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## NON-SOLVABLE GROUPS WITH A LARGE FRACTION OF INVOLUTIONS

by

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**Abstract.** — In this note we classify the non-solvable finite groups  $G$  such that the class number of  $G$  is at least  $|G|/16$ . Some consequences are derived as well.

C.T.C. Wall classified all finite groups in which the fraction of involutions exceeds  $1/2$  (see [1], Theorem 11.24). In this paper we classify all non-solvable finite groups in which the fraction of involutions is not less than  $1/4$ .

We recall some notation.

Let  $k(G)$  be the class number of  $G$ . Let  $i(G)$  denote the number of all involutions of  $G$ ,  $T(G) = \sum \chi(1)$  where  $\chi$  runs over the set  $\text{Irr}(G)$ . Now

$$\text{mc}(G) = k(G)/|G|, \quad f(G) = T(G)/|G|, \quad i_o(G) = i(G)/|G|.$$

It is well-known (see [1], chapter 11) that

$$i(G) < T(G), \quad i_o(G) < f(G), \quad f(G)^2 \leq \text{mc}(G)$$

(with equality if and only if  $G$  is abelian).

In this note we prove the following three theorems.

**Theorem 1.** — *Let  $G$  be a non-solvable group.*

*If  $\text{mc}(G) \geq 1/16$  then  $G = G'Z(G)$ , where  $G'$  is the commutator subgroup of  $G$ ,  $Z(G)$  is the centre of  $G$ ,  $G' \in \{PSL(2, 5), SL(2, 5)\}$ .*

**Theorem 2.** — *Let  $G$  be a non-solvable group.*

*If  $f(G) \geq 1/4$  then  $G = G'Z(G)$  and  $G' \in \{PSL(2, 5), SL(2, 5)\}$ .*

**Theorem 3.** — *Let  $G$  be a non-solvable group.*

*Then  $i_o(G) \geq 1/4$  if and only if  $G = PSL(2, 5) \times E$  with  $\exp E \leq 2$ .*

Lemma 1 contains some well-known results.

### Lemma 1

(a) *If  $G$  is simple and a non-linear  $\chi \in \text{Irr}(G)$  is such that  $\chi(1) < 4$ , then  $\chi(1) = 3$  and  $G \in \{PSL(2, 5), PSL(2, 7)\}$ ; see [2].*

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(b) (Isaacs; see [1], Theorem 14.19). If  $G$  is non-solvable, then  $|cdG| \geq 4$ ; here  $cdG = \{\chi(1) | \chi \in \text{Irr}(G)\}$ .

(c) (see, for example, [1], Chapter 11). If  $G$  is non-abelian then

$$mc(G) \leq 5/8, f(G) \leq 3/4.$$

**Lemma 2.** — Let  $G = G' > 1$ ,  $d \in \{4, 5, 6\}$ . If  $mc(G) \geq (1/d)^2$  then there exists a non-linear  $\chi \in \text{Irr}(G)$  such that  $\chi(1) < d$ .

*Proof.* — Suppose that  $G$  is a counterexample. Then by virtue of Lemma 1(b) one has

$$\begin{aligned} |G| &= \sum_{\chi} \chi(1)^2 \geq 1 + d^2(k(G) - 3) + (d + 1)^2 + (d + 2)^2 \\ &\geq 1 + d^2\left(\frac{|G|}{d^2} - 3\right) + 2d^2 + 6d + 5 = |G| - d^2 + 6d + 6 > |G| \end{aligned}$$

since  $d \in \{4, 5, 6\}$ , — a contradiction (here  $\chi$  runs over the set  $\text{Irr}(G)$ ).

Lemma 3 contains the complete classification of all groups  $G$  satisfying  $i_o(G) = 1/4$ .

