi_1^{-1})}{\theta(\eta^{s(s-\ell)}\kappa^{\ell-s}\xi_1^{s-\ell})} \times$$

$$\times \sum_{\sigma\in\mathbf{S}^\ell} \left[\!\left[\prod_{a=1}^{\ell} \frac{\theta(\eta^{1+(a-1)(a-1-\ell)}\kappa^{\ell-a+1}\xi_1^{a-1-\ell}t_{a-1}/t_a)}{\theta(\eta t_{a-1}/t_a)} \right]\!\right]_\sigma .$$

Here $t_0 = \eta^{-1}\xi_1 z_1$.

The proof is similar to the proof of Lemma A.4.

Proof of Lemma 2.31. The proof is similar to the proof of Lemma 2.21. The summation over the subgroup $\mathbf{S}^{\mathfrak{l}_1}\times\ldots\times\mathbf{S}^{\mathfrak{l}_n}\subset\mathbf{S}^\ell$ can be done explicitly using Corollary B.5. □

Let $W_{\mathfrak{l}}$, $\mathfrak{l}\in\mathcal{Z}_\ell^n$, be the elliptic weight functions. Define a matrix M^{ell} by the rule:

$$W_{\mathfrak{l}}(t,z) = \sum_{\mathfrak{m}\in\mathcal{Z}_\ell^n} M^{ell}_{\mathfrak{l}\mathfrak{m}}(z)\,G_{\mathfrak{m}}(t,z)\,, \qquad \mathfrak{l}\in\mathcal{Z}_\ell^n . \tag{B.7}$$

Set $d(n,m,\ell,s) = \sum\limits_{\substack{i,j\geqslant 0\\ i+j<\ell\\ i-j=s}} \binom{m-1+i}{m-1}\binom{n-m-1+j}{n-m-1}$.

(B.8) Theorem.

$$\det M^{ell}(z) = \Xi \prod_{s=1-\ell}^{\ell-1}\prod_{m=1}^{n-1} \theta\Big(\eta^s\kappa^{-1}\prod_{1\leqslant l\leqslant m}\xi_l^{-1}\prod_{m<l\leqslant n}\xi_l\Big)^{d(n,m,\ell,s)} \times$$

$$\times \prod_{m=1}^{n} (\xi_m^{-1}z_m)^{(m-n)\binom{n+\ell-1}{n}} \prod_{s=0}^{\ell-1}\prod_{1\leqslant l<m\leqslant n} \theta(\eta^s\xi_l^{-1}\xi_m^{-1}z_l/z_m)^{\binom{n+\ell-s-2}{n-1}}$$

where

$$\Xi = \left[(p)_\infty^{1-n^2}\prod_{m=1}^{n-1}\left(\frac{\theta(\omega^m)}{\omega^m-1}\right)^{n-m}\right]^{\binom{n+\ell-1}{n}} .$$

Proof. For any $\mathfrak{l}\in\mathcal{Z}_\ell^n$ define a function $J_{\mathfrak{l}}(t_1,\dots,t_\ell)$ by the rule:

$$W_{\mathfrak{l}}(t,z) = J_{\mathfrak{l}}(t_1,\dots,t_\ell)\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{\theta(\xi_m^{-1}t_a/z_m)}\prod_{1\leqslant a<b\leqslant\ell}\frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)}\,, \tag{B.9}$$

so that $J_{\mathfrak{l}}(t,z) = \sum_{\mathfrak{m}\in\mathcal{Z}_\ell^n} M^{ell}_{\mathfrak{l}\mathfrak{m}}(z)\,\Theta_{\mathfrak{m}}(t,z)\,, \quad \mathfrak{l}\in\mathcal{Z}_\ell^n$. Introduce new variables $A,\ x_1,\dots,x_n\,,\ y_1,\dots,y_n$:

$$A = \kappa\prod_{m=1}^{n} z_m\,, \qquad x_m = \xi_m z_m\,, \qquad y_m = \xi_m^{-1} z_m\,, \qquad m = 1,\dots,n\,,$$

and for any $x,y\in\mathbb{C}^n$, $\mathfrak{l}\in\mathcal{Z}_\ell^n$, define $x\triangleright\mathfrak{l}$, $y\triangleleft\mathfrak{l}$ by formulae (A.11).

(B.10) Lemma. *$J_{\mathfrak{l}}(x\triangleright\mathfrak{m}) = 0$ unless $\mathfrak{l}\leqq\mathfrak{m}$. $J_{\mathfrak{l}}(y\triangleleft\mathfrak{m}) = 0$ unless $\mathfrak{l}\geqq\mathfrak{m}$. Moreover,*

$$J_{\mathfrak{l}}(x\triangleright\mathfrak{l}) = \prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\Bigl(\theta\bigl(\eta^{s+1}A^{-1}\prod_{1\leqslant l<m}\eta^{\mathfrak{l}_l}y_l\prod_{m\leqslant l\leqslant n}\eta^{-\mathfrak{l}_l}x_l\bigr)\times$$
$$\times\prod_{1\leqslant l<m}\theta(\eta^{-s}x_m/y_l)\prod_{m<l\leqslant n}\theta(\eta^{\mathfrak{l}_l-s}x_m/x_l)\Bigr)\,,$$

$$J_{\mathfrak{l}}(y\triangleleft\mathfrak{l}) = \prod_{1\leqslant l<m\leqslant n}\eta^{\mathfrak{l}_l\mathfrak{l}_m}\prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\Bigl(\theta\bigl(\eta^{-s-1}A^{-1}\prod_{1\leqslant l\leqslant m}\eta^{\mathfrak{l}_l}y_l\prod_{m<l\leqslant n}\eta^{-\mathfrak{l}_l}x_l\bigr)\times$$
$$\times\prod_{1\leqslant l<m}\theta(\eta^{s-\mathfrak{l}_l}y_m/y_l)\prod_{m<l\leqslant n}\theta(\eta^{s}y_m/x_l)\Bigr)\,.$$

The proof is straightforward.

The rest of the proof is similar to the proof of Theorem A.7. Using Lemma B.10 we calculate the determinants $\det\bigl[J_{\mathfrak{l}}(x\triangleright\mathfrak{m})\bigr]$, $\det\bigl[J_{\mathfrak{l}}(y\triangleleft\mathfrak{m})\bigr]$, cf. Lemma A.13. Then by the separation of variables we obtain the following formulae

$$\text{(B.11)}\quad \det\bigl[\Theta_{\mathfrak{l}}(x\triangleright\mathfrak{m})\bigr]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} = K\,\eta^{n(1-n)/2\cdot\binom{n+\ell-1}{n+1}}\prod_{m=1}^{n}(-x_m)^{(m-1)\binom{n+\ell-1}{n}}\times$$
$$\times\prod_{s=0}^{\ell-1}\theta\bigl(\eta^{-s}A^{-1}\prod_{m=1}^{n}x_m\bigr)^{\binom{n+s-1}{n-1}}\prod_{s=1-\ell}^{\ell-1}\ \prod_{1\leqslant l<m\leqslant n}\theta(\eta^{s}x_l/x_m)^{D(n,\ell,s)},$$

$$\det\bigl[\Theta_{\mathfrak{l}}(y\triangleleft\mathfrak{m})\bigr]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n} = K\,\eta^{n(n-1)/2\cdot\binom{n+\ell-1}{n+1}}\prod_{m=1}^{n}y_m^{(n-m)\binom{n+\ell-1}{n}}\times$$
$$\times\prod_{s=0}^{\ell-1}\theta\bigl(\eta^{s}A^{-1}\prod_{m=1}^{n}y_m\bigr)^{\binom{n+s-1}{n-1}}\prod_{s=1-\ell}^{\ell-1}\ \prod_{1\leqslant l<m\leqslant n}\theta(\eta^{s}y_m/y_l)^{D(n,\ell,s)}$$

as well as the required formula for $\det M^{ell}(z)$ with the constant Ξ replaced by K^{-1}. Here functions $\Theta_{\mathfrak{l}}$ are given by formula (B.2), K is a nonzero constant which does not depend on $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ and

$$D(n,\ell,s) \;=\; \sum_{\substack{r\in\mathbb{Z}_{\geqslant 0}\\ 2r\leqslant \ell-|s|-1}} \binom{n+\ell-|s|-2r-3}{n-2}.$$

To calculate the constant K we consider its dependence on η. Any of formulae (B.11) shows that K is a holomorphic function in η on $\mathbb{C}^\times$ and $K(p\eta)=K(\eta)$. Hence, K does not depend on η at all.

Let $\omega=\exp(2\pi i/n)$. Take $\eta=1$, $x_m=y_m=\omega^{m-1}x_1$, $m=1,\ldots,n$. In this case we have

$$\Theta_{\mathfrak{l}}(x\triangleright\mathfrak{m}) \;=\; Q_{\mathfrak{l}}(x\triangleright\mathfrak{m})\prod_{m=1}^{n}\left(x_1^{1-m}\vartheta_m(x_1)\right)^{\mathfrak{l}_m}$$

for any $\mathfrak{l},\mathfrak{m}\in\mathcal{Z}^n_\ell$, where the function $Q_{\mathfrak{l}}$ is defined by (A.6). Hence, comparing formulae (A.14) and (B.11) we find that $K=\Xi^{-1}$. In calculations we use Lemma B.12 and the equality

$$\sum_{s=1-\ell}^{\ell-1} D(n,\ell,s) \;=\; \binom{n+\ell-1}{n}$$

following from (G.1). The theorem is proved. □

(B.12) Lemma. *Let* $\alpha^n=p$. *Then* $\displaystyle\prod_{l=1}^{n}\prod_{m=1}^{n}\theta(\alpha^l\omega^m u) \;=\; (p)_\infty^{n^2-1}\,\theta(u^n)$.

The proof is straightforward using the definition $\theta(u)=(u)_\infty(p/u)_\infty(p)_\infty$.

Proof of Lemma 2.38. We will indicate explicitly the dependence of the elliptic weight functions on κ. Namely, the elliptic weight function $W_{\mathfrak{l}}[\kappa]$ is an element of the elliptic hypergeometric space of a fiber $\mathcal{F}_{ell}[\kappa,\sum_{m=1}^{n}\mathfrak{l}_m](z)$.

Under the assumptions of the lemma, the functions $W_{\mathfrak{l}}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m]$, $\mathfrak{l}\in\mathcal{Z}^n_\ell$, form a basis of the space $\mathcal{F}_{ell}(z)=\mathcal{F}_{ell}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m;\ell](z)$ and the functions $W_{\mathfrak{l}}[\eta^{-\ell}\prod_{m=1}^{n}\xi_m]$, $\mathfrak{l}\in\mathcal{Z}^n_{\ell-1}$, form a basis of the space $\mathcal{F}_{ell}[\eta^{-\ell}\prod_{m=1}^{n}\xi_m;\ell-1](z)$.

