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RANDOM PERTURBATIONS OF NONUNIFORMLY EXPANDING MAPS

by

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Abstract. — We give both sufficient conditions and necessary conditions for the stochastic stability of nonuniformly expanding maps either with or without critical sets. We also show that the number of probability measures describing the statistical asymptotic behaviour of random orbits is bounded by the number of SRB measures if the noise level is small enough. As an application of these results we prove the stochastic stability of certain classes of nonuniformly expanding maps introduced in [**Vi1**] and [**ABV**].

1. Introduction

Dynamical systems theory has, among its main goals, the description of the typical behaviour of orbits as time goes to infinity, and understanding how this behaviour is modified under small perturbations of the system. This work refers to the study of the latter problem from a probabilistic point of view.

Given a map f from a manifold M into itself, let $(x_n)_{n\geqslant 1}$ be the orbit of a given point $x_0 \in M$, that is $x_{n+1} = f(x_n)$ for every $n \geqslant 1$. Consider the sequence of time averages of Dirac measures δ_{x_j} along the orbit of x_0 from time 0 to n. A special interest lies on the study of the convergence of such time averages for a "large" set of points $x_0 \in M$ and the properties of their limit measures. In this direction, we refer the work of Sinai [**Si**] for Anosov diffeomorphisms, later extended by Ruelle and Bowen [**BR, Ru**] for Axiom A diffeomorphisms and flows. In the context of systems with no uniform hyperbolic structure Jakobson [**Ja**] proved the existence of such measures for certain quadratic transformations of the interval exhibiting chaotic behaviour.

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Another important contribution on this subject was given by Benedicks and Young [**BY1**], based on the previous work of Benedicks and Carleson [**BC1, BC2**], where this kind of measures were constructed for Hénon two dimensional maps exhibiting strange attractors. The recent work of Alves, Bonatti and Viana [**ABV**] shows that such measures exist in great generality for systems exhibiting some nonuniformly expanding behaviour.

The notion of stability that most concerns us can be formulated in the following way. Assume that, instead of time averages of Dirac measures supported on the iterates of $x_0 \in M$, we consider time averages of Dirac measures δ_{x_j}, where at each iteration we take x_{j+1} close to $f(x_j)$ with a controlled error. One is interested in studying the existence of limit measures for these time averages and their relation to the analogous ones for unperturbed orbits, that is, the stochastic stability of the initial system.

Systems with some uniformly hyperbolic structure are quite well understood and stability results have been established in general by Kifer and Young; see [**Ki1, Ki2**] and [**Yo**]. The knowledge of the stochastic behaviour of systems that do not exhibit such uniform expansion/contraction is still very incomplete. Important results on this subject were obtained by Katok, Kifer [**KK**], Benedicks, Young [**BY1**], Baladi and Viana [**BV**] for certain quadratic maps of the interval. Another important contribution is the announced work of Benedicks and Viana for Hénon-like strange attractors. As far as we know these are the only results of this type for systems with no uniform expanding behaviour.

In this work we present both sufficient conditions and necessary conditions for the stochastic stability of nonuniformly expanding dynamical systems. As an application of these results we prove that the classes of nonuniformly expanding maps introduced in [**Vi1**] and [**ABV**] are stochastically stable.

1.1. Statement of results. — Let $f : M \to M$ be a smooth map defined on a compact riemannian manifold M. We fix some normalized riemannian volume form m on M that we call *Lebesgue measure.*

Given μ an f-invariant Borel probability measure on M, we say that μ is an *SRB measure* if, for a positive Lebesgue measure set of points $x \in M$, the averaged sequence of Dirac measures along the orbit $(f^n(x))_{n\geqslant 0}$ converges in the weak* topology to μ, that is,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi\big(f^n(x)\big) = \int \varphi \, d\mu \tag{1}$$

for every continuous map $\varphi : M \to \mathbb{R}$. We define the *basin* of μ as the set of those points x in M for which (1) holds for all continuous φ. The maps to be considered in this work will only have a finite number of SRB measures whose basins cover the whole manifold M, up to a set of zero Lebesgue measure.

We are interested in studying random perturbations of the map f. For that, we take a continuous map

$$\begin{aligned} \Phi : T &\longrightarrow C^2(M, M) \\ t &\longmapsto f_t \end{aligned}$$

from a metric space T into the space of C^2 maps from M to M, with $f = f_{t^*}$ for some fixed $t^* \in T$. Given $x \in M$ we call the sequence $\big(f_{\underline{t}}^n(x)\big)_{n\geqslant 1}$ a *random orbit* of x, where $\underline{t}$ denotes an element $(t_1, t_2, t_3, \dots)$ in the product space $T^{\mathbb{N}}$ and

$$f_{\underline{t}}^n = f_{t_n} \circ \cdots \circ f_{t_1} \quad \text{for } n \geqslant 1.$$

We also take a family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on T such that $(\operatorname{supp}\theta_\varepsilon)_{\varepsilon>0}$ is a nested family of connected compact sets and $\operatorname{supp}\theta_\varepsilon \to \{t^*\}$ when $\varepsilon \to 0$. We will also assume some quite general nondegeneracy conditions on Φ and $(\theta_\varepsilon)_{\varepsilon>0}$ (see the beginning of Section 3) and refer to $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ as a *random perturbation* of f.

In the context of random perturbations of a map we say that a Borel probability measure μ^ε on M is *physical* if for a positive Lebesgue measure set of points $x \in M$, the averaged sequence of Dirac probability measures $\delta_{f_{\underline{t}}^n(x)}$ along random orbits $\big(f_{\underline{t}}^n(x)\big)_{n\geqslant 0}$ converges in the weak* topology to μ^ε for $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. That is,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f_{\underline{t}}^n(x)\big) = \int \varphi \, d\mu^\varepsilon \quad \text{for all continuous } \varphi\colon M \to \mathbb{R} \tag{2}$$

and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. We denote the set of points $x \in M$ for which (2) holds by $B(\mu^\varepsilon)$ and call it the *basin of* μ^ε. The map $f\colon M \to M$ is said to be *stochastically stable* if the weak* accumulation points (when $\varepsilon > 0$ goes to zero) of the physical probability measures of f are convex linear combinations of the (finitely many) SRB measures of f.