**Lemma 3.** — If  $i_o(G) = 1/4$  then one and only one of the following assertions holds:

- (a)  $G \cong A_4$ , the alternating group of degree 4.
- (b)  $G \cong PSL(2, 5)$ .
- (c)  $G$  is a Frobenius group with kernel of index 4.
- (d)  $G$  is a non-cyclic abelian group of order 12.
- (e)  $G$  contains a normal subgroup  $R$  of order 3 such that  $G/R \cong S_3 \times S_3$ ; if  $x$  is an involution in  $G$  then  $|C_G(x)| = 12$  (here  $S_3$  is the symmetric group of degree 3).

*Proof.* — By the assumption  $|G|$  is even.  $i(G)$  is therefore odd by the Sylow Theorem and  $|G| = 4i(G)$ ,  $P \in \text{Syl}_2(G)$  has order 4.

(i) Suppose that  $G$  has no a normal 2-complement. Then  $P$  is abelian of type  $(2, 2)$  and by the Frobenius normal  $p$ -complement Theorem  $G$  contains a minimal non-nilpotent subgroup  $F = C(3^a) \cdot P$  (here  $C(m)$  is a cyclic group of order  $m$  and  $A \cdot B$  is a semi-direct product of  $A$  and  $B$  with kernel  $B$ ). Since all involutions are conjugate in  $F$ , all involutions are conjugate in  $G$ . Hence  $C_G(x) = P$  for  $x \in P^\# = P - \{1\}$ ,  $a = 1$ . If  $G$  is simple then by the Brauer-Suzuki-Wall Theorem (see [1], Theorem 5.20) one has

$$|G| = (2^2 - 1)2^2(2^2 + 1) = 60.$$

Now we assume that  $G$  is not simple. Take  $H$ , a non-trivial normal subgroup of  $G$ . If  $|G : H|$  is odd, then

$$\begin{aligned} i(G) &= i(H), i_o(H) = i(H)/|H| = i(G)/|H| = \\ &|G|i_o(G)/|H| = |G : H|i_o(G) = |G : H|/4. \end{aligned}$$

Therefore  $|G : H| = 3$  and  $i_o(H) = 3/4$ . Now  $f(H) > i_o(H)$ , hence  $H$  is abelian (Lemma 1(c)) and  $f(H) = 1$ . It is easy to see that  $H$  is an elementary abelian 2-group,  $H = P$ . Now  $|P| = 4$  implies  $|G| = 12$ ,  $F = G \cong A_4$ .

Now suppose that  $H$  has even index. Since  $G$  is not 2-nilpotent (= has no a normal 2-complement) then  $|H|$  is odd. In view of  $|C_G(x)| = 4$  for  $x \in P^\#$  one obtains that  $PH$  is a Frobenius group with kernel  $H$ ,  $P$  is cyclic — a contradiction.

- (ii)  $G$  has a normal 2-complement  $K$ .

First assume that  $P$  is cyclic. Then all involutions are conjugate in  $G$ , and for the involution  $x \in P$  one has  $C_G(x) = P$ . Then  $G$  is a Frobenius group with kernel  $K$  of index 4.

Assume that  $P = \langle \alpha \rangle \times \langle \beta \rangle$  is not cyclic. We have  $P = \{1, \alpha, \beta, \alpha\beta\}$ , and all elements from  $P^\#$  are not pairwise conjugate in  $G$ . Thus

$$|G : C_G(\alpha)| + |G : C_G(\beta)| + |G : C_G(\alpha\beta)| = i(G) = |G : P|.$$

Note that  $C_G(\alpha) = P \cdot C_K(\alpha)$ , and similarly for  $\beta$  and  $\alpha\beta$ . Therefore

$$(1) \quad |C_K(\alpha)|^{-1} + |C_K(\beta)|^{-1} + |C_K(\alpha\beta)|^{-1} = 1.$$

Since  $|K| > 1$  is odd then (1) implies

$$(2) \quad |C_K(\alpha)| = |C_K(\beta)| = |C_K(\alpha\beta)| = 3.$$

By the Brauer Formula (see [1], Theorem 15.47) one has

$$(3) \quad |K||C_K(P)|^2 = |C_K(\alpha)||C_K(\beta)||C_K(\alpha\beta)| = 3^3.$$

If  $C_K(P) > 1$  then (3) implies  $|K| = 3$  and  $G = P \times K$  is an abelian non-cyclic group of order 12.