Since

$$W_{\mathfrak{l}}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m](t_1,\dots,t_\ell) =$$

$$= \frac{1}{\mathfrak{l}_2!\dots\mathfrak{l}_n!}\,\frac{\theta(\eta)}{\theta(\eta^{\mathfrak{l}_1})}\sum_{\sigma\in\mathbf{S}^\ell}\Big[\!\Big[W_{\mathfrak{l}-\mathfrak{e}(1)}[\eta^{-\ell}\prod_{m=1}^{n}\xi_m](t_2,\dots,t_\ell)\Big]\!\Big]_\sigma\,,$$

for any $\mathfrak{l}\in\mathcal{Z}^n_\ell$ such that $\mathfrak{l}_1>0$, the functions $W_{\mathfrak{l}}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m]$, $\mathfrak{l}_1>0$, $\mathfrak{l}\in\mathcal{Z}^n_\ell$, form a basis of the boundary subspace $\mathcal{Q}(z)\subset\mathcal{F}_{ell}(z)$, and the equivalence classes of the functions $W_{\mathfrak{l}}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m]$, $\mathfrak{l}_1=0$, $\mathfrak{l}\in\mathcal{Z}^n_\ell$, form a basis of the quotient space $\mathcal{F}_{ell}(z)/\mathcal{Q}(z)$.

Recall that the space $\mathcal{F}_{ell}[\alpha;l](z)$ is naturally isomorphic to the space $\mathcal{E}_l[A]$ of symmetric functions in variables $t_1,\dots,t_l$ which are holomorphic on $\mathbb{C}^{\times l}$ and have the property

$$f(t_1,\dots,pt_a,\dots,t_l) = A(-t_a)^{-n}f(t_1,\dots,t_l)\,,$$

where $A=\alpha\prod_{m=1}^{n}z_m$. Using this isomorphism we observe that the rank of the map

$$\mathcal{F}_{ell}[\eta^{-\ell}\prod_{m=1}^{n}\xi_m;z_1,\dots,z_n;\xi_1,\dots,\xi_n;\ell-1](z)\to$$

$$\to\mathcal{F}_{ell}[\eta^{1-\ell}\prod_{m=1}^{n}\xi_m;z_1,\dots,z_n;\xi_1,\dots,\xi_n;\ell](z)\,,$$

$$W(t_1,\dots,t_{\ell-1})\mapsto\sum_{\sigma\in\mathbf{S}^\ell}\Big[\!\Big[W(t_2,\dots,t_\ell)\Big]\!\Big]_\sigma\,,$$

does not depend on $\xi_1,\dots,\xi_n$. Therefore, $\dim\mathcal{Q}(z)=\binom{n+\ell-2}{n-1}$ and $\dim\mathcal{F}_{ell}(z)/\mathcal{Q}(z)=\binom{n+\ell-2}{n-2}$ provided that $\eta^r\neq p^s$ for any $r=2,\dots,\ell$, $s\in\mathbb{Z}$. Lemma 2.38 is proved. □

Proof of Lemma 2.39. The proof is similar to the proof of Lemma 2.38. □

Proof of Lemma 2.36. Let $f(t)=\sum_{\sigma\in\mathbf{S}^\ell}\Big[\!\Big[W(t_2,\dots,t_\ell)\Big]\!\Big]_\sigma$ be an element of the boundary subspace $\mathcal{Q}(z)$. One can see that the term of the sum corresponding to a permutation σ does not contribute to $\operatorname{Res} f(t)\big|_{t=x\triangleright\mathfrak{m}}$ unless

$$\sigma_{\mathfrak{m}^k+1}>\dots>\sigma_{\mathfrak{m}^{k+1}}>1\,,\qquad k=0,\dots,n-1\,.$$

Here $\mathfrak{m}^i = \sum_{j=1}^{i} \mathfrak{m}_j$. Since no permutation satisfies above conditions, the first claim of the lemma is proved.

Consider the determinant $\det\bigl[\operatorname{Res} G_{\mathfrak{l}}(t)\big|_{t=x\triangleright\mathfrak{m}}\bigr]_{\mathfrak{l},\mathfrak{m}\in\mathcal{Z}_\ell^n}$ where functions $G_{\mathfrak{l}}$ are given by (B.3). Under assumptions of the lemma the first of formulae (B.11) implies that the determinant as a function of κ is not zero for generic κ and it has a zero of multiplicity $\binom{n+\ell-2}{n-1}$ at $\kappa = \eta^{1-\ell}\prod_{m=1}^{n}\xi_m$. Hence, for $\kappa = \eta^{1-\ell}\prod_{m=1}^{n}\xi_m$ we have

$$\dim\{f\in\mathcal{F}_{ell}(z) \mid \operatorname{Res} f(t)\big|_{t=x\triangleright\mathfrak{m}} = 0,\ \mathfrak{m}\in\mathcal{Z}_\ell^n\} \leqslant \binom{n+\ell-2}{n-1}.$$

Let $\kappa = \eta^{1-\ell}\prod_{m=1}^{n}\xi_m$. We have proved already that

$$\mathcal{Q}(z)\subset\{f\in\mathcal{F}_{ell}(z) \mid \operatorname{Res} f(t)\big|_{t=x\triangleright\mathfrak{m}} = 0,\ \mathfrak{m}\in\mathcal{Z}_\ell^n\}.$$

Since $\dim\mathcal{Q}(z) = \binom{n+\ell-2}{n-1}$ by Lemma 2.38, the second claim of Lemma 2.36 is proved. □

Proof of Lemma 2.37. The proof is similar to the proof of Lemma 2.36. □

C. The Shapovalov pairings of the hypergeometric spaces of fibers

In this appendix we define pairings of the hypergeometric spaces of fibers which provides a geometric interpretation for the coefficients $c_{\mathfrak{l}}^{\tau}$, cf. (4.9), used in the definition of the tensor coordinates on the elliptic hypergeometric space of a fiber, see formula (C.5).

In this appendix we always suppose that assumptions (2.13) – (2.15) hold. Set

(C.1) $$\Omega_{\mathit{ell}}(t_1,\dots,t_\ell) = \\ = \prod_{a=1}^{\ell} t_a^{-1} \prod_{m=1}^{n}\prod_{a=1}^{\ell} \frac{\theta(\xi_m^{-1}t_a/z_m)}{\theta(\xi_m t_a/z_m)} \prod_{1\leqslant a<b\leqslant \ell} \frac{\theta(\eta t_a/t_b)}{\eta\theta(\eta^{-1}t_a/t_b)}\,.$$

Let $x \triangleright \mathfrak{l}$ and $y \triangleleft \mathfrak{l}$, $\mathfrak{l} \in \mathcal{Z}_\ell^n$, be the points defined by (2.35). Recall that we define the multiple residue $\operatorname{Res} f(t)\big|_{t=t^*}$ by formula (2.34).

Consider the elliptic hypergeometric spaces of a fiber $\mathcal{F}_{\mathit{ell}}(z) = \mathcal{F}_{\mathit{ell}}[\kappa;\ell](z)$ and $\widetilde{\mathcal{F}}_{\mathit{ell}}(z) = \mathcal{F}_{\mathit{ell}}[\kappa^{-1};\ell](z)$. For any functions $W \in \mathcal{F}_{\mathit{ell}}(z)A$ and $\widetilde{W} \in \widetilde{\mathcal{F}}_{\mathit{ell}}(z)$ set

(C.2) $$S_{\mathit{ell}}(W,\widetilde{W}) = \sum_{\mathfrak{m}\in\mathcal{Z}_\ell^n} \operatorname{Res}\big(\Omega_{\mathit{ell}}(t)\,W(t)\,\widetilde{W}(t)\big)\big|_{t=x\triangleright\mathfrak{m}}\,.$$

(C.3) Lemma. *For any functions $W \in \mathcal{F}_{\mathit{ell}}(z)$ and $\widetilde{W} \in \widetilde{\mathcal{F}}_{\mathit{ell}}(z)$ we have that*

$$S_{\mathit{ell}}(W,\widetilde{W}) = (-1)^\ell \sum_{\mathfrak{m}\in\mathcal{Z}_\ell^n} \operatorname{Res}\big(\Omega_{\mathit{ell}}(t)\,W(t)\,\widetilde{W}(t)\big)\big|_{t=y\triangleleft\mathfrak{m}}\,.$$

The statement follows from Lemma C.11.

The pairing $S_{\mathit{ell}} : \mathcal{F}_{\mathit{ell}}(z) \otimes \widetilde{\mathcal{F}}_{\mathit{ell}}(z) \to \mathbb{C}$ is called the *Shapovalov pairing* of the elliptic hypergeometric spaces of a fiber.

We will indicate explicitly the dependence of the elliptic weight functions on κ. Namely, the elliptic weight function $W_{\mathfrak{l}}^{\tau}[\kappa]$ is an element of the elliptic hypergeometric space of a fiber $\mathcal{F}_{\mathit{ell}}[\kappa;\ell](z)$.

(C.4) Theorem. *Let $\omega \in \mathbf{S}^n$ be the permutation of the maximal length. Let (2.13) – (2.15) hold. Then for any permutation $\tau \in \mathbf{S}^n$ and any $\mathfrak{l},\mathfrak{m} \in \mathcal{Z}_\ell^n$ we have that*

$$S_{\mathit{ell}}\big(W_{\mathfrak{l}}^{\tau}[\kappa], W_{\mathfrak{m}}^{\tau\omega}[\kappa^{-1}]\big) = \delta_{\mathfrak{l}\mathfrak{m}}\,N_{\mathfrak{l}}^{\tau}\,.$$

Here $\delta_{\mathfrak{l}\mathfrak{m}}$ *is the Kronecker symbol and* $N^\tau_{\mathfrak{l}} = N_{{}^\tau\mathfrak{l}}(\xi_{\tau_1}, \dots, \xi_{\tau_n})$ *where* ${}^\tau\mathfrak{l} = (\mathfrak{l}_{\tau_1}, \dots, \mathfrak{l}_{\tau_n})$ *and*

$$N_{\mathfrak{l}}(\xi_1, \dots, \xi_n) = \prod_{m=1}^{n} \prod_{s=1}^{\mathfrak{l}_m} \Big(\frac{\theta(\eta)}{\theta'(1)\,\theta(\eta^s)\,\theta(\eta^{1-s}\xi_m^2)} \times$$

$$\times\, \theta\big(\eta^s \kappa^{-1} \prod_{1 \leqslant l < m} \eta^{\mathfrak{l}_l} \xi_l^{-1} \prod_{m \leqslant l \leqslant n} \eta^{-\mathfrak{l}_l} \xi_l\big)\, \theta\big(\eta^s \kappa \prod_{1 \leqslant l \leqslant m} \eta^{-\mathfrak{l}_l} \xi_l \prod_{m < l \leqslant n} \eta^{\mathfrak{l}_l} \xi_l^{-1}\big)\Big),$$

$$\theta'(1) = \frac{d}{du}\theta(u)\big|_{u=1} = -(p)^3_\infty .$$

Proof. For $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}^n_\ell$ say that $\mathfrak{l} \leqq \mathfrak{m}$ if $\sum_{i=1}^{m} \mathfrak{l}_i \leqslant \sum_{i=1}^{m} \mathfrak{m}_i$ for any $m = 1, \dots, n-1$. Formulae (2.31) and (2.40) for the elliptic weight functions imply that

$$\operatorname{Res}\big(\Omega_{ell}(t)\, W^\tau_{\mathfrak{l}}[\kappa](t)\, W^{\tau\omega}_{\mathfrak{m}}[\kappa^{-1}](t)\big)\big|_{t=x\triangleright\mathfrak{n}} = 0,$$

$$\operatorname{Res}\big(\Omega_{ell}(t)\, W^\tau_{\mathfrak{m}}[\kappa](t)\, W^{\tau\omega}_{\mathfrak{l}}[\kappa^{-1}](t)\big)\big|_{t=y\triangleleft\mathfrak{n}} = 0,$$

unless $\mathfrak{l} \leqq \mathfrak{n} \leqq \mathfrak{m}$, and

$$\operatorname{Res}\big(\Omega_{ell}(t)\, W^\tau_{\mathfrak{l}}[\kappa](t)\, W^{\tau\omega}_{\mathfrak{l}}[\kappa^{-1}](t)\big)\big|_{t=x\triangleright\mathfrak{l}} = N^\tau_{\mathfrak{l}} .$$