1.1.1. Local diffeomorphisms. — Let $f : M \to M$ be a C^2 local diffeomorphism of the manifold M. We say that f is *nonuniformly expanding* if there is some constant $c > 0$ for which

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leqslant -c < 0 \tag{3}$$

for Lebesgue almost every $x \in M$. It was proved in [**ABV**] that for a nonuniformly expanding local diffeomorphism f the following holds:

(P) *There is a finite number of ergodic absolutely continuous* (SRB) *f-invariant probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure subset of M. Moreover, every absolutely continuous f-invariant probability measure μ may be written as a convex linear combination of $\mu_1, \dots, \mu_p$: there are real numbers $w_1, \dots, w_p \geqslant 0$ with $w_1 + \cdots + w_p = 1$ for which $\mu = w_1\mu_1 + \cdots + w_p\mu_p$.*

The proof of the previous result was based on the existence of α-hyperbolic times for the points in M: given $0<\alpha<1$, we say that $n\in\mathbb{Z}^+$ is a α-*hyperbolic time* for the point $x\in M$ if

$$\prod_{j=n-k}^{n-1}\|Df(f^j(x))^{-1}\|\leqslant\alpha^k\quad\text{for every}\quad 1\leqslant k\leqslant n. \tag{4}$$

The existence of (a positive frequency of) α-hyperbolic times for points $x\in M$ is a consequence of the hypothesis of nonuniform expansion of the map f and permits us to define a map $h: M\to\mathbb{Z}^+$ giving the first hyperbolic time for m almost every $x\in M$.

In the context of random perturbations of a nonuniformly expanding map we are also able to prove a result on the finitness of physical measures.

Theorem A. — *Let $f\colon M\to M$ be a C^2 nonuniformly expanding local diffeomorphism. If $\varepsilon>0$ is sufficiently small, then there are physical measures $\mu_1^\varepsilon,\dots,\mu_\ell^\varepsilon$ (with ℓ not depending on ε) such that:*

(1) *for each $x\in M$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t}\in T^{\mathbb{N}}$, the average of Dirac measures $\delta_{f_{\underline{t}}^n(x)}$ converges in the weak* topology to some μ_i^ε with $1\leqslant i\leqslant\ell$;*

(2) *for each $1\leqslant i\leqslant\ell$ we have*

$$\mu_i^\varepsilon=w^*\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int\big(f_{\underline{t}}^j\big)_*\big(m\|B(\mu_i^\varepsilon)\big)\,d\theta_\varepsilon^{\mathbb{N}}(\underline{t}),$$

where $m\|B(\mu_i^\varepsilon)$ is the normalization of the Lebesgue measure restricted to $B(\mu_i^\varepsilon)$;

(3) *if f is topologically transitive, then $\ell=1$.*

We say that the map f is *nonuniformly expanding for random orbits* if there is some constant $c>0$ such that for $\varepsilon>0$ small enough

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f_{\underline{t}}^j(x))^{-1}\|\leqslant -c<0, \tag{5}$$

for $\theta_\varepsilon^{\mathbb{N}}\times m$ almost every $(\underline{t},x)\in T^{\mathbb{N}}\times M$. Similarly to the deterministic situation, condition (5) permits us to introduce a notion of α-hyperbolic times for points in $T^{\mathbb{N}}\times M$ and define a map

$$h_\varepsilon\colon T^{\mathbb{N}}\times M\longrightarrow\mathbb{Z}^+$$

by taking $h_\varepsilon(\underline{t},x)$ the first α-hyperbolic time for the point $(\underline{t},x)\in T^{\mathbb{N}}\times M$ (see Section 2). Assuming that h_ε is integrable with respect to $\theta_\varepsilon^{\mathbb{N}}\times m$, then

$$\|h_\varepsilon\|_1=\sum_{k=0}^{\infty}k\,(\theta_\varepsilon^{\mathbb{N}}\times m)\big(\{(\underline{t},x)\colon h_\varepsilon(\underline{t},x)=k\}\big)<\infty. \tag{6}$$

We say that the family $(h_\varepsilon)_{\varepsilon>0}$ has *uniform L^1-tail*, if the series in (6) converges uniformly to $\|h_\varepsilon\|_1$ (as a series of functions of the variable ε).

Theorem B. — *Let $f\colon M\to M$ be a nonuniformly expanding C^2 local diffeomorphism.*

(1) *If f is stochastically stable, then f is nonuniformly expanding for random orbits.*

(2) *If f is nonuniformly expanding for random orbits and $(h_\varepsilon)_\varepsilon$ has uniform L^1-tail, then f is stochastically stable.*

We should emphasize that we do not know if condition (2) in Theorem B is really necessary. No example of a stochastically stable map which does not satisfy the uniform L^1-tail property is known.

1.1.2. Maps with critical sets. — Similar results to those presented for random perturbations of local diffeomorphisms will also be obtained for maps with critical sets in the sense of [**ABV**]. We start by describing the class of maps that we are going to consider. Let $f\colon M\to M$ be a continuous map of the compact manifold M that fails to be a C^2 local diffeomorphism on a critical set $\mathcal{C}\subset M$ with zero Lebesgue measure. We assume that f *behaves like a power of the distance* close to the critical set $\mathcal{C}$: there are constants $B>1$ and $\beta>0$ for which

(S1) $\displaystyle \frac{1}{B}\mathrm{dist}(x,\mathcal{C})^{\beta}\leqslant\frac{\|Df(x)v\|}{\|v\|}\leqslant B\mathrm{dist}(x,\mathcal{C})^{-\beta}$;

(S2) $\displaystyle \big|\log\|Df(x)^{-1}\|-\log\|Df(y)^{-1}\|\,\big|\leqslant B\frac{\mathrm{dist}(x,y)}{\mathrm{dist}(x,\mathcal{C})^{\beta}}$;