Assume  $C_K(P) = 1$ . Then  $|K| = 3^3$ . Now (2) implies that  $K$  is not cyclic. By analogy, (2) implies that  $\exp K = 3$ . From  $\exp P = 2$  follows that  $G$  is supersolvable. Therefore  $R$ , a minimal normal subgroup of  $G$ , has order 3. Applying the Brauer Formula to  $G/R$ , one obtains  $G/R \cong S_3 \times S_3$ , and we obtain group (e).

**Proof of Theorem 1.** — Denote by  $S = S(G)$  the maximal normal solvable subgroup of  $G$ .

(i) If  $G$  is non-abelian simple then  $G \cong \text{PSL}(2, 5)$ .

*Proof.* — Take  $d = 4$  in Lemma 2. Then there exists  $\chi \in \text{Irr}(G)$  with  $\chi(1) = 3$ . Now Lemma 1(a) implies  $G \in \{\text{PSL}(2, 5), \text{PSL}(2, 7)\}$ . Since

$$\text{mc}(\text{PSL}(2, 7)) = 1/28 < 1/16$$

then  $G \cong \text{PSL}(2, 5)$  (note that  $\text{mc}(\text{PSL}(2, 5)) = 1/12$ ).

(ii) If  $G$  is semi-simple then  $G \cong \text{PSL}(2, 5)$ .

*Proof.* — Take in  $G$  a minimal normal subgroup  $D$ . Then  $D = D_1 \times \dots \times D_s$  where the  $D_i$ 's are isomorphic non-abelian simple groups. Since (see [1], Chapter 11)  $\text{mc}(D_1) \geq \text{mc}(G) \geq 1/16$ ,  $D \cong \text{PSL}(2, 5)$  by (i) and so  $\text{mc}(D_1) = 1/12$ . Now

$$\text{mc}(D) = \text{mc}(D_1)^s = (1/12)^s \geq 1/16$$

implies that  $s = 1$ . Therefore  $D \cong \text{PSL}(2, 5)$ . Since  $G/C_G(D)$  is isomorphic to a subgroup of  $\text{Aut} D \cong S_5$ ,  $\text{mc}(S_5) = 7/120 < 1/16$ , then  $G/C_G(D) \cong \text{PSL}(2, 5)$ . Because  $D \cap C_G(D) = 1$ ,  $G = D \times C_G(D)$ . Now

$$1/16 \leq \text{mc}(G) = \text{mc}(C_G(D))\text{mc}(D) = (1/12)\text{mc}(C_G(D))$$

implies that  $\text{mc}(C_G(D)) \geq 3/4 > 5/8$ ,  $C_G(D)$  is abelian (Lemma 1(c)),  $C_G(D) = 1$  (since  $G$  is semi-simple), and  $G \cong \text{PSL}(2, 5)$ .

(iii)  $G/S \cong \text{PSL}(2, 5)$ .

This follows from  $mc(G/S) \geq mc(G)$  (P.Gallagher; see [1], Theorem 7.46) and (ii).

(iv) If  $G = G'$  then  $G \in \{PSL(2, 5), SL(2, 5)\}$ .

*Proof.* — By virtue of (iii) we may assume that  $S > 1$ .

Suppose that (iv) is true for all proper epimorphic images of  $G$ . Take in  $S$  a minimal normal subgroup  $R$  of  $G$ , and put  $|R| = p^n$ . Then by the Gallagher Theorem and induction one has  $G/R \in \{PSL(2, 5), SL(2, 5)\}$ .

(1iv)  $G/R \cong PSL(2, 5)$ , i.e.  $R = S$ .

If  $Z(G) > 1$  then  $R = Z(G)$  is isomorphic to a subgroup of the Schur multiplier of  $G/R$  so  $|R| = 2$  and  $G \cong SL(2, 5)$  (Schur). In the sequel we suppose that  $Z(G) = 1$ .