Therefore, by formula (C.2) we have that $S_{ell}\big(W^\tau_{\mathfrak{l}}[\kappa], W^{\tau\omega}_{\mathfrak{m}}[\kappa^{-1}]\big) = 0$, unless $\mathfrak{l} \leqq \mathfrak{m}$ and

$$S_{ell}\big(W^\tau_{\mathfrak{l}}[\kappa], W^{\tau\omega}_{\mathfrak{l}}[\kappa^{-1}]\big) = N^\tau_{\mathfrak{l}} .$$

Similarly, by formula (C.3) we have that $S_{ell}\big(W^\tau_{\mathfrak{l}}[\kappa], W^{\tau\omega}_{\mathfrak{m}}[\kappa^{-1}]\big) = 0$, unless $\mathfrak{l} \geqq \mathfrak{m}$. The theorem is proved. □

Remark. The coefficients $c_{\mathfrak{l}}(\xi_1, \dots, \xi_n)$ defined by formula (4.9) are inverse to the coefficients $N_{\mathfrak{l}}(\xi_1, \dots, \xi_n)$ defined in Theorem C.4 up to a common factor. Namely,

$$c_{\mathfrak{l}}(\xi_1, \dots, \xi_n) = \frac{\eta^{\ell(1-\ell)/2}\kappa^\ell}{(p)^{3\ell}_\infty\, N_{\mathfrak{l}}(\xi_1, \dots, \xi_n)} \prod_{m=1}^{n} \xi_m^\ell . \tag{C.5}$$

The Shapovalov pairing of the trigonometric hypergeometric spaces of a fiber is defined similarly to the Shapovalov pairing of the elliptic hypergeometric spaces of a fiber. Assumptions (2.13) – (2.15) can be replaced by weaker assumptions

$$\text{(C.6)} \qquad \eta^r \neq 1\,, \qquad r = 1, \dots, \ell\,,$$

$$\xi_m^2 \neq \eta^r\,, \qquad m = 1, \dots, n\,, \qquad r = 1-\ell, \dots, \ell-1\,,$$

$$\xi_l^{\pm 1} \xi_m^{\pm 1} z_l / z_m \neq \eta^r\,, \qquad l, m = 1, \dots, n\,, \quad l \neq m\,, \qquad r = 1-\ell, \dots, \ell-1\,.$$

Let

$$\text{(C.7)} \qquad \Omega(t_1, \dots, t_\ell) \,=\, \prod_{a=1}^{\ell} t_a^{-2} \prod_{m=1}^{n} \prod_{a=1}^{\ell} \frac{t_a - \xi_m z_m}{\xi_m t_a - z_m} \prod_{1 \leqslant a < b \leqslant \ell} \frac{\eta t_a - t_b}{t_a - \eta t_b}\,.$$

For any functions $w, \tilde{w}$ in the trigonometric hypergeometric space of a fiber $\mathcal{F}(z)$ set

$$S(w, \tilde{w}) \,=\, \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \operatorname{Res}\bigl(\Omega(t)\, w(t)\, \tilde{w}(t)\bigr)\big|_{t = x \triangleright \mathfrak{m}}\,.$$

(C.8) Lemma. *For any functions $w, \tilde{w} \in \mathcal{F}(z)$ we have that*

$$S(w, \tilde{w}) \,=\, (-1)^\ell \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \operatorname{Res}\bigl(\Omega(t)\, w(t)\, \tilde{w}(t)\bigr)\big|_{t = y \triangleleft \mathfrak{m}}\,.$$

Proof. Due to Lemma 2.22 it suffices to prove the statement if both functions $w, \tilde{w}$ are trigonometric weight functions. Moreover, since $S(w_{\mathfrak{l}}, w_{\mathfrak{m}})$ depends analytically on parameters η, $\xi_1, \dots, \xi_n$, $z_1, \dots, z_n$, it is enough to prove the lemma under the assumptions: $|\eta| > 1$ and $|z_m| = 1$, $|\xi_m| < 1$, $m = 1, \dots, n$.

Consider the integral $\displaystyle\int_{\mathbb{T}^\ell} \Omega(t)\, w(t)\, \tilde{w}(t)\; d^\ell t$ where $\mathbb{T}^\ell = \{ t \in \mathbb{C}^\ell \mid |t_1| = 1, \dots, |t_\ell| = 1 \}$. Similarly to Theorems F.1 and F.2 one can show that under the above assumptions

$$\int_{\mathbb{T}^\ell} \Omega(t)\, w(t)\, \tilde{w}(t)\; d^\ell t \,=\, (2\pi i)^\ell\, \ell! \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \operatorname{Res}\bigl(\Omega(t)\, w(t)\, \tilde{w}(t)\bigr)\big|_{t = x \triangleright \mathfrak{m}}$$

and

$$\int_{\mathbb{T}^\ell} \Omega(t)\, w(t)\, \tilde{w}(t)\; d^\ell t \,=\, (-2\pi i)^\ell\, \ell! \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \operatorname{Res}\bigl(\Omega(t)\, w(t)\, \tilde{w}(t)\bigr)\big|_{t = y \triangleleft \mathfrak{m}}\,.$$

The lemma is proved. □

(C.9) Theorem. *Let $\omega \in \mathbf{S}^n$ be the permutation of the maximal length. Let assumptions* (C.6) *hold. Then for any permutation $\tau \in \mathbf{S}^n$ and any $\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n$ we have that*

$$S(w_{\mathfrak{l}}^{\tau}, w_{\mathfrak{m}}^{\tau\omega}) = \delta_{\mathfrak{l}\mathfrak{m}} \prod_{m=1}^{n} \prod_{s=1}^{\mathfrak{l}_m} \frac{1-\eta}{z_m (1-\eta^s)(\xi_m^2 - \eta^{s-1})}$$

where $\delta_{\mathfrak{l}\mathfrak{m}}$ is the Kronecker symbol.

The proof is similar to the proof of Theorem C.4.

The pairing $S : \mathcal{F}(z) \otimes \mathcal{F}(z) \to \mathbb{C}$ is called the *Shapovalov pairing* of the trigonometric hypergeometric spaces of a fiber.

The trigonometric Shapovalov pairing S can be considered as a degeneration either of the elliptic Shapovalov pairing S_{ell} or the hypergeometric pairing $I : \mathcal{F}_{ell}(z) \otimes \mathcal{F}(z) \to \mathbb{C}$. Namely, let $p \to 0$ and after that $\kappa \to \infty$. Then in this limit we have that

$$(u)_\infty \to 1-u\,, \qquad \theta(u) \to 1-u\,, \qquad (p)_\infty \to 1\,,$$

the elliptic weight functions tend to the respective trigonometric weight functions:

$$W_{\mathfrak{l}}[\kappa](t) \to w_{\mathfrak{l}}(t)\,(-1)^\ell \prod_{a=1}^{\ell} t_a^{-1} \prod_{m=1}^{n} z_m^{\mathfrak{l}_m} \prod_{1\leqslant l<m\leqslant n} \xi_l^{\mathfrak{l}_m}\,,$$

$$\kappa^{-\ell}\, W_{\mathfrak{l}}[\kappa^{-1}](t) \to w_{\mathfrak{l}}(t) \prod_{1\leqslant l<m\leqslant n} \xi_m^{\mathfrak{l}_l}\,,$$

the function Ω_{ell} defined in (C.1) and the short phase function Φ defined in (5.2) turn into the function Ω given by (C.7):

$$\Omega_{ell}(t) \to \Omega(t) \prod_{a=1}^{\ell} t_a \prod_{m=1}^{n} \xi_m^{-\ell}\,, \qquad \Phi(t) \to \Omega(t)\, \eta^{\ell(\ell-1)/2} \prod_{a=1}^{\ell} t_a^2 \prod_{m=1}^{n} \xi_m^{-\ell}\,,$$

and both the elliptic Shapovalov pairing and the hypergeometric pairing become the trigonometric Shapovalov pairing:

$$\kappa^{-\ell}\, S_{ell}\big(W_{\mathfrak{l}}[\kappa], W_{\mathfrak{m}}[\kappa^{-1}]\big) \to S(w_{\mathfrak{l}}, w_{\mathfrak{m}})\,(-1)^\ell \prod_{m=1}^{n} z_m^{\mathfrak{l}_m} \prod_{1\leqslant l<m\leqslant n} \xi_m^{\mathfrak{m}_l-\mathfrak{l}_l}\,,$$

$$I\big(W_{\mathfrak{l}}[\kappa], w_{\mathfrak{m}}\big) \to S(w_{\mathfrak{l}}, w_{\mathfrak{m}})\,(-1)^\ell \eta^{\ell(\ell-1)/2} \prod_{m=1}^{n} z_m^{\mathfrak{l}_m} \prod_{1\leqslant l<m\leqslant n} \xi_m^{-\mathfrak{l}_l}\,.$$

In the rest of the appendix we formulate and prove Lemma C.11 which implies Lemma C.3.

Let $u_1, \dots, u_k$ be nonzero complex numbers. Consider a function $F(t_1, \dots, t_\ell)$ which is quasiperiodic with respect to each of the variables $t_1, \dots, t_\ell$:

$$F(t_1, \dots, pt_a, \dots, t_\ell) = p^{-1} F(t_1, \dots, t_\ell),$$

and which has the form

$$\text{(C.10)} \qquad f(t_1, \dots, t_\ell) \prod_{j=1}^{k} \prod_{a=1}^{\ell} \frac{1}{\theta(t_k/u_j)} \prod_{a=1}^{\ell} \prod_{\substack{b=1 \\ b \neq a}}^{\ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)}$$

where $f(t_1, \dots, t_\ell)$ is a symmetric holomorphic function on $\mathbb{C}^{\times \ell}$. We call the points $u_1, \dots, u_k$ the "root" singularities of the function F.

For any $\mathfrak{l} \in \mathbb{Z}^k_{\geqslant 0}$, $\sum_{j=1}^{k} \mathfrak{l}_m = \ell$, introduce the points $u \triangleright \mathfrak{l}$, $u \triangleleft \mathfrak{l} \in \mathbb{C}^{\times \ell}$ by the rule:

$$u \triangleright \mathfrak{l} = (\eta^{1-\mathfrak{l}_1} u_1, \eta^{2-\mathfrak{l}_1} u_1, \dots, u_1, \eta^{1-\mathfrak{l}_2} u_2, \dots, u_2, \dots, \eta^{1-\mathfrak{l}_k} u_k, \dots, u_k),$$

$$u \triangleleft \mathfrak{l} = (\eta^{\mathfrak{l}_1 - 1} u_1, \eta^{\mathfrak{l}_1 - 2} u_1, \dots, u_1, \eta^{\mathfrak{l}_2 - 1} u_2, \dots, u_2, \dots, \eta^{\mathfrak{l}_k - 1} u_k, \dots, u_k),$$

cf. (2.35). For any $i = 1, \dots, k$ let

$$Z_i^+ = \{ \mathfrak{l} \in \mathbb{Z}^k_{\geqslant 0} \mid \mathfrak{l}_{i+1} = \dots = \mathfrak{l}_k = 0, \ \sum_{j=1}^{k} \mathfrak{l}_j = \ell \}$$

and

$$Z_i^- = \{ \mathfrak{l} \in \mathbb{Z}^k_{\geqslant 0} \mid \mathfrak{l}_1 = \dots = \mathfrak{l}_i = 0, \ \sum_{j=1}^{k} \mathfrak{l}_j = \ell \}.$$

In particular, $Z_0^+ = Z_k^- = \varnothing$ and $Z_k^+ = Z_0^- = \{ \mathfrak{l} \in \mathbb{Z}^k_{\geqslant 0} \mid \sum_{j=1}^{k} \mathfrak{l}_j = \ell \}$.