(S3) $\displaystyle \big|\log|\det Df(x)^{-1}|-\log|\det Df(y)^{-1}|\,\big|\leqslant B\frac{\mathrm{dist}(x,y)}{\mathrm{dist}(x,\mathcal{C})^{\beta}}$;

for every $x,y\in M\smallsetminus\mathcal{C}$ with $\mathrm{dist}(x,y)<\mathrm{dist}(x,\mathcal{C})/2$ and $v\in T_xM$. Given $\delta>0$ we define the *δ-truncated distance* from $x\in M$ to $\mathcal{C}$

$$\mathrm{dist}_\delta(x,\mathcal{C})=\begin{cases}1 & \text{if } \mathrm{dist}(x,\mathcal{C})\geqslant\delta,\\ \mathrm{dist}(x,\mathcal{C}) & \text{otherwise.}\end{cases}$$

Assume that f is a nonuniformly expanding map, in the sense that there is $c>0$ such that the limit in (3) holds for Lebesgue almost every $x\in M$ (recall that we are taking $\mathcal{C}$ with zero Lebesgue measure) and, moreover, suppose that the orbits of f have *slow approximation to the critical set*: given small $\gamma>0$ there is $\delta>0$ such that

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}-\log\mathrm{dist}_\delta(f^j(x),\mathcal{C})\leqslant\gamma \tag{7}$$

for Lebesgue almost every $x\in M$. The results in [**ABV**] show that in this situation we obtain the same conclusion on the finiteness of SRB measures for such an f, also holding property (P).

In order to prove the stochastic stability of maps with critical sets we need to restrict the class of perturbations we are going to consider: we take maps f_t with the same critical set $\mathcal{C}$ and impose that

$$Df_t(x)=Df(x)\quad\text{for every } x\in M\smallsetminus\mathcal{C} \text{ and } t\in T. \tag{8}$$

This may be implemented, for instance, in parallelizable manifolds (with an additive group structure, e.g. tori $\mathbb{T}^d$ or cylinders $\mathbb{T}^{d-k} \times \mathbb{R}^k$) by considering

$$T = \{t \in \mathbb{R}^d \colon \|t\| \leqslant \varepsilon_0\}$$

for some $\varepsilon_0 > 0$, θ_ε the normalized Lebesgue measure on the ball of radius $\varepsilon \leqslant \varepsilon_0$, and taking $f_t = f + t$; that is, adding at each step a random noise to the unperturbed dynamics.

For the case of maps with critical sets we also need to impose an analog of condition (7) for random orbits; we assume *slow approximation of random orbits to the critical set*: given any small $\gamma > 0$ there is $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f_{\underline{t}}^j(x), \mathcal{C}) \leqslant \gamma \tag{9}$$

for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ and small $\varepsilon > 0$. Results similar to those presented for local diffeomorphisms on the finiteness of physical measures can also be obtained in this case.

Theorem C. — *Let $f\colon M \to M$ be a C^2 nonuniformly expanding map behaving like a power of the distance close to the critical set $\mathcal{C}$, and whose orbits have slow approximation to $\mathcal{C}$. If f is nonuniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}$, then we arrive at the same conclusions as in Theorem A.*

The property of nonuniform expansion for random orbits, together with the slow approximation of random orbits to the critical set permit us to introduce a notion of (α, δ)-hyperbolic times for points in $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ and define a map

$$h_\varepsilon \colon T^{\mathbb{N}} \times M \longrightarrow \mathbb{Z}^+,$$

by taking $h_\varepsilon(\underline{t}, x)$ the first (α, δ)-hyperbolic time for the point $(\underline{t}, x) \in T^{\mathbb{N}} \times M$, see Section 2. Assuming that h_ε is integrable with respect to $\theta_\varepsilon \times m$, then we obtain an analog to (6), which enables us to define a notion of *uniform L^1-tail* exactly in the same way as before.

Due to the fact that $\log \|Df^{-1}\|$ is not a continuous map (it is not even everywhere defined) we are not able to present in this setting a result similar to Theorem B in all its strength. However, we obtain the same kind of conclusion of the second item of Theorem B.

Theorem D. — *Let $f\colon M \to M$ be nonuniformly expanding C^2 map behaving like a power of the distance close to its critical set $\mathcal{C}$ and whose orbits have slow approximation to $\mathcal{C}$. Assume that f is nonuniformly expanding for random orbits and random orbits have slow approximation to $\mathcal{C}$. If $(h_\varepsilon)_\varepsilon$ has uniform L^1-tail, then f is stochastically stable.*

As a major application of the previous theorem we are thinking of a class of maps on the cylinder $S^1 \times \mathbb{R}$ introduced in [**Vi1**]. Subsequent works [**Al**] and [**AV**] showed that such systems are topologically mixing (thus transitive) and have a unique SRB measure. The work [**AV**] also shows that these SRB measures vary continuously with the map, which means that time averages of continuous functions are only slightly affected when the system is perturbed. Although this points in a direction of statistical stability, this does not imply the stochastic stability of such systems as we defined above.

The class of nonuniformly expanding maps (with critical sets) introduced by M. Viana can be described as follows. Let $a_0 \in (1,2)$ be such that the critical point $x=0$ is pre-periodic for the quadratic map $Q(x)=a_0-x^2$. Let $S^1=\mathbb{R}/\mathbb{Z}$ and $b: S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(s)=\sin(2\pi s)$. For fixed small $\alpha>0$, consider the map

$$\begin{aligned} \widehat{f}: S^1\times\mathbb{R} &\longrightarrow \quad S^1\times\mathbb{R} \\ (s,x) &\longmapsto \big(\widehat{g}(s),\widehat{q}(s,x)\big) \end{aligned}$$

where $\widehat{g}$ is the uniformly expanding map of the circle defined by $\widehat{g}(s)=ds \pmod{\mathbb{Z}}$ for some $d\geqslant 16$, and $\widehat{q}(s,x)=a(s)-x^2$ with $a(s)=a_0+\alpha b(s)$. It is easy to check that for $\alpha>0$ small enough there is an interval $I\subset(-2,2)$ for which $\widehat{f}(S^1\times I)$ is contained in the interior of $S^1\times I$. Thus, any map f sufficiently close to $\widehat{f}$ in the C^0 topology has $S^1\times I$ as a forward invariant region. We consider from here on these maps f close to $\widehat{f}$ restricted to $S^1\times I$. Taking into account the expression of $\widehat{f}$ it is not difficult to check that $\widehat{f}$ (and any map f close to $\widehat{f}$ in the C^2 topology) behaves like a power of the distance close to the critical set.