Then  $C_G(R) = R$ , so  $n > 1$ . If  $x \in R^\#$  then  $|G : C_G(x)| \geq 5$ , since index of any proper subgroup of  $PSL(2, 5)$  is at least 5. Let  $k_G(M)$  denote the number of conjugacy classes of  $G$  ( $= G$ -classes), containing elements from  $M$ . Then

$$k_G(R) \leq 1 + |R^\#|/5 = (p^n + 4)/5.$$

If  $x \in G - R$  then  $Z(G) = 1$ , and the structure of  $G/R$  imply  $|G : C_G(x)| \geq 12p$  (indeed,  $x$  does not centralize  $R$  and  $|G/R : C_{G/R}(xR)| \geq 12$ ). Hence

$$k_G(G - R) = k(G) - k_G(R) = |G|mc(G) - k_G(R) \geq 60p^n/16 - (p^n + 4)/5 = (71p^n - 16)/20.$$

Now

- (1)  $|G - R| = 59p^n \geq 12pk_G(G - R) \geq 12p(71p^n - 16)/20,$
- (2)  $5 \times 59p^{n-1} = 295p^{n-1} \geq 213p^n - 48 \geq 426p^{n-1} - 48 \Rightarrow 131p^{n-1} \leq 48,$

a contradiction.

(2iv)  $G/R \cong SL(2, 5)$ .

*Proof.* — Suppose that  $R_1 \neq R$  is a minimal normal subgroup of  $G$ . Then (by induction)

$$RR_1 = R \times R_1 = S, |R_1| = 2, G/R_1 \cong SL(2, 5)$$

and  $G' < G$ , since the multiplier of  $SL(2, 5)$  is trivial, a contradiction. Therefore  $R$  is a unique minimal normal subgroup of  $G$ . Similarly, one obtains  $Z(G) = 1$ .

Let  $p > 2$ . Then  $C_G(R) = R$ . In this case  $Z(S) < R$ , so  $Z(S) = 1$  and  $S$  is a Frobenius group with kernel  $R$  of index 2. As in (1iv) one has

$$k_G(S) = k_G(S - R) + k_G(R) \leq 1 + (p^n + 4)/5 = (p^n + 9)/5.$$

If  $x \in G - S$  then  $|G : C_G(x)| \geq 12p$  and

$$\begin{aligned} k_G(G - S) &= k(G) - k_G(S) = |G|mc(G) - k_G(S) \geq \\ &120p^n/16 - (p^n + 9)/5 = (73p^n - 18)/10, \\ |G - S| &= 118p^n \geq 12pk_G(G - S) \geq 6p(73p^n - 18)/5, \\ 295p^{n-1} &\geq 219p^n - 54 \geq 657p^{n-1} - 54, \\ &54 \geq 362p^{n-1}, \end{aligned}$$

a contradiction.

Let  $p = 2$ . Since  $R$  is the only minimal normal subgroup of  $G$  and  $Z(G) = 1$  then,

$$\begin{aligned} k_G(S) &\leq 1 + (2^{n+1} - 1)/5 = (2^{n+1} + 4)/5, \\ k_G(G - S) &\geq 120 \cdot 2^n / 16 - (2^{n+1} + 4)/5 = (71 \cdot 2^n - 8)/10, \\ 59 \cdot 2^{n+1} = |G - S| &\geq 24k_G(G - S) \geq 24(71 \cdot 2^n - 8)/10, \\ 295 \cdot 2^n &\geq 426 \cdot 2^n - 48, \\ 48 &\geq 131 \cdot 2^n, \end{aligned}$$

a contradiction.

(v) If  $D$  is the last term of the derived series of  $G$  then  $D \in \{\text{PSL}(2, 5), \text{SL}(2, 5)\}$ .

*Proof.* — Since  $D = D'$  and  $\text{mc}(D) \geq \text{mc}(G) \geq 1/16$  the result follows from (iv).