(C.11) Lemma. *For any $i = 1, \dots, k$ we have that*

$$\sum_{\mathfrak{m} \in Z_i^+} \operatorname{Res} F(t) \Big|_{t = u \triangleright \mathfrak{m}} = (-1)^\ell \sum_{\mathfrak{m} \in Z_i^-} \operatorname{Res} F(t) \Big|_{t = u \triangleleft \mathfrak{m}},$$

provided that the singularity hyperplanes

$$t_a = u_j, \qquad a = 1, \dots, \ell, \quad j = 1, \dots, k,$$

$$t_a = \eta t_b, \qquad a, b = 1, \dots, \ell, \quad a \neq b,$$

have no multiple intersections at the points $u \triangleright \mathfrak{m}$, $\mathfrak{m} \in Z_i^+$, *and at the points* $u \triangleleft \mathfrak{m}$, $\mathfrak{m} \in Z_i^-$.

Proof. We prove the lemma by induction with respect to ℓ. For $\ell = 1$ the statement is standard, which provides the base of induction.

Let $\ell > 1$. To avoid complicated notations we first give the idea of the proof in the general case and then explain technical details for the example $\ell = 3$, $k = 3$, $i = 1$.

The function F considered as a function of $t_2, \dots, t_\ell$ for a fixed t_1 has a form similar to (C.10) with the "root" singularities at the points ηt_1, $\eta^{-1} t_1$, $u_1, \dots, u_k$. Using the induction assumption we apply the lemma in this case taking in the left hand side of the formula the sum of residues corresponding to the "root" singularities $u_1, \dots, u_i$, $\eta^{-1} t_1$ and taking in the right hand side of the formula the sum of residues corresponding to the "root" singularities $u_{i+1}, \dots, u_k$, ηt_1. To complete the proof we apply the lemma to the function of t_1 given by the sum of residues in the left hand side of the formula obtained at the previous step.

Consider the example $\ell = 3$, $k = 3$, $i = 1$. At the first step we get

$$\operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_1}\big)\big|_{t_2=\eta^{-1}u_1} + \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_1}\big)\big|_{t_2=\eta^{-1}t_1} + {}$$
$$+ \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=\eta^{-1}t_1}\big)\big|_{t_2=\eta^{-2}t_1} =$$
$$= \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_2}\big)\big|_{t_2=\eta u_2} + \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_3}\big)\big|_{t_2=\eta u_3} + {}$$
$$+ \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_3}\big)\big|_{t_2=u_2} + \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_2}\big)\big|_{t_2=\eta t_1} + {}$$
$$+ \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_3}\big)\big|_{t_2=\eta t_1} + \operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=\eta t_1}\big)\big|_{t_2=\eta^2 t_1}.$$

Denote by $G(t_1)$ the sum of residues in the left hand side of the above formula, or equivalently, the sum of residues in the right hand side of the above formula. The function G can have poles only at the following points: $p^s \eta^{-r} u_1$, $p^s \eta^r u_2$, $p^s \eta^r u_3$, $r = 0, 1, 2$, $s \in \mathbb{Z}$, because at any other point at least one of the defining expressions for the function G has no pole. Hence, we have that

$$-\sum_{s=0}^{2} \operatorname{Res} G(t_1)\big|_{t_1=\eta^{-s}u_1} = \sum_{s=0}^{2} \big(\operatorname{Res} G(t_1)\big|_{t_1=\eta^s u_2} + \operatorname{Res} G(t_1)\big|_{t_1=\eta^s u_3}\big).$$

Substituting respectively the left (right) hand side definition of the function G into the left (right) hand side of the above formula, we obtain that

$$-\begin{Bmatrix} t_3 = u_1 \\ t_2 = \eta^{-1}u_1 \\ t_1 = \eta^{-2}u_1 \end{Bmatrix} - \begin{Bmatrix} t_3 = u_1 \\ t_2 = \eta^{-1}t_1 \\ t_1 = \eta^{-1}u_1 \end{Bmatrix} - \begin{Bmatrix} t_3 = \eta^{-1}t_1 \\ t_2 = \eta^{-2}t_1 \\ t_1 = u_1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} t_3 = u_2 \\ t_2 = \eta u_2 \\ t_1 = \eta^2 u_2 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_2 \\ t_2 = \eta u_2 \\ t_1 = u_3 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_3 \\ t_2 = \eta u_3 \\ t_1 = u_2 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_3 \\ t_2 = \eta u_3 \\ t_1 = \eta^2 u_3 \end{Bmatrix} +$$

$$+ \begin{Bmatrix} t_3 = u_3 \\ t_2 = u_2 \\ t_1 = \eta u_2 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_3 \\ t_2 = u_2 \\ t_1 = \eta u_3 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_2 \\ t_2 = \eta t_1 \\ t_1 = \eta u_2 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_2 \\ t_2 = \eta t_1 \\ t_1 = u_3 \end{Bmatrix} +$$

$$+ \begin{Bmatrix} t_3 = u_3 \\ t_2 = \eta t_1 \\ t_1 = u_2 \end{Bmatrix} + \begin{Bmatrix} t_3 = u_3 \\ t_2 = \eta t_1 \\ t_1 = \eta u_3 \end{Bmatrix} + \begin{Bmatrix} t_3 = \eta t_1 \\ t_2 = \eta^2 t_1 \\ t_1 = u_2 \end{Bmatrix} + \begin{Bmatrix} t_3 = \eta t_1 \\ t_2 = \eta^2 t_1 \\ t_1 = u_3 \end{Bmatrix}$$

where we use the notation

$$\begin{Bmatrix} t_3 = c \\ t_2 = b \\ t_1 = a \end{Bmatrix} = \operatorname{Res}\big(\operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=c}\big)\big|_{t_2=b}\big)\big|_{t_1=a}.$$

To complete the proof we have to transform the multiple residues to the form (2.34). This is straightforward under the assumptions of the lemma because at each step we have to calculate residues only at simple poles and the function F is symmetric. The transformation can be done term by term, for instance,

$$\operatorname{Res}\big(\operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_2}\big)\big|_{t_2=\eta t_1}\big)\big|_{t_1=u_3} =$$

$$= \operatorname{Res}\big(\operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_2}\big)\big|_{t_1=u_3}\big)\big|_{t_2=\eta u_3} =$$

$$= \operatorname{Res}\big(\operatorname{Res}\big(\operatorname{Res} F(t)\big|_{t_3=u_3}\big)\big|_{t_2=\eta u_3}\big)\big|_{t_1=u_2}.$$

The lemma is proved. □

D. The q-Selberg integral

In this appendix we give proofs of formulae (5.13) and (D.9). The last formula is equivalent to the formula for the q-Selberg integral, see [AK, Theorem 3.2].

Denote by $F(t_1,\dots,t_\ell;a,b,c)$ the integrand of the integral (5.13):

$$F(t_1,\dots,t_\ell;a,b,c) = \prod_{k=1}^{\ell} \frac{\theta(ct_k)}{t_k\,(at_k)_\infty\,(b/t_k)_\infty} \prod_{j=1}^{\ell}\prod_{\substack{k=1\\k\neq j}}^{\ell} \frac{(t_j/t_k)_\infty}{(xt_j/t_k)_\infty}\,. \tag{D.1}$$

***Proof of formula* (5.13).** Consider the integral in the left hand side of (5.13) as a function of c and denote it by $S(c)$. Let $f(c)$ be the ratio of $S(c)$ and the right hand side of formula (5.13).

The function $S(c)$ satisfies a difference equation

$$S(pc) = S(c)\prod_{s=0}^{\ell-1}\frac{1-x^s a/c}{1-x^s bc}\,, \tag{D.2}$$

cf. Corollary D.6. The right hand side of formula (5.13) solves the same difference equation with respect to c. Therefore, $f(c)$ is a p-periodic function: $f(pc)=f(c)$.

$S(c)$ is a holomorphic function of c on $\mathbb{C}^\times$, since the integrand $F(t_1,\dots,t_\ell;a,b,c)$ is a holomorphic function of c on $\mathbb{C}^\times$ and the integration contour is compact. So, the function $f(c)$ is regular in the annulus $|pa|<|c|<|b|^{-1}$ of width greater than $|p|$. Hence, $f(c)$ is a holomorphic function of c on $\mathbb{C}^\times$, and therefore, $f(c)$ is a constant function.

We will show that $f=1$ by induction with respect to ℓ. We will indicate the dependence of ℓ explicitly, that is $S(c)=S_\ell(c)$ and $f=f_\ell$. The trivial case $\ell=0$ provides the base of the induction.

Formulae (2.30) and (B.5) after a suitable change of notation give the following identity:

$$\prod_{k=1}^{\ell}\theta(ct_k) = \prod_{s=1}^{\ell}\frac{\theta(x)}{\theta(x^s)}\sum_{\sigma\in\mathbf{S}^\ell}\Bigl(\prod_{k=1}^{\ell}\theta(x^{\ell-2k+1}ct_{\sigma_k})\prod_{1\leqslant j<k\leqslant\ell}\frac{\theta(xt_{\sigma_k}/t_{\sigma_j})}{\theta(t_{\sigma_k}/t_{\sigma_j})}\Bigr).$$

Replace the product $\prod_{k=1}^{\ell}\theta(ct_k)$ in the integrand $F(t_1,\dots,t_\ell)$ by the right hand side of the identity. Since the rest of the integrand is a symmetric function in $t_1,\dots,t_\ell$ and the integration contour is invariant with respect to permutations of the variables $t_1, \dots, t_\ell$, we can keep in the sum only the term corresponding to the identity permutation and then multiply the result of the integration by $\ell!$. Therefore, we have that

$$S_\ell(c) = \ell! \prod_{s=1}^{\ell} \frac{\theta(x)}{\theta(x^s)} \times$$
$$\times \int_{\mathbb{T}^\ell} \prod_{k=1}^{\ell} \frac{\theta(x^{\ell-2k+1} c t_k)}{t_k\,(a t_k)_\infty\,(b/t_k)_\infty} \prod_{1\leqslant j<k\leqslant \ell} \frac{(1-t_j/t_k)\,(p x^{-1} t_j/t_k)_\infty}{(x t_j/t_k)_\infty}\, d^\ell t\,.$$

Let $c = p x^{1-\ell} b^{-1}$. Then the integrand of the above integral is regular in the punctured disk $0<|t_1|\leqslant 1$ and has a simple pole at $t_1=0$. Perfoming the integration with respect to t_1 we obtain

$$S_\ell(p x^{1-\ell} b^{-1}) = 2\pi i\,\ell\,(p)_\infty\, \frac{\theta(x)}{\theta(x^\ell)}\, S_{\ell-1}(p x^{-\ell} b^{-1})\,.$$

The right hand side of formula (5.13) satisfies the same recurrence relation with respect to ℓ. Hence, $f_\ell = f_{\ell-1}$, which completes the proof. □

(D.3) Lemma. *Let* $X_k = \int_{\mathbb{T}^\ell} t_1 \dots t_k\, F(t_1, \dots, t_\ell; a, b, c)\, d^\ell t, \quad k = 0, \dots, \ell\,.$

The following recurrence relation holds:

$$X_k = X_{k-1}\, \frac{k\,(1-x^{\ell-k+1})\,(p - x^{k-1} b c)}{(\ell-k+1)\,(1-x^k)\,(p x^{\ell-k} a - c)}\,, \qquad k = 1, \dots, \ell\,.$$

Proof. Consider the integrals

$$\text{(D.4)} \qquad \widetilde{X}_k = \int_{\mathbb{T}^\ell} (1 - a t_1)\, t_2, \dots, t_k \prod_{j=2}^{\ell} \frac{x t_1 - t_j}{t_1 - t_j}\, F(t_1, \dots, t_\ell; a, b, c)\, d^\ell t\,,$$

$k = 1, \dots, \ell$. Notice that the integrands are regular on $\mathbb{T}^\ell$ since $F(t_1, \dots, t_\ell; a, b, c)$ vanishes at all diagonals $t_i = t_j$.