Theorem E. — *If f is sufficiently close to $\widehat{f}$ in the C^3 topology then f is nonuniformly expanding and its orbits have slow approximation to the critical set. Moreover, if the noise level of a random perturbation of f is sufficiently small, then*

(1) *f is nonuniformly expanding for random orbits;*
(2) *random orbits have slow approximation to the critical set;*
(3) *the family of hyperbolic time maps $(h_\varepsilon)_\varepsilon$ has uniform L^1-tail.*

As an immediate consequence of Theorems C, D and E we have that Viana maps are stochastically stable. An application of Theorems A and B will also be given in Section 6 for an open class of local diffeomorphisms introduced in [**ABV**, Appendix A].

2. Distortion bounds

In this section we generalize some of the results in [**Al**] and [**ABV**] for the setting of stochastic perturbations of a nonuniformly expanding map. These results will be proved in the setting of maps with critical sets. Then everything follows in the same way for local diffeomorphisms if we think of $\mathcal{C}$ as being equal to the empty set, with the only exception of a particular point that we clarify in Remark 2.4 below (due to the fact that we are not assuming condition (8) for maps with no critical sets). For the next definition we take $0 < b < \min\{1/2, 1/(2\beta)\}$.

Definition 2.1. — Given $0 < \alpha < 1$ and $\delta > 0$, we say that $n \in \mathbb{Z}^+$ is a (α, δ)-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ if

$$\prod_{j=n-k}^{n-1} \|Df_{t_{j+1}}(f_{\underline{t}}^j(x))^{-1}\| \leqslant \alpha^k \quad \text{and} \quad \operatorname{dist}_\delta(f_{\underline{t}}^{n-k}(x), \mathcal{C}) \geqslant \alpha^{bk}$$

for every $1 \leqslant k \leqslant n$.

The following lemma, due to Pliss [**Pl**], provides the main tool in the proof of the existence of hyperbolic times for points with nonuniform expansion on random orbits.

Lemma 2.2. — *Let $H \geqslant c_2 > c_1 > 0$ and $\zeta = (c_2 - c_1)/(H - c_1)$. Given real numbers $a_1, \dots, a_N$ satisfying*

$$\sum_{j=1}^{N} a_j \geqslant c_2 N \quad \textit{and} \quad a_j \leqslant H \ \textit{ for all } \ 1 \leqslant j \leqslant N,$$

there are $\ell > \zeta N$ and $1 < n_1 < \cdots < n_\ell \leqslant N$ such that

$$\sum_{j=n+1}^{n_i} a_j \geqslant c_1 \cdot (n_i - n) \ \textit{ for each } \ 0 \leqslant n < n_i, \ i = 1, \dots, \ell.$$

Proof. — See [**ABV**, Lemma 3.1]. □

Proposition 2.3. — *There are $\alpha > 0$ and $\delta > 0$ for which $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ has some (α, δ)-hyperbolic time.*

Proof. — Let $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ be a point satisfying (5). For large N we have

$$-\sum_{j=0}^{N-1} \log \left\| Df(f_{\underline{t}}^j(x))^{-1} \right\| \geqslant \frac{c}{2} N > 0,$$

by definition of nonuniform expansion on random orbits. Fixing $\rho > \beta$ we see that condition (S1) implies

$$\left|\log \left\|Df(x)^{-1}\right\|\right| \leqslant \rho \left|\log \operatorname{dist}(x, \mathcal{C})\right| \tag{10}$$

for every x in a neighborhood V of $\mathcal{C}$. Now we take $\gamma_1 > 0$ so that $\rho\gamma_1 \leqslant c/10$ and let $\delta_1 > 0$ be small enough to get

$$-\sum_{j=0}^{N-1} \log \operatorname{dist}_{\delta_1}(f_{\underline{t}}^j(x), S) \leqslant \gamma_1 N \quad \text{for large } N, \tag{11}$$

which is possible after property (7) of slow approximation to $\mathcal{C}$. Moreover, fixing $H \geqslant \rho|\log\delta|$ sufficiently large in order that it be also an upper bound for for the set $\{-\log\|Df_{\underline{t}}^{-1}\| : t \in T,\ x \in M \setminus V\}$, then the set

$$E = \{1 \leqslant j \leqslant N : -\log\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\| > H\}$$

is such that $f_{\underline{t}}^{j-1}(x) \in V$ for all $j \in E$ and

$$\rho\left|\log\operatorname{dist}(f_{\underline{t}}^{j-1}(x),\mathcal{C})\right| > -\log\left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| > H \geqslant \rho|\log\delta|$$

i.e., $\operatorname{dist}(f_{\underline{t}}^{j-1}(x),\mathcal{C}) < \delta_1$, in particular $\operatorname{dist}_{\delta_1}(f_{\underline{t}}^{j-1}(x),\mathcal{C}) = \operatorname{dist}(f_{\underline{t}}^{j-1}(x),\mathcal{C}) < \delta_1$ for all $j \in E$. Hence, defining

$$a_j = \begin{cases} -\log\left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| & \text{if } j \notin E \\ 0 & \text{if } j \in E \end{cases}$$

it holds $a_j \leqslant H$ for $1 \leqslant j \leqslant N$, and (10) and (11) imply

$$-\sum_{j\in E}\log\left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| \leqslant \rho\sum_{j\in E}\left|\log\operatorname{dist}(f_{\underline{t}}^{j-1}(x),\mathcal{C})\right| \leqslant \rho\gamma_1 N.$$