(vi) The subgroup  $D$  from (v) coincides with  $G'$ .

*Proof.* — We have  $D \in \{\text{PSL}(2, 5), \text{SL}(2, 5)\}$  by (v). Since  $Z(G) < D$  we may, by virtue of the Gallagher Theorem [1], Theorem 7.46, assume that  $Z(D) = 1$ . Then  $D \cong \text{PSL}(2, 5)$ . Since

$$\text{Aut}D \cong S_5, \text{mc}(S_5) = 7/120 < 1/16$$

then

$$G/C_G(D) \cong \text{PSL}(2, 5), G = D \times C_G(G),$$

and  $C_G(D)$  is abelian (see (ii)). So  $D = G'$ .

(vii)  $G = SG'$ .

This follows from (iii) and (vi).

(viii)  $|S'| \leq 2$ . In particular,  $S$  is nilpotent and all its Sylow subgroups of odd orders are abelian.

*Proof.* — In fact,  $S' \leq S \cap G' \leq Z(G')$ .

(ix)  $G = S * G'$ , a central product.

*Proof.* — Take an element  $x$  of order 5 in  $G'$ . Since  $G' \cap S \leq Z(G)$ , then

$$G/G' \cap S = G'/G' \cap S \times S/S \cap G'$$

implies that  $\langle x, S \rangle$  is nilpotent. Hence  $\langle S, x \rangle = P \times A$  where  $P \in \text{Syl}_2(S)$  and  $A$  is abelian. As  $x \in A$  then  $x \in C_G(S)$ . Since  $G' = \langle x \in G' | x^5 = 1 \rangle$  it follows that  $G = SG' = S * G'$ .

(x)  $S$  is abelian.

*Proof.* — We have  $G = (S \times G')/Z$  where  $|Z| \leq 2$ . For  $G' \cong \text{PSL}(2, 5)$  our assertion is evident. Now let  $G' \cong \text{SL}(2, 5)$ . Then  $|Z| = 2, Z \geq S'$ . Suppose that  $S$  is non-abelian. Then  $Z = S'$ .

Take  $\chi \in \text{Irr}(G)$ . We consider  $\chi$  as a character of  $G' \times S$  such that  $Z \leq \ker \chi$ . Then  $\chi = \tau\vartheta$  where  $\tau \in \text{Irr}(G'), \vartheta \in \text{Irr}(S)$  and  $\chi_Z = \chi(1)1_Z = \tau(1)\vartheta(1)1_Z$ . Now  $\tau_Z = \tau(1)\lambda, \vartheta_Z = \vartheta(1)\mu$  where  $\lambda, \mu \in \text{Irr}(Z), \lambda\mu = 1_Z$ . Noting that  $|Z| = 2$ , one has

$\lambda = \mu$  and  $\tau_Z = \tau(1)\lambda$ ,  $\vartheta_Z = \vartheta(1)\lambda$ . Since  $S$  is non-abelian then  $\text{cd}S = \{1, m\}$  where  $m^2 = |S : Z(S)|$ .

Suppose that  $\lambda = 1_Z$ .  $\text{Irr}(G')$  has exactly 5 characters containing  $Z$  in their kernels, so for  $\tau$  we have exactly 5 possibilities. Since  $Z \leq \ker \vartheta$  then  $\vartheta \in \text{Lin}(S)$ , and for  $\vartheta$  we have exactly  $|\text{Lin}(S)| = |S|/2$  possibilities. Hence for  $\chi$  we have exactly  $5|S|/2$  possibilities if  $\lambda = 1_Z$ .

Suppose that  $\lambda \neq 1_Z$ . Then  $Z$  is not contained in  $\ker \tau$ , so for  $\tau$  we have exactly  $|\text{Irr}(G')| - |\text{Irr}(G'/Z)| = 9 - 5 = 4$  possibilities. Since  $S' = Z$  is not contained in  $\ker \vartheta$ , then  $\vartheta$  is not linear, and for  $\vartheta$  we have exactly  $(|S| - |S/S'|)/m^2 = |S|/2m^2$  possibilities. For  $\chi$  we have, in this case, exactly  $4|S|/2m^2 = 2|S|/m^2$  possibilities.