Replacing t_1 by t_1/p in the integrand and using the explicit formula for $F(t_1, \dots, t_\ell; a, b, c)$ we obtain that

$$\widetilde{X}_k = \int_{\mathbb{T}_p \times \mathbb{T}^{\ell-1}} p^{-1} c\,(b - t_1)\, t_2, \dots, t_k \prod_{j=2}^{\ell} \frac{t_1 - x t_j}{t_1 - t_j}\, F(t_1, \dots, t_\ell; a, b, c)\, d^\ell t$$

where $\mathbb{T}_p = \{t_1 \in \mathbb{C} \mid |t_1| = p\}$. The integrand considered as a function of t_1 is regular in the annulus $p \leqslant |t_1| \leqslant 1$. Therefore, we can replace the integration contour $\mathbb{T}_p \times \mathbb{T}^{\ell-1}$ in the above integral by $\mathbb{T}^\ell$ without changing the integral:

$$\text{(D.5)} \qquad \widetilde{X}_k = \int\limits_{\mathbb{T}^\ell} p^{-1} c(b-t_1)\, t_2, \dots, t_k \prod_{j=2}^{\ell} \frac{t_1 - x t_j}{t_1 - t_j}\, F(t_1, \dots, t_\ell; a, b, c)\, d^\ell t\,.$$

Since the integration contour $\mathbb{T}^\ell$ is invariant with respect to permutations of the variables $t_1, \dots, t_\ell$, we can symmetrize the integrands in the formulae for $X_0, \dots, X_\ell$ and $\widetilde{X}_1, \dots, \widetilde{X}_\ell$. Then formula (D.4) and the first two identities (D.7) imply that

$$(1-x)\widetilde{X}_k = \frac{1 - x^{\ell-k+1}}{\ell - k + 1} X_{k-1} - \frac{x^{\ell-k} - x^\ell}{k} a X_k\,, \qquad k = 1, \dots, \ell\,,$$

and formula (D.5) and the last two identities (D.7) imply that

$$(1-x)\widetilde{X}_k = \frac{x^{k-1} - x^\ell}{\ell - k + 1} p^{-1} bc X_{k-1} - \frac{1 - x^k}{k} p^{-1} c X_k\,, \qquad k = 1, \dots, \ell\,,$$

because the function $F(t_1, \dots, t_\ell; a, b, c)$ is symmetric in the variables $t_1, \dots, t_\ell$. The rest of the proof is obvious. The lemma is proved. □

(D.6) Corollary. *The difference equation* (D.2) *holds.*

Proof. The statement is clear since $S(c) = X_0$ and $S(c/p) = (-c/p)^\ell X_\ell$. □

(D.7) Lemma. *The following identities hold*:

$$k(1-x) \sum_{\sigma \in \mathbf{S}^\ell} \Big(t_{\sigma_1} \dots t_{\sigma_k} \prod_{j=2}^{\ell} \frac{x t_{\sigma_1} - t_{\sigma_j}}{t_{\sigma_1} - t_{\sigma_j}} \Big) = (x^{\ell-k} - x^\ell) \sum_{\sigma \in \mathbf{S}^\ell} t_{\sigma_1} \dots t_{\sigma_k}\,,$$

$$(\ell - k)(1-x) \sum_{\sigma \in \mathbf{S}^\ell} \Big(t_{\sigma_1} \dots t_{\sigma_k} \prod_{j=1}^{\ell-1} \frac{x t_{\sigma_1} - t_{\sigma_j}}{t_{\sigma_1} - t_{\sigma_j}} \Big) = (1 - x^{\ell-k}) \sum_{\sigma \in \mathbf{S}^\ell} t_{\sigma_1} \dots t_{\sigma_k}\,,$$

$$k(1-x) \sum_{\sigma \in \mathbf{S}^\ell} \Big(t_{\sigma_1} \dots t_{\sigma_k} \prod_{j=2}^{\ell} \frac{t_{\sigma_1} - x t_{\sigma_j}}{t_{\sigma_1} - t_{\sigma_j}} \Big) = (1 - x^k) \sum_{\sigma \in \mathbf{S}^\ell} t_{\sigma_1} \dots t_{\sigma_k}\,,$$

$$(\ell - k)(1-x) \sum_{\sigma \in \mathbf{S}^\ell} \Big(t_{\sigma_1} \dots t_{\sigma_k} \prod_{j=1}^{\ell-1} \frac{t_{\sigma_1} - x t_{\sigma_j}}{t_{\sigma_1} - t_{\sigma_j}} \Big) = (x^k - x^\ell) \sum_{\sigma \in \mathbf{S}^\ell} t_{\sigma_1} \dots t_{\sigma_k}\,.$$

Proof. The left hand sides of the above formulae are homogeneous symmetric polynomials in the variables $t_1, \dots, t_\ell$ of the homogeneous degree k and of degree one in each of the variables $t_1, \dots, t_\ell$. Hence, they are proportional to $\sum_{\sigma\in\mathbf{S}^\ell} t_{\sigma_1}\dots t_{\sigma_k}$.

Restrict the polynomials in question to the line $t_j = x^{j-1}t_1$, $j = 1, \dots, \ell$, and use the following identity

$$\text{(D.8)} \qquad \sum_{0\leqslant r_1<\dots<r_k\leqslant \ell} x^{r_1+\dots+r_k} = \prod_{s=0}^{k-1} \frac{x^s - x^\ell}{1-x^{s+1}}\,.$$

Then the calculation of the proportionality coefficients is straighforward. Identity (D.8) can be proved by induction with respect to ℓ. □

Let

$$S(t_1, \dots, t_\ell) = \prod_{k=1}^{\ell} \frac{(pt_k)_\infty}{(\alpha t_k)_\infty} \prod_{1\leqslant j<k\leqslant \ell} \frac{(1-t_k/t_j)\,(px^{-1}t_k/t_j)_\infty}{(xt_k/t_j)_\infty}\,.$$

(D.9) Theorem. *Let* $|u| < \min\,(1, |x^{\ell-1}|)$. *Then*

$$\sum_{k=1}^{\ell}\sum_{r_k=0}^{\infty} u^{\sum_{i=1}^{\ell}(\ell-i+1)r_i}\, x^{-\sum_{i=1}^{\ell}(i-1)(\ell-i+1)r_i}\, S(p^{r_1}, p^{r_1+r_2}x, \dots, p^{r_1+\dots+r_\ell}x^{\ell-1}) =$$

$$= \prod_{s=0}^{\ell-1} \frac{(x)_\infty\,(x^s\alpha u)_\infty\,(p)_\infty}{(x^{s+1})_\infty\,(x^s\alpha)_\infty (x^{-s}u)_\infty}$$

provided the parameters α *and* x *are such that all the terms of the sum are regular.*

Proof. The sum in the left hand side of the formula is absolutely convergent and, hence, defines an analytic function of the parameters α, u, x. Therefore, it suffices to prove the formula under the assumption $|\alpha| < 1$, $|x| < 1$.

Under this assumption formula (D.9) follows from formula (5.13) and Lemma D.10, since the sum in formula (D.9) is proportional term by term to the sum in formula (D.10) if we identify $\alpha = ab$, $u = pb^{-1}c^{-1}$ and the proportionality coefficient equals $\prod_{s=0}^{\ell-1} \big((p)_\infty^2/\theta(x^{-s}u)\big)$. □

(D.10) Lemma. *Let* $|a| < 1$, $|b| < 1$, $|x| < 1$ *and* $|x^{\ell-1}bc| > |p|$. *Then*

$$\int_{\mathbb{T}^\ell} F(t_1,\dots,t_\ell;a,b,c)\,d^\ell t \;=\; (2\pi i)^\ell\,\ell!\;\times$$

$$\times \sum_{k=1}^{\ell}\sum_{r_k=0}^{\infty} \operatorname{Res}\big(\operatorname{Res}\big(\dots \operatorname{Res} F(t_1,\dots,t_\ell;a,b,c)\big|_{t_\ell=p^{r_\ell}xt_{\ell-1}} \dots\big)\big|_{t_2=p^{r_2}xt_1}\big)\big|_{t_1=p^{r_1}b}\,.$$

Proof. We begin the proof with the next identity which follows from formulae (B.5), (B.6) after a suitable change of notations:

$$\prod_{k=1}^{\ell}\frac{\theta(ct_k)}{\theta(t_k/b)} \;=\; \prod_{s=0}^{\ell-1}\frac{\theta(px^s bc)}{\theta(x^{s(s-\ell)}(bc)^{s-\ell})}\;\times$$

$$\times\sum_{\sigma\in\mathbf{S}^\ell}\Big(\prod_{k=1}^{\ell}\frac{\theta(x^{1+(k-1)(k-1-\ell)}(bc)^{k-1-\ell}t_{\sigma_{k-1}}/t_{\sigma_k})}{\theta(xt_{\sigma_{k-1}}/t_{\sigma_k})}\prod_{1\leqslant j<k\leqslant\ell}\frac{\theta(xt_{\sigma_j}/t_{\sigma_k})}{\theta(t_{\sigma_j}/t_{\sigma_k})}\Big)$$

where $\sigma_0 = 0$. Here and below we set $t_0 = x^{-1}b$. Using the identity we obtain that

$$\int_{\mathbb{T}^\ell} F(t_1,\dots,t_\ell;a,b,c)\,d^\ell t \;=\; \ell!\int_{\mathbb{T}^\ell}\widetilde F(t_1,\dots,t_\ell)\,d^\ell t\,, \tag{D.11}$$

$$\widetilde F(t_1,\dots,t_\ell) \;=\; (-b)^{-\ell}\,(p)_\infty^\ell\prod_{s=0}^{\ell-1}\frac{\theta(px^s bc)}{\theta(x^{s(s-\ell)}(bc)^{s-\ell})}\;\times$$

$$\times\prod_{k=1}^{\ell}\frac{\theta(x^{1+(k-1)(k-1-\ell)}(bc)^{k-1-\ell}t_{k-1}/t_k)\,(pt_k/b)_\infty}{\theta(xt_{k-1}/t_k)\,(at_k)_\infty}\;\times$$

$$\times\prod_{1\leqslant j<k\leqslant\ell}\frac{(1-t_k/t_j)\,(px^{-1}t_k/t_j)_\infty}{(xt_k/t_j)_\infty}\,.$$

This step is similar to the transformation of the integral in the proof of formula (5.13).