Since $\rho\gamma_1 \leqslant c/10$ we deduce

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N}\Big(-\log\left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\|\Big) - \sum_{j\in E}\Big(-\log\left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\|\Big) \geqslant \frac{2}{5}cN.$$

By the previous arguments we may apply Lemma 2.2 to the sequence a_j with $c_1 = c/5$ and $c_2 = 2c/5$ (we may suppose $H > c_1$ too by increasing H if needed). Thus there are $\zeta_1 > 0$ and $\ell_1 > \zeta_1 N$ times $1 \leqslant q_1 < \cdots < q_{\ell_1} \leqslant N$ such that

$$\sum_{j=n+1}^{q_i} -\log\left\|Df(f_{\underline{t}}^{j-1}(x))^{-1}\right\| \geqslant \sum_{j=n+1}^{q_i} a_j \geqslant \frac{c}{2}(q_i - n) \tag{12}$$

for every $0 \leqslant n < q_i$, $i = 1,\dots,\ell_1$. We observe that (12) is just the first part of the requirements on (α,δ)-hyperbolic times for $(\underline{t},x)$ if $\alpha = \exp(c/5)$.

Now we apply again Lemma 2.2, this time to the sequence $a_j = \log\operatorname{dist}_{\delta_2}(f_{\underline{t}}^{j-1}(x),\mathcal{C})$, where $\delta_2 > 0$ is small enough so that for $\gamma_2 > 0$ with $2\gamma_2(bc)^{-1} < \zeta_1$ we have by assumption (7)

$$\sum_{j=0}^{N-1}\log\operatorname{dist}_{\delta_2}(f_{\underline{t}}^{j}(x),\mathcal{C}) \geqslant -\gamma_2 N \quad \text{for large } N.$$

Defining $c_1 = bc/2$, $c_2 = -\gamma_2$, $H = 0$ and

$$\zeta_2 = \frac{c_2 - c_1}{H - c_1} = 1 - \frac{2\gamma_2}{bc},$$

Lemma 2.2 ensures that there are $\ell_2 \geqslant \zeta_2 N$ times $1 \leqslant r_1 < \cdots < r_{\ell_2} \leqslant N$ satisfying

$$\sum_{j=n+1}^{r_i} \log \operatorname{dist}_{\delta_2}(f_{\underline{t}}^{j+1}(x), \mathcal{C}) \geqslant \frac{bc}{2}(r_i - n) \tag{13}$$

for every $0 \leqslant n < r_i$, $i = 1, \dots, \ell_2$. Let us note that the condition on γ_2 assures $\zeta_1 + \zeta_2 > 1$. So if $\zeta = \zeta_1 + \zeta_2 - 1$, then there must be $\ell = (\ell_1 + \ell_2 - N) \geqslant \zeta N$ and $1 \leqslant n_1 < \cdots < n_\ell \leqslant N$ for which (12) and (13) both hold. This means that for $1 \leqslant i \leqslant \ell$ and $1 \leqslant k \leqslant n_i$ we have

$$\prod_{j=n_i-k}^{n_i} \left\| Df(f_{\underline{t}}^j(x))^{-1} \right\| \leqslant \alpha^k \quad \text{and} \quad \operatorname{dist}_{\delta_2}(f_{\underline{t}}^{n_i-k}(x), \mathcal{C}) \geqslant \alpha^{bk},$$

and hence these n_i are (α, δ)-hyperbolic times for $(\underline{t}, x)$, with $\delta = \delta_2$ and $\alpha = \exp(c/5)$. It follows that for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ there are (positive frequency of) times $n \in \mathbb{Z}^+$ for which

$$\prod_{j=n-k}^{n-1} \|Df(f_{\underline{t}}^j(x))^{-1}\| \leqslant \alpha^k \quad \text{and} \quad \operatorname{dist}_\delta(f_{\underline{t}}^{n-k}(x), \mathcal{C}) \geqslant \alpha^{bk} \tag{14}$$

for every $1 \leqslant k \leqslant n$. Now the conclusion of the lemma is a direct consequence of assumption (8). □

Remark 2.4. — In the setting of random perturbations of a local diffeomorphism f we may also derive from the first part of (14) the existence of hyperbolic times for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ without assuming condition (8). Actually, let $(\underline{t}, x)$ be a point in $T^{\mathbb{N}} \times M$ for which the first part of (14) holds. Taking the perturbations f_t in a sufficiently small C^1-neighborhood of f, then

$$\|Df_t(y)^{-1}\| \leqslant \frac{1}{\sqrt{\alpha}} \|Df(y)^{-1}\|$$

for every $y \in M$, which together with (14) gives

$$\prod_{j=n-k}^{n-1} \|Df_t(f_{\underline{t}}^j(x))^{-1}\| \leqslant \prod_{j=n-k}^{n-1} \frac{1}{\sqrt{\alpha}} \|Df(f_{\underline{t}}^j(x))^{-1}\| \leqslant \alpha^{k/2}.$$

In the context of maps with no critical sets this n may be defined as a $\sqrt{\alpha}$-hyperbolic time for $(\underline{t}, x)$ and all the results that we present below hold with $\sqrt{\alpha}$-hyperbolic times replacing (α, δ)-hyperbolic times for maps with critical sets.

Proposition 2.3 allows us to introduce a map

$$h_\varepsilon \colon T^{\mathbb{N}} \times M \longrightarrow \mathbb{Z}^+,$$

by taking $h_\varepsilon(\underline{t}, x)$ as the first (α, δ)-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$. We assume henceforth that the family $(h_\varepsilon)_{\varepsilon > 0}$ has uniform L^1-tail. For the next lemma we fix $\delta_1 > 0$ in such a way that $4\delta_1 < \min\{\delta, \delta^\beta |\log \alpha|\}$.