Finally,

$$k(G) = 5|S|/2 + 2|S|/m^2$$

and

$$\text{mc}(G) = k(G)/|G| = k(G)/60|S| = 1/24 + 1/30m^2.$$

Since  $m > 1$  then

$$\text{mc}(G) \leq 1/24 + 1/120 = 1/20 < 1/16,$$

a contradiction. Therefore  $S$  is abelian,  $S = Z(G)$  and  $G = G'Z(G)$ . In this case  $\text{mc}(G) \in \{1/12, 3/40\}$ . The theorem is proved.

Let now  $f(G) \geq 1/4$ . Then  $\text{mc}(G) > f(G)^2 \geq 1/16$ , and Theorem 2 is a corollary of Theorem 1. It is easy to see that in this case  $f(G) = f(G') \in \{4/15, 1/4\}$ .

**Proof of Theorem 3.** — In view of Lemma 3 we may assume that  $i_o(G) > 1/4$ . Since

$$\text{mc}(G) \geq f(G)^2 > i_o(G)^2 > 1/16$$

we may apply Theorem 1. By this theorem  $G = G'Z(G)$  where

$$G' \in \{\text{PSL}(2, 5), \text{SL}(2, 5)\}.$$

If  $G' = G$  then  $G \cong \text{PSL}(2, 5)$  since  $i_o(\text{SL}(2, 5)) = 1/120 < 1/4$ . Now let  $G' < G$ .

Suppose that  $\exp(G/G') > 2$ . Let  $M/G'$  be the subgroup generated by all involutions of  $G/G'$ . Then  $i(M) = i(G)$ ,

$$\begin{aligned} i_o(M) &= i(M)/|M| = |G : M|i(G)/|G| = \\ &|G : M|i_o(G) \geq |G : M|/4 \geq 1/2, \end{aligned}$$

and  $M$  is solvable by [1] Theorem 11.24 (since  $f(M) > i_o(M) \geq 1/2$ ), a contradiction. Thus  $\exp(G/G') = 2$ .

If  $G' = \text{PSL}(2, 5)$  then  $G = G' \times Z(G)$ . If  $\exp Z > 2$  and  $M = G' \times \Omega_1(Z(G))$  then

$$i(G) = i(M), \quad i_o(M) = |G : M|i_o(G) > |G : M|/4 \geq 1/2,$$

and  $M$  is solvable (see [1], Theorem 11.24) — a contradiction. Hence if  $G' \cong \text{PSL}(2, 5)$  then  $G = \text{PSL}(2, 5) \times E$  with  $\exp E \leq 2$ .

Now suppose that  $G = G'Z(G)$ ,  $G' \cong \text{SL}(2, 5)$  and  $Z(G)$  is a 2-subgroup. Set  $\langle z \rangle = Z(G')$ .

If  $\exp Z(G) = 2$  then  $Z(G) = \langle z \rangle \times E$ ,  $G = G' \times E$ , and  $i_o(G) < 1/4$ . Assume that  $\exp Z(G) = 4$ . Then

$$G' \cap Z(G) = \langle z \rangle = \Phi(G)$$

where  $\Phi(G)$  is the Frattini subgroup of  $G$ .

Let  $s$  be an element of order 4 in  $Z(G)$ . Then  $Z(G) = \langle s \rangle \times E$  and

$$G = (G' \langle s \rangle) \times E, \exp E \leq 2.$$

Let us calculate  $i_o(H)$  where

$$H = G' \langle s \rangle, Z(H) = \langle s \rangle, o(s) = 4.$$

Take  $P \in \text{Syl}_2(G')$ . Then  $P \cong Q(8)$  contains exactly three distinct cyclic subgroups  $\langle a \rangle, \langle b \rangle, \langle c \rangle$  of order 4, and  $a^2 = b^2 = c^2 = s^2 = z$ . Hence