Suppose that the variables $t_1,\dots,t_{\ell-1}$ are fixed and $|t_k| = 1$, $k = 1, \dots, \ell-1$. Then poles of the function $\widetilde{F}(t_1,\dots,t_\ell)$ in the punctured disk $0 < |t_\ell| < 1$ form the following set: $\{p^r b \mid r \in \mathbb{Z}_{\geqslant 0}\}$. Hence,

$$\int\limits_{\mathbb{T}^\ell} \widetilde{F}(t_1,\dots,t_\ell)\, d^\ell t \;=\; \int\limits_{\mathbb{T}^{\ell-1}\times\mathbb{T}_{p^s x\varepsilon}} \widetilde{F}(t_1,\dots,t_\ell)\, d^\ell t \;+$$

$$+\; 2\pi i \sum_{r=0}^{s-1} \int\limits_{\mathbb{T}^{\ell-1}} \operatorname{Res} \widetilde{F}(t_1,\dots,t_\ell)\big|_{t_\ell = p^r x t_{\ell-1}}\, d^{\ell-1}t$$

where $\mathbb{T}_y = \{t_\ell \in \mathbb{C} \mid |t_\ell| = y\}$, ε is an arbitrary positive number between $|p|$ and 1, and the residues are calculated with respect to the variable t_ℓ, all other variables being fixed. Due to the functional relation

$$\frac{\widetilde{F}(t_1,\dots,t_{\ell-1},pt_\ell)}{\widetilde{F}(t_1,\dots,t_\ell)} \;=\; x^{1-\ell}(bc)^{-1}\,\frac{1-at_\ell}{1-pb^{-1}t_\ell}\prod_{k=1}^{\ell-1}\frac{(t_k-pt_\ell)(t_k-xt_\ell)}{(t_k-t_\ell)(t_k-px^{-1}t_\ell)}$$

we see that $\widetilde{F}(t_1,\dots,t_\ell) = O\big((x^{\ell-1}bc)^{-s}\big)$ uniformly at $\mathbb{T}^{\ell-1}\times\mathbb{T}_{p^s x\varepsilon}$ and

$$\operatorname{Res} \widetilde{F}(t_1,\dots,t_\ell)\big|_{t_\ell = p^r x t_{\ell-1}} \;=\; O\big(p^s (x^{\ell-1}bc)^{-s}\big)$$

as $s \longrightarrow \infty$. Therefore,

$$\int\limits_{\mathbb{T}^\ell} \widetilde{F}(t_1,\dots,t_\ell)\, d^\ell t \;=\; 2\pi i \sum_{r=0}^{\infty} \int\limits_{\mathbb{T}^{\ell-1}} \operatorname{Res} \widetilde{F}(t_1,\dots,t_\ell)\big|_{t_\ell = p^r x t_{\ell-1}}\, d^{\ell-1}t\,.$$

Similarly to the previous consideration we transform the integrals in the right hand side of the above formula and obtain that

$$\int\limits_{\mathbb{T}^\ell} \widetilde{F}(t_1,\dots,t_\ell)\, d^\ell t \;=\; (2\pi i)^2\,\times$$

$$\times \sum_{r_\ell=0}^{\infty}\sum_{r_{\ell-1}=0}^{\infty} \int\limits_{\mathbb{T}^{\ell-2}} \operatorname{Res}\big(\operatorname{Res} \widetilde{F}(t_1,\dots,t_\ell)\big|_{t_\ell = p^{r_\ell} x t_{\ell-1}}\big)\big|_{t_{\ell-1} = p^{r_{\ell-1}} x t_{\ell-2}}\, d^{\ell-2}t\,.$$

It is clear that the order of summation is irrelevant. Repeating the procedure ℓ times we get the following formula

$$\int_{\mathbb{T}^\ell} \widetilde{F}(t_1,\ldots,t_\ell)\, d^\ell t =$$

$$= (2\pi i)^\ell \sum_{k=1}^{\ell} \sum_{r_k=0}^{\infty} \operatorname{Res}\big(\operatorname{Res}\big(\ldots \operatorname{Res} \widetilde{F}(t_1,\ldots,t_\ell)\big|_{t_\ell=p^{r_\ell}xt_{\ell-1}} \ldots\big)\big|_{t_2=p^{r_2}xt_1}\big)\big|_{t_1=p^{r_1}b}$$

which is equivalent to formula (D.10) because of relation (D.11) and the equality

$$\operatorname{Res}\big(\operatorname{Res}\big(\ldots \operatorname{Res} F(t_1,\ldots,t_\ell;a,b,c)\big|_{t_\ell=p^{r_\ell}xt_{\ell-1}} \ldots\big)\big|_{t_2=p^{r_2}xt_1}\big)\big|_{t_1=p^{r_1}b} =$$

$$= \operatorname{Res}\big(\operatorname{Res}\big(\ldots \operatorname{Res} \widetilde{F}(t_1,\ldots,t_\ell)\big|_{t_\ell=p^{r_\ell}xt_{\ell-1}} \ldots\big)\big|_{t_2=p^{r_2}xt_1}\big)\big|_{t_1=p^{r_1}b} .$$

The lemma is proved. ☐

E. The multidimensional Askey-Roy formula and Askey's conjecture

In this appendix we give proofs of formula (5.14) and Askey's conjecture [As, Conjecture 8], see formula (E.8).

***Proof of formula* (5.14).** Let $n = 2$ and $\kappa = p^{-1}\eta^{\ell-1}\xi_1^{-1}\xi_2^{-1}$. Assume that parameters ξ_1, ξ_2, z_1, z_2 and η are generic. Consider formula (5.11). It takes the form

$$\text{(E.1)}\quad I(W_0, w) = \prod_{s=0}^{\ell-1} \frac{(\eta^{-1})_\infty\,(p\eta^{s+2-\ell}\xi_1^2\xi_2^2)_\infty\,(\eta^s\xi_2^{-2})_\infty\,(\eta^s\xi_1^{-1}\xi_2^{-1}z_1/z_2)_\infty}{(\eta^{-s-1})_\infty\,(p)_\infty\,(\eta^{-s}\xi_1^2)_\infty\,(\eta^s\xi_1\xi_2 z_1/z_2)_\infty}\,.$$

Here and after we use the simplified notations: $w_0 = w_{(\ell,0)}$, $W_k = W_{(\ell-k,k)}$, $k = 0, \dots, \ell$, and I denotes the hypergeometric integral (5.1) as usual.

In the case in question the quotient space $\mathcal{F}_{ell}(z)/\mathcal{Q}'(z)$ is one-dimensional and is spanned by the equivalence class of the function W_0, see Lemma 2.39. Let $y \in \mathbb{C}^{\times\ell}$ be the following point:

$$y = (\eta^{\ell-1}\xi_1^{-1}z_1, \eta^{\ell-2}\xi_1^{-1}z_1, \dots, \xi_1^{-1}z_1)\,.$$

Lemma 2.37 means that for any element $W(t)$ of the elliptic hypergeometric space $\mathcal{F}_{ell}(z)$ of a fiber the function $W(t)\,W_0(y) - W_0(t)\,W(y)$ is an element of the boundary subspace $\mathcal{Q}'(z)$. Therefore, by Lemma 5.8 for any $W \in \mathcal{F}_{ell}(z)$ we have that

$$\text{(E.2)}\qquad I(W, w_0) = I(W_0, w_0)\,\frac{W(y)}{W_0(y)}\,.$$

Let ζ be a nonzero complex number. The next function is an element of the space $\mathcal{F}_{ell}(z)$:

$$\text{(E.3)}\qquad W(t_1, \dots, t_\ell) =$$

$$= \prod_{a=1}^{\ell} \frac{\theta(p\eta^{1-\ell}\xi_1\xi_2\zeta^{-1}t_a/z_1)\,\theta(\zeta t_a/z_2)}{\theta(\xi_1^{-1}t_a/z_1)\,\theta(\xi_2^{-1}t_a/z_2)} \prod_{1\leqslant a<b\leqslant \ell} \frac{\theta(t_a/t_b)}{\theta(\eta t_a/t_b)}\,.$$

In particular, for $\zeta = \xi_2^{-1}$ we get the function W_0.

Let $|z_1| = |z_2| = 1$, $|\xi_1| < 1$, $|p\xi_2| < 1$ and $|\eta| > 1$. Under these assumptions for any function $W \in \mathcal{F}_{ell}(z)$ the hypergeometric integral $I(W, w_0)$ is given by formula (5.3):

$$I(W, w_0) = \int_{\mathbb{T}^\ell} \Phi(t)\, w_0(t)\, W(t)\, (dt/t)^\ell .$$

Calculating the hypergeometric integral $I(W, w_0)$ for a function W of the form (E.3) via formulae (E.1) and (E.2) we obtain formula (5.14) for generic values of parameters $a, b, c, \alpha, \beta, x$ up to a change of notations:

$$\text{(E.4)} \qquad \begin{aligned} a &= \xi_1/z_1\,, & b &= \xi_2/z_2\,, & c &= pz_2/\zeta\,, \\ \alpha &= \xi_1 z_1\,, & \beta &= p\xi_2 z_2\,, & x &= \eta^{-1}\,. \end{aligned}$$

Formula (5.14) extends to arbitrary values of parameters $a, b, c, \alpha, \beta, x$ by the analytic continuation. □

Formula (5.14) admits the following modifications.

(E.5) Lemma. *Let* $|a| < 1$, $|b| < 1$, $|\alpha| < 1$, $|\beta| < 1$, $|x| < 1$. *Then*

$$\int_{\mathbb{T}^\ell} \prod_{k=1}^{\ell} \frac{\theta(pt_k/c)\,\theta(x^{\ell-1}abct_k)}{t_k\,(at_k)_\infty\,(bt_k)\,(\alpha/t_k)_\infty\,(\beta/t_k)_\infty} \prod_{1\leqslant j<k\leqslant\ell} \frac{\theta(t_j/t_k)_\infty}{(xt_j/t_k)_\infty\,(pxt_k/t_j)_\infty}\, d^\ell t =$$

$$= (2\pi i)^\ell\,(p)_\infty^{\ell(\ell-3)/2} \prod_{s=0}^{\ell-1} \frac{(px)_\infty\,(x^{\ell+s-1}ab\alpha\beta)_\infty\,\theta(x^s ac)\,\theta(x^s bc)}{(px^{s+1})_\infty\,(x^s a\alpha)_\infty\,(x^s a\beta)_\infty\,(x^s b\alpha)_\infty\,(x^s b\beta)_\infty}\,.$$

(E.6) Lemma. *Let* $|a| < 1$, $|b| < 1$, $|\alpha| < 1$, $|\beta| < 1$, $|x| < 1$. *Then*

$$\int_{\mathbb{T}^\ell} \prod_{k=1}^{\ell} \frac{\theta(pt_k/c)\,\theta(x^{\ell-1}abct_k)}{t_k\,(at_k)_\infty\,(bt_k)\,(\alpha/t_k)_\infty\,(\beta/t_k)_\infty} \prod_{1\leqslant j<k\leqslant\ell} \frac{\theta(t_k/t_j)_\infty}{(xt_k/t_j)_\infty\,(pxt_j/t_k)_\infty}\, d^\ell t =$$

$$= (2\pi i)^\ell\,(p)_\infty^{\ell(\ell-3)/2} \prod_{s=0}^{\ell-1} \frac{(px)_\infty\,(x^{\ell+s-1}ab\alpha\beta)_\infty\,\theta(x^s ac)\,\theta(x^s bc)}{(px^{s+1})_\infty\,(x^s a\alpha)_\infty\,(x^s a\beta)_\infty\,(x^s b\alpha)_\infty\,(x^s b\beta)_\infty}\,.$$

Proof of Lemma E.5. Denote by $f(t_1,\dots,t_\ell)$ the integrand in formula (5.14). Then the integrand in the first formula of the lemma equals

$$f(t_1,\dots,t_\ell)\,(p)_\infty^{\ell(\ell-1)/2}\prod_{1\leqslant j<k\leqslant\ell}\frac{t_j-xt_k}{t_j-t_k}\,.$$

Since the integration contour $\mathbb{T}^\ell$ is invariant with respect to permutations of the variables $t_1,\dots,t_\ell$, the first formula follows from formula (5.14) and the next identity:

$$\sum_{\sigma\in\mathbf{S}^\ell}\prod_{1\leqslant j<k\leqslant\ell}\frac{t_{\sigma_j}-xt_{\sigma_k}}{t_{\sigma_j}-t_{\sigma_k}}=\prod_{s=1}^{\ell}\frac{1-x^s}{1-x}\,.$$

The identity is equivalent to Corollary A.2 up to a change of notations. □

The proof of Lemma E.6 is similar.