Lemma 2.5. — *Given any $1 \leqslant j \leqslant n$, we have*

$$\|Df(y)^{-1}\| \leqslant \alpha^{-1/2}\|Df(f_{\underline{t}}^{n-j}(x))^{-1}\|$$

for every y in the ball of radius $2\delta_1\alpha^{j/2}$ around $f_{\underline{t}}^{n-j}(x)$.

Proof. — We are assuming $\mathrm{dist}_\delta(f_{\underline{t}}^{n-j}(x), \mathcal{C}) \geqslant \alpha^j$ since n is a (α, δ)-hyperbolic time for $(\underline{t}, x)$. This means that

$$\mathrm{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C}) = \mathrm{dist}_\delta(f_{\underline{t}}^{n-j}(x), \mathcal{C}) \geqslant \alpha^{bj} \text{ or else } \mathrm{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C}) \geqslant \delta.$$

Either way it holds $\mathrm{dist}(y, f_{\underline{t}}^{n-j}(x)) \geqslant \mathrm{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C})/2$ because $b < 1/2$ and $\delta_1 < \delta/4 < 1/4$ for all y in the ball of radius $2\delta_1\alpha^{j/2}$ around $f_{\underline{t}}^{n-j}(x)$. Therefore condition (S2) implies

$$\log \frac{\|Df(y)^{-1}\|}{\|Df(f_{\underline{t}}^{n-j}(x))^{-1}\|} \leqslant B\frac{\mathrm{dist}(f_{\underline{t}}^{n-j}(x), y)}{\mathrm{dist}(f_{\underline{t}}^{n-j}(x), \mathcal{C})^\beta} \leqslant B\frac{2\delta_1\alpha^{j/2}}{\min\{\alpha^{b\beta j}, \delta^\beta\}}.$$

But $\alpha, \delta < 1$ and $b\beta < 1/2$ so $\alpha^{j/2} < \alpha^{b\beta j}$ and thus the right hand side of the last expression is bounded from above by $2B\delta_1\delta^{-\beta}$. The assumptions on δ_1 assure this last bound to be smaller than $\log\alpha^{-1/2}$, which implies the statement. □

Proposition 2.6. — *There is $\delta_1 > 0$ such that if n is (α, δ)-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times M$, then there is a neighborhood $V_n(\underline{t}, x)$ of x in M such that*

(1) *$f_{\underline{t}}^n$ maps $V_n(\underline{t}, x)$ diffeomorphically onto the ball of radius δ_1 around $f_{\underline{t}}^n(x)$;*
(2) *for every $1 \leqslant k \leqslant n$ and $y, z \in V_k(\underline{t}, x)$*

$$\mathrm{dist}(f_{\underline{t}}^{n-k}(y), f_{\underline{t}}^{n-k}(z)) \leqslant \alpha^{k/2}\mathrm{dist}(f_{\underline{t}}^n(y), f_{\underline{t}}^n(z)).$$

Proof. — The proof will be by induction on $j \geqslant 1$. First we show that there is a well defined branch of f^{-j} on a ball of small enough radius around $f_{\underline{t}}^j(x)$. Now we observe that Lemma 2.5 gives for $j = 1$

$$\|Df(y)^{-1}\| \leqslant \alpha^{-1/2}\|Df(f_{\underline{t}}^{n-1}(x))^{-1}\| \leqslant \alpha^{1/2},$$

because n is a (α, δ)-hyperbolic time for $(\underline{t}, x)$. This means that f is a $\alpha^{-1/2}$-dilation in the ball of radius $2\delta_1\alpha^{1/2}$ around $f_{\underline{t}}^{n-1}(x)$. Consequently there is some neighborhood $V_1(\underline{t}, x)$ of $f_{\underline{t}}^{n-1}(x)$ inside the ball of radius $2\delta_1\alpha^{1/2}$ that is diffeomorphic to the ball of radius δ_1 around $f_{\underline{t}}^n(x)$ through f_{t_n}, when f is a map with critical set satisfying (8).

For $j \geqslant 1$ let us suppose that we have obtained a neighborhood $V_j(\underline{t}, x)$ of $f_{\underline{t}}^{n-j}(x)$ such that $f_{t_n} \circ \cdots \circ f_{t_{n-j+1}} \mid V_j(\underline{t}, x)$ is a diffeomorphism onto the ball of radius δ_1 around $f_{\underline{t}}^n(x)$ with

$$(15) \qquad \|Df(f_{t_{n-j+i+1}} \circ \cdots \circ f_{t_{n-j+1}}(z))^{-1}\| \leqslant \alpha^{-1/2}\|Df(f_{\underline{t}}^{n-j+i+1}(x))^{-1}\|$$

for all $z \in V_j(\underline{t}, x)$ and $0 \leqslant i < j$. Then, by Lemma 2.5 and under the assumption that n is a (α, δ)-hyperbolic time for x,

$$\begin{aligned}
\left\| D\big(f_{t_n} \circ \cdots \circ f_{t_{n-j}}(y)\big)^{-1} \right\| &\leqslant \prod_{i=0}^{j} \left\| Df_{t_{n-j+i}}\big(f_{t_{n-j+i-1}} \circ \cdots \circ f_{t_{n-j}}(y)\big)^{-1} \right\| \\
&\leqslant \prod_{i=0}^{j} \alpha^{-1/2} \left\| Df_{t_{n-j+i}}\big(f_{\underline{t}}^{n-j+i-1}(x)\big)^{-1} \right\| \\
&\leqslant (\alpha^{-1/2})^{j+1} \cdot \alpha^{j+1} = \alpha^{(j+1)/2}
\end{aligned}$$

for every y on the ball of radius $2\delta_1 \alpha^{(j+1)/2}$ around $f_{\underline{t}}^{n-j-1}(x)$ whose image $f_{t_{n-j}}(y)$ is in $V_j(\underline{t}, x)$ (above we convention $f_{t_{n-j+i-1}} \circ \cdots \circ f_{t_{n-j}}(y) = y$ for $i = 0$).