$$(as)^2 = (bs)^2 = (cs)^2 = 1$$

and it is easy to see that  $i_o(\langle P, s \rangle) = 7$ . Now

$$\begin{aligned} \langle P, s \rangle \in \text{Syl}_2(H), |H : N_H(\langle P, s \rangle)| &= 5, \\ \langle P, s \rangle \cap \langle P, s \rangle^x &= \langle s \rangle \end{aligned}$$

for all  $x \in H - N_H(\langle P, s \rangle)$ . Thus

$$\begin{aligned} i_o(H) &= |H : N_H(\langle P, s \rangle)| i_o(\langle P, s \rangle) - \\ &(|H : N_H(\langle P, s \rangle)| - 1) i_o(\langle s \rangle) = 5 \times 7 - 4 = 31. \end{aligned}$$

Since

$$G = H \times E, |E| = 2^\alpha, \exp E \leq 2,$$

then

$$\begin{aligned} i(G) &= i(H)|E| + |E| - 1 = 31 \cdot 2^\alpha + 2^\alpha - 1 = 32 \cdot 2^\alpha - 1, \\ i_o(G) &= i(G)/|G| = (32 \cdot 2^\alpha - 1)/240 \cdot 2^\alpha < 2/15 < 1/4, \end{aligned}$$

a contradiction. Therefore  $G' \not\cong \text{SL}(2, 5)$  and the theorem is proved.

**Question.** — Find all non-solvable groups  $G$  with  $i_o(G) = 2^{-n}$ ,  $n > 2$ .

There exist four multiplication tables for two-element subsets of group elements (see [3]). These multiplication tables afford the following  $2 \times 2$  squares:

$$\begin{array}{cc|cc|cc|cc} A & B & A & B & A & B & A & B \\ B & A & B & C & C & A & C & D \end{array}$$

Here distinct letters denote distinct elements of a group.

Let us calculate the number  $P(1)$  of the squares of the first type in a finite group  $G$ . If a pair  $\{a, b\}$  of elements of  $G$  affords a square of the first type, then  $a^2 = b^2$ ,  $ab = ba$ . Then  $(a^{-1}b)^2 = 1$ , so  $i = a^{-1}b$  is the involution commuting with  $a$  and  $b$ . If  $i \in \text{Inv}(G)$  (the set of all involutions of  $G$ ),  $x \in C_G(i)$ , then the pair  $(x, xi)$  affords the square of the first type. Therefore  $i \in \text{Inv}(G)$  affords exactly  $|C_G(i)|$  squares of the first type. Let

$$\text{Inv}(G) = K(1) \cup \dots \cup K(r),$$



where  $K(1), \dots, K(r)$  are distinct conjugacy classes of  $G$ . Then

$$P(1) = \sum_{i \in \text{Inv}(G)} |C_G(i)| = \sum_{j=1}^r \sum_{i \in K(j)} |C_G(i)| = r|G|.$$

Thus  $P(1) = r|G|$ , where  $r$  is the number of conjugacy classes of involutions in  $G$ .

By analogy, we may prove that the number  $P(1, 2)$  of commutative squares in the multiplicative table of  $G$  is equal to  $k(G)|G|$ . The number  $P(2)$  of squares of the second type in the multiplicative table of  $G$  is therefore equal to  $P(2) = P(1, 2) - P(1) = (k(G) - r)|G|$ . If  $p(n)$  is the fraction of squares of the  $n$ -th type in the multiplicative table of  $G$  then

$$p(1) = r/|G|, \quad p(2) = (k(G) - r)/|G| = \text{mc}(G) - p(1).$$

It is easy to see that the number  $P(1) + P(3)$  of squares of the first and the third type in the multiplicative table of  $G$  is equal to  $|G|s$  where  $s$  is the number of real classes (a class  $K$  of  $G$  is said to be real if  $x \in K \Rightarrow x^{-1} \in K$ ). Thus

$$P(4) \equiv 0 \pmod{|G|}.$$

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