Introduce points $v_s\in\mathbb{C}$, $s\in\mathbb{Z}$, by the rule:

(E.7) $$v_s=p^sb\quad\text{for}\quad s\geqslant 0\,,\qquad v_s=p^{-s-1}a\quad\text{for}\quad s<0\,.$$

Set

$$A(u_1,\dots,u_\ell;x)=\prod_{k=1}^{\ell}\frac{u_k\,(pu_k/a)_\infty\,(pu_k/b)_\infty}{(\alpha u_k)_\infty\,(\beta u_k)_\infty}\prod_{1\leqslant j<k\leqslant\ell}\frac{(px^{-1}u_k/u_j)_\infty}{(pxu_k/u_j)_\infty}\,.$$

(E.8) Theorem. *Let m be a nonnegative integer. Then*

$$\sum_{k=1}^{\ell}\sum_{r_k\in\mathbb{Z}}\operatorname{sgn}(r_k)\,A(v_{r_1},\dots,v_{r_\ell};p^m)\prod_{k=1}^{\ell}v_{r_k}^{2m(\ell-k)}=$$

$$=p^{m^2\binom{\ell}{3}-\binom{m}{2}\binom{\ell}{2}}\prod_{s=0}^{\ell-1}\frac{(p^{m+1})_\infty\,(p^{m(\ell+s-1)}ab\alpha\beta)_\infty\,(-ab)^{ms}\,b\,\theta(a/b)}{(p^{m(s+1)+1})_\infty\,(p^{ms}a\alpha)_\infty\,(p^{ms}a\beta)_\infty\,(p^{ms}b\alpha)_\infty\,(p^{ms}b\beta)_\infty}$$

provided the parameters a,b,α,β are such that all the terms of the sum are regular. Here $\operatorname{sgn}(r)=1$ *for* $r\geqslant 0$, *and* $\operatorname{sgn}(r)=-1$ *for* $r<0$.

Remark. The above formula was conjectured by Askey [As, Conjecture 8]. There is the following correspondence of notations:

$$p=q\,,\qquad a=-c\,,\qquad b=d\,,\qquad \alpha=-q^x/c\,,\qquad \beta=q^y/d\,,\qquad m=k\,,\qquad \ell=n$$

where notations in the left hand side are from this paper and notations in the right hand sides are from [As]. The above formula differs from the conjectured formula in [As] by the factor $p^{m^2\binom{\ell}{3}-\binom{m}{2}\binom{\ell}{2}}$.

Remark. After this paper was written we found out that formula (E.8) was proved in [E].

Proof of Theorem E.8. The sum in the left hand side of the formula is absolutely convergent since

$$A(v_{r_1},\dots,v_{r_\ell};p^m)\prod_{k=1}^{\ell} v_{r_k}^{2m(\ell-k)} = O(p^{|r_1|+\ldots+|r_\ell|})$$

as $|r_1|+\ldots+|r_\ell|\to\infty$. Hence, the sum defines an analytic function of the parameters a,b,α,β. Therefore, it suffices to prove the formula under the assumptions $|a|<1$, $|b|<1$, $|\alpha|<1$, $|\beta<1|$.

Consider the integral in the left hand side of formula (E.6) for $x=p^m$, and denote the integrand by $f(t_1,\dots,t_\ell)$. The poles of the integrand are located at the hyperplanes

$$t_j=p^{-s}a^{-1}, \qquad t_j=p^{-s}b^{-1}, \qquad t_j=p^s\alpha, \qquad t_j=p^s\beta,$$

$j=1,\dots,\ell$, $s\in\mathbb{Z}_{\geqslant 0}$. Due to Lemma E.6 the formula (E.8) is equivalent to the next formula

$$\text{(E.9)}\qquad \int_{\mathbb{T}^\ell} f(t_1,\dots,t_\ell)\,d^\ell t = (-2\pi i)^\ell \times$$

$$\times \sum_{k=1}^{\ell}\sum_{r_k\in\mathbb{Z}} \operatorname{Res}\bigl(\operatorname{Res}\bigl(\dots\operatorname{Res} f(t_1,\dots,t_\ell)\big|_{t_\ell=v_{r_\ell}^{-1}}\cdots\bigr)\big|_{t_2=v_{r_2}^{-1}}\bigr)\big|_{t_1=v_{r_1}^{-1}},$$

since the sum of residues in the right hand side coincides with the sum in the left hand side of formula (E.8). Formula (E.9) can be proved by standard arguments, cf. definition (E.7) of points v_s, $s\in\mathbb{Z}$. The theorem is proved. □

Theorem E.8 admits the following generalization. Set

$$\tilde{A}(u_1,\dots,u_\ell;x) =$$

$$= \prod_{k=1}^{\ell} \frac{u_k\,(pu_k/a)_\infty\,(pu_k/b)_\infty}{(\alpha u_k)_\infty\,(\beta u_k)_\infty} \prod_{1\leqslant j<k\leqslant\ell} \frac{(1-u_k/u_j)\,(px^{-1}u_k/u_j)_\infty}{(xu_k/u_j)_\infty}.$$

(E.10) Theorem. *Let* $|px^{\ell-1}|<1$. *Then*

$$\sum_{j=0}^{\ell}\sum_{k=1}^{\ell}\sum_{r_k=0}^{\infty}(-1)^j\times$$

$$\times\, x^{\sum_{i=1}^{\ell}(\ell-i)(\ell-i+1)r_i-(\ell-j-1)(\ell-j)(1+2\sum_{i=1}^{j}r_i)/2}\prod_{s=0}^{\ell-j-1}\frac{\theta(x^{j+s}a/b)}{\theta(x^{j-s}a/b)}\times$$

$$\times\,\tilde{A}(p^{r_1}a,p^{r_1+r_2}xa,\dots,p^{r_1+\dots+r_j}x^{j-1}a,p^{r_{j+1}}b,\dots,p^{r_{j+1}+\dots+r_\ell}x^{\ell-j-1}b;x)=$$

$$=\prod_{s=0}^{\ell-1}\frac{(x)_\infty\,(x^{\ell+s-1}ab\alpha\beta)_\infty\,b\theta(x^sa/b)}{(x^{s+1})_\infty\,(x^sa\alpha)_\infty\,(x^sa\beta)_\infty\,(x^sb\alpha)_\infty\,(x^sb\beta)_\infty}$$

provided the parameters a,b,α,β,x *are such that all the terms of the sum are regular.*

Proof. The proof is similar to the proof of the previous Theorem. The terms of the sum behave as $O\big((px^{\ell-1})^{r_1+\dots+r_\ell}\big)$ for $r_1+\dots+r_\ell$ going to infinity. Hence, the sum is absolutely convergent and defines an analytic function of the parameters a,b,α,β,x. Therefore, it suffices to prove the formula under the assumptions $|a|<1$, $|b|<1$, $|\alpha|<1$, $|\beta<1|$, $|x|<1$.

Consider the integral in the left hand side of formula (5.14) and denote the integrand by $f(t_1,\dots,t_\ell)$. The poles of the integrand are located at the hyperplanes

$$t_j=p^{-s}a^{-1}\,,\qquad t_j=p^{-s}b^{-1}\,,\qquad t_j=p^s\alpha\,,\qquad t_j=p^s\beta\,,$$

$j=1,\dots,\ell$, $s\in\mathbb{Z}_{\geqslant 0}$, and at the hyperplanes

$$t_j=p^sxt_k\,,\qquad j,k=1,\dots,\ell,\quad j\neq k\,,\qquad s\in\mathbb{Z}_{\geqslant 0}\,.$$

Moreover, the integrand vanishes at the hyperplanes

$$t_j=p^st_k\,,\qquad j,k=1,\dots,\ell,\quad j\neq k\,,\qquad s\in\mathbb{Z}\,,$$

and it is a symmetric function of the integration variables $t_1,\dots,t_\ell$. Using these properties of the integrand by standard arguments we obtain the following formula

$$\text{(E.11)} \quad \int\limits_{\mathbb{T}^\ell} f(t_1, \ldots, t_\ell)\, d^\ell t \;=\; (2\pi i)^\ell\, \ell! \;\times$$

$$\times \sum_{j=0}^{\ell} \sum_{k=1}^{\ell} \sum_{r_k=0}^{\infty} \operatorname{Res}\Big(\ldots \operatorname{Res}\Big(\operatorname{Res}\Big(\ldots \operatorname{Res} f(t_1, \ldots, t_\ell)\big|_{t_\ell = p^{-r_\ell} x^{-1} t_{\ell-1}} \ldots \Big)\Big|_{t_{j+1} = p^{-r_{j+1}} b^{-1}} \Big)\Big|_{t_j = p^{-r_j} x^{-1} t_{j-1}} \ldots \Big)\Big|_{t_1 = p^{-r_1} a^{-1}} \Big),$$

Formula (E.11) is a particular case of formula (F.2). We will give a detailed proof of a more general formula in Appendix F.

Due to formula (5.14) the formula (E.10) is equivalent to formula (E.11) since the sum of residues in the right hand side of formula (E.11) coisides with the sum in the left hand side of formula (E.10). □

F. The Jackson integrals via the hypergeometric integrals

In this appendix we present two theorems which connect the hypergeometric integrals described in this paper with symmetric A-type Jackson integrals.