This shows that the derivative of $f_{t_n} \circ \cdots \circ f_{t_{n-j}}$ is a $\alpha^{-(j+1)/2}$-dilation on the intersection of $f_{t_{n-j}}^{-1}\big(V_j(\underline{t}, x)\big)$ with the ball of radius $2\delta_1 \alpha^{(j+1)/2}$ around $f_{\underline{t}}^{n-j-1}(x)$, and hence there is an inverse branch of $f_{t_n} \circ \cdots \circ f_{t_{n-j}}$ defined on the ball of radius δ_1 around $f_{\underline{t}}^{n}(x)$. Thus we may define $V_{j+1}(\underline{t}, x)$ as the image of the ball of radius δ_1 around $f_{\underline{t}}^{n}(x)$ under this inverse branch, and recover the induction hypothesis for $j+1$. In this manner we get neighborhoods $V_j(\underline{t}, x)$ of $f_{\underline{t}}^{n-j}(x)$ as above for all $1 \leqslant j \leqslant n$. □

Corollary 2.7. — *There is a constant $C_1 > 0$ such that if $\underline{t} \in T^{\mathbb{N}}$, n is a (α, δ)-hyperbolic time for $x \in M$ and $y, z \in V_n(\underline{t}, x)$, then*

$$\frac{1}{C_1} \leqslant \frac{|\det Df_{\underline{t}}^{n}(y)|}{|\det Df_{\underline{t}}^{n}(z)|} \leqslant C_1.$$

Proof. — For $1 \leqslant k \leqslant n$ the distance between $f_{\underline{t}}^{k}(x)$ and either $f_{\underline{t}}^{k}(y)$ or $f_{\underline{t}}^{k}(z)$ is smaller than $\alpha^{(n-k)/2}$ which is smaller than $\alpha^{b(n-k)} \leqslant \operatorname{dist}(f_{\underline{t}}^{k}(x), \mathcal{C})$. So, by (S3) we have

$$\begin{aligned}
\log \frac{|\det Df_{\underline{t}}^{n}(y)|}{|\det Df_{\underline{t}}^{n}(z)|} &= \sum_{k=0}^{n-1} \log \frac{|\det Df_{t_{k+1}}(f_{\underline{t}}^{k}(y))|}{|\det Df_{t_{k+1}}(f_{\underline{t}}^{k}(z))|} \\
&\leqslant \sum_{k=1}^{n-1} \log \frac{|\det Df(f_{\underline{t}}^{k}(y))|}{|\det Df(f_{\underline{t}}^{k}(z))|} \\
&\leqslant \sum_{k=0}^{n-1} 2B \frac{\alpha^{(n-k)/2}}{\alpha^{b\beta(n-k)}},
\end{aligned}$$

and it is enough to take $C_1 \leqslant \exp\big(\sum_{i=1}^{\infty} 2B\alpha^{(1/2-b\beta)i}\big)$, recalling that $b\beta < 1/2$ and also (8). □

3. Stationary measures

As mentioned before, we will assume the random perturbations of the nonuniformly expanding map f satisfy some *nondegeneracy conditions*: there exists $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon < \varepsilon_0$ we may take $n_0 = n_0(\varepsilon) \in \mathbb{N}$ for which the following holds:

(1) there is $\xi = \xi(\varepsilon) > 0$ such that $\left\{f^n_{\underline{t}}(x) \colon \underline{t} \in (\operatorname{supp}\theta_\varepsilon)^{\mathbb{N}}\right\}$ contains the ball of radius ξ around $f^n(x)$ for all $x \in M$ and $n \geqslant n_0$;

(2) $(f^n_x)_*\theta^{\mathbb{N}}_\varepsilon \ll m$ for all $x \in M$ and $n \geqslant n_0$.

Here $(f^n_x)_*\theta^{\mathbb{N}}_\varepsilon$ is the push-forward of $\theta^{\mathbb{N}}_\varepsilon$ to M via $f^n_x : T^{\mathbb{N}} \to M$, defined as $f^n_x(\underline{t}) = f^n_{\underline{t}}(x)$. Condition (1) means that perturbed iterates cover a full neighborhood of the unperturbed ones after a threshold for all sufficiently small noise levels. Condition (2) means that sets of perturbation vectors of positive $\theta^{\mathbb{N}}_\varepsilon$ measure must send any point $x \in M$ onto subsets of M with positive Lebesgue measure after a finite number of iterates.

In [**Ar**, Examples 1 & 2] it was shown that given any smooth map $f : M \to M$ of a compact manifold we can always construct a random perturbation satisfying the nondegeneracy conditions (1) and (2), if we take $T = \mathbb{R}^p$, $t^* = 0$ and θ_ε is equal to the normalized restriction of the Lebesgue measure to the ball of radius ε around 0, for a sufficiently big number $p \in \mathbb{N}$ of parameters. For parallelizable manifolds the random perturbations which consist in adding at each step a random noise to the unperturbed dynamics, as described in the Introduction, clearly satisfy nondegeneracy conditions (1) and (2) for $n_0 = 1$.

In the context of random perturbations of a map, we say that a set $A \subset M$ is *invariant* if $f_t(A) \subset A$, at least for $t \in \operatorname{supp}(\theta_\varepsilon)$ with $\varepsilon > 0$ small. The usual invariance of a measure with respect to a transformation is replaced by the following one: a probability measure μ is said to be *stationary*, if for every continuous $\varphi : M \to \mathbb{R}$ it holds

$$\int \varphi \, d\mu = \iint \varphi\big(f_t(x)\big) \, d\mu(x) \, d\theta_\varepsilon(t). \tag{16}$$

Remark 3.1. — If $(\mu^\varepsilon)_{\varepsilon>0}$ is a family of stationary measures having μ_0 as a weak* accumulation point when ε goes to 0, then it follows from (16) and the convergence of $\operatorname{supp}(\theta_\varepsilon)$ to $\{t^*\}$ that μ_0 must be invariant by $f = f_{t^*}$.