Let $\mathbf{s} = (s_1, \dots, s_\ell)$ be a vector with integer components. For any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ define the points $x \triangleright (\mathfrak{l}, \mathbf{s})$, $y \triangleleft (\mathfrak{l}, \mathbf{s}) \in \mathbb{C}^{\times \ell}$ as follows:

$$\begin{aligned} x \triangleright (\mathfrak{l}, \mathbf{s}) \,=\, \bigl(& p^{s_1+\ldots+s_{\mathfrak{l}_1}} \eta^{1-\mathfrak{l}_1} \xi_1 z_1, p^{s_2+\ldots+s_{\mathfrak{l}_1}} \eta^{2-\mathfrak{l}_1} \xi_1 z_1, \dots, p^{s_{\mathfrak{l}_1}} \xi_1 z_1, \\ & p^{s_{\mathfrak{l}_1+1}+\ldots+s_{\mathfrak{l}_1+\mathfrak{l}_2}} \eta^{1-\mathfrak{l}_2} \xi_2 z_2, \dots, p^{s_{\mathfrak{l}_1+\mathfrak{l}_2}} \xi_2 z_2, \ \dots, \\ & p^{s_{\ell-\mathfrak{l}_n+1}+\ldots+s_\ell} \eta^{1-\mathfrak{l}_n} \xi_n z_n, \dots, p^{s_\ell} \xi_n z_n \bigr), \end{aligned}$$

$$\begin{aligned} y \triangleleft (\mathfrak{l}, \mathbf{s}) \,=\, \bigl(& p^{s_1+\ldots+s_{\mathfrak{l}_1}} \eta^{\mathfrak{l}_1-1} \xi_1^{-1} z_1, p^{s_2+\ldots+s_{\mathfrak{l}_1}} \eta^{\mathfrak{l}_1-2} \xi_1^{-1} z_1, \dots, p^{s_{\mathfrak{l}_1}} \xi_1^{-1} z_1, \\ & p^{s_{\mathfrak{l}_1+1}+\ldots+s_{\mathfrak{l}_1+\mathfrak{l}_2}} \eta^{\mathfrak{l}_2-1} \xi_2^{-1} z_2, \dots, p^{s_{\mathfrak{l}_1+\mathfrak{l}_2}} \xi_2^{-1} z_2, \ \dots, \\ & p^{s_{\ell-\mathfrak{l}_n+1}+\ldots+s_\ell} \eta^{\mathfrak{l}_n-1} \xi_n^{-1} z_n, \dots, p^{s_\ell} \xi_n^{-1} z_n \bigr), \end{aligned}$$

cf. (2.35). In particular, for $\mathbf{s} = (0, \dots, 0)$ we have $x \triangleright (\mathfrak{l}, \mathbf{s}) = x \triangleright \mathfrak{l}$ and $y \triangleleft (\mathfrak{l}, \mathbf{s}) = y \triangleleft \mathfrak{l}$.

Recall, that the short phase function $\Phi(t, z)$ is given by formula (5.2) and we define the multiple residue by formula (2.34). Set

$$\widetilde{\Phi}(t, z) \,=\, t_1^{-1} \ldots t_\ell^{-1} \Phi(t, z) \,.$$

(F.1) Theorem. *Let* $|p\kappa \prod_{m=1}^{n} \xi_m^{-1}| < \min\,(1, |\eta|^{1-\ell})$. *Let* (2.13) – (2.15) *hold. Then for any functions* $w \in \mathcal{F}(z)$ *and* $W \in \mathcal{F}_{ell}(z)$ *we have that*

$$I(W, w) \,=\, (2\pi i)^\ell \ell! \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \sum_{\mathbf{s} \in \mathbb{Z}_{\geqslant 0}^\ell} \operatorname{Res} \bigl(\widetilde{\Phi}(t)\, w(t)\, W(t) \bigr) \bigr|_{t = x \triangleright (\mathfrak{m}, \mathbf{s})} \,.$$

(F.2) Theorem. *Let* $|\kappa \prod_{m=1}^{n} \xi_m| > \max\,(1, |\eta|^{\ell-1})$. *Let* (2.13) – (2.15) *hold. Then for any functions* $w \in \mathcal{F}(z)$ *and* $W \in \mathcal{F}_{ell}(z)$ *we have that*

$$I(W, w) \,=\, (-2\pi i)^\ell \ell! \sum_{\mathfrak{m} \in \mathcal{Z}_\ell^n} \sum_{\mathbf{s} \in \mathbb{Z}_{\leqslant 0}^\ell} \operatorname{Res} \bigl(\widetilde{\Phi}(t)\, w(t)\, W(t) \bigr) \bigr|_{t = y \triangleleft (\mathfrak{m}, \mathbf{s})} \,.$$

Proof of Theorem F.1. We prove the theorem assuming that the functions w and W are the trigonometric and elliptic weight function, respectively. For arbitrary functions w and W the statement holds by linearity.

Let $\zeta = \max\,(1, |\eta|^{\ell-1})\,|p\kappa \prod_{m=1}^{n} \xi_m^{-1}|$, that is $0 < \zeta < 1$ under the assumptions of the theorem. Using the functional relation

$$\text{(F.3)} \qquad \frac{\Phi(t_1,\dots,pt_a,\dots,t_\ell)\,W(t_1,\dots,pt_a,\dots,t_\ell)}{\Phi(t_1,\dots,t_\ell)\,W(t_1,\dots,t_\ell)} = $$

$$= \kappa \prod_{m=1}^{n} \frac{\xi_m t_a - z_m}{t_a - \xi_m z_m} \prod_{a<b\leqslant\ell} \frac{t_a - \eta t_b}{\eta t_a - t_b} \prod_{1\leqslant b<a} \frac{t_a - p\eta t_b}{p\eta t_a - t_b}\,,$$

$a = 1,\dots,\ell$, we estimate the residues:

$$\operatorname{Res}\big(\widetilde{\Phi}(t)\,w(t)\,W(t)\big)\big|_{t=x\triangleright(\mathrm{m},\mathrm{s})} = O(\zeta^{s_1+\ldots+s_\ell})\,,$$

and obtain that the sum is absolutely convergent. Therefore, the sum defines an analytic function of the parameters η, $\xi_1,\dots,\xi_n$, $z_1,\dots,z_n$, and it suffices to prove the formula under the assumptions that $|\eta| > 1$ and $|z_m| = 1$, $|\xi_m| < 1$, $m = 1,\dots,n$.

Under these assumptions the hypergeometric integral $I(W,w)$ is given by formula (5.3) and the claim of the theorem is equivalent to Lemma F.4 for $f = 1$. Theorem F.1 is proved. □

(F.4) Lemma. *Let* $|\eta| > 1$ *and* $|\xi_l z_l| < |\xi_m^{-1} z_m|$ *for all* $l, m = 1,\dots,n$. *Let* $|p\kappa \prod_{m=1}^{n} \xi_m^{-1}| < 1$. *Let* (2.13) – (2.15) *hold. Let* α *be a real number such that* $\max\,(|\xi_1 z_1|,\dots,|\xi_n z_n|) < \alpha < \min\,(|\xi_1^{-1} z_1|,\dots,|\xi_n^{-1} z_n|)$. *Then,*

$$\int_{\mathbb{T}^\ell_\alpha} \Phi(t)\,w(t)\,W(t)\,f(t)\,(dt/t)^\ell =$$

$$= (2\pi i)^\ell\,\ell! \sum_{\mathrm{m}\in\mathcal{Z}^n_\ell}\ \sum_{\mathrm{s}\in\mathbb{Z}^\ell_{\geqslant 0}} \operatorname{Res}\big(\widetilde{\Phi}(t)\,w(t)\,W(t)\,f(t)\big)\big|_{t=x\triangleright(\mathrm{m},\mathrm{s})}$$

for any symmetric function $f(t)$ *regular inside the torus*

$$\mathbb{T}^\ell_\alpha = \{t \in \mathbb{C}^\ell \mid |t_1| = \alpha,\ \dots\ , |t_\ell| = \alpha\}\,.$$

Proof. We prove the lemma by induction with respect to ℓ, the number of integrations.

Notice that the integral does not change if we replace the integration contour in the integral by $\mathbb{T}_{\alpha_1}\times\ldots\times\mathbb{T}_{\alpha_\ell}$, where $\alpha_1,\ldots,\alpha_\ell$ are pairwise distinct real numbers close to α.

Similar to the proof of Theorem F.1 we see that the sum of residues is absolutely convergent. Hence, the sum defines a holomorphic function of the parameters η, $\xi_1,\ldots,\xi_n$, $z_1,\ldots,z_n$, and the same does the integral in the left hand side of formula (F.4). Therefore, without loss of generality we can assume that all the numbers $|p^s\alpha_m|$, $|p^s\xi_m z_m|$, $m=1,\ldots,n$, $s\in\mathbb{Z}_{\geqslant 0}$, are pairwise distinct.

Replace the integration contour in the integral by $\mathbb{T}_{\varepsilon\alpha_1}\times\ldots\times\mathbb{T}_{\varepsilon\alpha_\ell}$ for real ε, so that at $\varepsilon=1$ we have the initial integral. If ε is decreasing starting from 1, the integral is not changing until the integration contour touches one of the hyperplanes where the integrand has a pole. And every time when the integration contour crosses the singularity hyperplane we pick up the integral of dimension $\ell-1$ of the corresponding residue. Notice that during the described deformation the integration contour can touch only the singularity hyperplanes of the form $t_a=p^s\xi_m z_m$, $s\in\mathbb{Z}_{\geqslant 0}$. Finally, for any positive integer r we get

$$\int\limits_{\mathbb{T}^\ell_\alpha}\Phi(t)\,w(t)\,W(t)\,f(t)\;(dt/t)^\ell\;=\int\limits_{\mathbb{T}^\ell_{p^r\alpha}}\Phi(t)\,w(t)\,W(t)\,f(t)\;(dt/t)^\ell\;+$$

$$+\;2\pi i\sum\nolimits_{(r)}\int\operatorname{Res}\bigl(t_a^{-1}\Phi(t)\,w(t)\,W(t)\,f(t)\bigr)\big|_{t_a=p^s\xi_m z_m}(dt/t)^{\ell-1},$$

where each term of the sum $\sum_{(r)}$ corresponds to a passing of the integration contour through a singularity hyperplane when ε goes from 1 to p^r.

Using relation (F.3) it is easy to show that

$$\int\limits_{\mathbb{T}^\ell_{p^r\alpha}}\Phi(t)\,w(t)\,W(t)\,f(t)\;(dt/t)^\ell\;=\;O\bigl(|p\kappa\prod_{m=1}^{n}\xi_m^{-1}|^{\ell r}\bigr)$$

as $r\to\infty$. Hence, the integral disappears as $r\to\infty$.

All integrals of smaller dimension in the sum $\sum_{(r)}$ are of the same form as the initial integral and can be replaced by the sums of residues according to formula (F.4) because of the induction assumption. Since $\widetilde{\Phi}(t)\,w(t)\,W(t)\,f(t)$ is a symmetric function of $t_1,\ldots,t_\ell$, it is straightforward to show that the resulting sum of residues can be transformed to the sum in the right hand side of formula (F.4). The lemma is proved. □

The proof of Theorem F.2 is similar to the proof of Theorem F.1.

The sums of residues in formulae (F.1) and (F.2) coincide with symmetric A-type Jackson integrals, see for example [AK]. This means that the transition functions between asymptotic solutions of the qKZ equation coincide with the connection matrices for symmetric A-type Jackson integrals. The Gauss decomposition of the connection matrices studied in [AK] is related to Lemmas B.10 and 7.6 in this paper.

G. One useful identity

Particular specializations of the following identity:

$$\sum_{a=0}^{l} \binom{j+a}{j} \binom{j+k+a}{k} \binom{l+m-a}{m} = \tag{G.1}$$

$$= \binom{j+k}{k} \binom{j+k+l+m+1}{j+k+m+1},$$

are often used in the paper. The identity can be proved by induction with respect to l and m.

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