It is not difficult to see (cf. [**Ar**]) that a stationary measure μ satisfies

$$x \in \operatorname{supp}(\mu) \implies f_t(x) \in \operatorname{supp}(\mu) \text{ for all } t \in \operatorname{supp}(\theta_\varepsilon)$$

just by continuity of Φ. This means that if μ is a stationary measure, then $\operatorname{supp}(\mu)$ is an invariant set. Nondegeneracy condition (1) ensures that the interior of $\operatorname{supp}(\mu)$ is nonempty.

Let us write $\operatorname{supp}(\mu)$ as a disjoint union $\bigcup_i C_i$ of connected components and consider only those C_i for which $m(C_i) > 0$ — this collection is nonempty since

$\operatorname{supp}(\mu)$ contains open sets. Moreover each f_t must permute these components for $t \in \operatorname{supp}(\theta_\varepsilon)$, because $f_t(C_i)$ is connected by continuity, $f_t(C_i) \subset \operatorname{supp}(\mu)$ by invariance, and $m(f_t(C_i)) > 0$ since we have $(f_t)_* m \ll m$.

The connectedness of C_i and continuity of Φ guarantee that the above-mentioned perturbation of the components C_i induced by f_t does not depend on $t \in \operatorname{supp}(\theta_\varepsilon)$. Indeed, supposing that $t, t' \in \operatorname{supp}(\theta_\varepsilon)$ are such that

$$f_t(C_i) \subset C_j \quad \text{and} \quad f_{t'}(C_i) \subset C_{j'},$$

then fixing some $z \in C_i$ we have that $\{f_t(z)\colon t \in \operatorname{supp}(\theta_\varepsilon)\}$ is a connected set intersecting both C_j and $C_{j'}$ inside $\operatorname{supp}(\mu)$, and so $C_j = C_{j'}$.

We will show that these connected components are periodic under the action induced by f_t with $t \in \operatorname{supp}(\theta_\varepsilon)$. After this, we may use nondegeneracy condition (1) to conclude that each component contains a ball of uniform radius and thus that each component satisfies $m(C_i) > \text{const} > 0$. Hence there existing only a finite number of such components.

At this point it is useful to introduce the skew-product map

$$\begin{aligned} F : T^{\mathbb{N}} \times M &\longrightarrow \quad T^{\mathbb{N}} \times M \\ (\underline{t}, z) \quad &\longmapsto \big(\sigma(\underline{t}), f_{t_1}(z)\big) \end{aligned}$$

where σ is the left shift on sequences $\underline{t} = (t_1, t_2, \dots) \in T^{\mathbb{N}}$. It is easy to check that the product measure $\theta_\varepsilon^{\mathbb{N}} \times \mu$ is F-invariant, as so is the set $\operatorname{supp}(\theta_\varepsilon^{\mathbb{N}} \times \mu) = \operatorname{supp}(\theta_\varepsilon)^{\mathbb{N}} \times \operatorname{supp}(\mu)$.

Lemma 3.2. — *The support of a stationary measure μ contains a finite number of connected components arranged in cycles permuted by the action of f_t for $t \in \operatorname{supp}(\theta_\varepsilon)$.*

Proof. — Is is enough to obtain that each connected component C_i is periodic under the action of f_t for $t \in \operatorname{supp}(\theta_\varepsilon)$, in the sense that $f_{\underline{t}}^p(C_i) \subset C_i$ for some $p \in \mathbb{N}$ and all $\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$. There are components C_i with nonempty interior, since the interior of $\operatorname{supp}(\mu)$ is nonempty. So we may take a component C_i that contains some ball B. Then we have $m(B) > 0$ and so $(\theta_\varepsilon^{\mathbb{N}} \times \mu)(\operatorname{supp}(\theta_\varepsilon^{\mathbb{N}}) \times B) > 0$. Poincaré Recurrence Theorem now guarantees there is $(\underline{t}, x) \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}}) \times B$ such that the F-orbit of $(\underline{t}, x)$ has the same $(\underline{t}, x)$ as an accumulation point. We see that there must exist some $p \in \mathbb{N}$ such that $f_{\underline{t}}^p(x) \in B \subset C_i$. In view of the independence of the permutation on the choice of $\underline{t}$, we conclude that C_i is sent inside itself by $f_{\underline{t}}^p$ for all $\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$. □

It is clear that the cycles obtained above are invariant sets. We are now ready to decompose μ into some simpler measures. For that we need the following result.

Lemma 3.3. — *The normalized restriction of a stationary measure to an invariant set is a stationary measure.*

Proof. — See [**Ar**, Lemma 8.2]. □

We define an *invariant domain* in M as a finite collection $(U_0, \ldots, U_{p-1})$ of pairwise separated open sets, that is, $\overline{U}_i \cap \overline{U}_j = \varnothing$ if $i \neq j$, such that $f_{\underline{t}}^k(U_i) \subset U_{(k+i) \bmod p}$ for all $k \geqslant 1$, $i = 0, \ldots, p-1$ and $\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$.

In order to get the separation of the connected components in a cycle, we may unite those components C_i and C_j such that $\overline{C}_i \cap \overline{C}_j \neq \varnothing$ and observe that the permutation now induced in the new sets by f_t also does not depend on the choice of $t \in \operatorname{supp}(\theta_\varepsilon)$. In this manner we construct invariant domains inside the support of any stationary probability measure.

The next step is to look for *minimal invariant domains* with respect to the natural order relation of inclusion of sets. Let $D = (U_0, \ldots, U_{p-1})$ and $D' = (W_0, \ldots, W_{q-1})$ be invariant domains. On the one hand, $D = D'$ if there are $i, j \in \mathbb{N}$ such that $U_{(i+k) \bmod p} = W_{(j+k) \bmod q}$ for all $k \geqslant 1$, which implies $p = q$ because the open sets that form each invariant domain are pairwise disjoint. On the other hand, we say $D \prec D'$ if there are $i, j \in \mathbb{N}$ such that $U_{i \bmod p} \subsetneq W_{j \bmod q}$ and $U_{(i+k) \bmod p} \subset W_{(j+k) \bmod q}$ for all $k \geqslant 1$.

Lemma 3.4. — *In the partially ordered family of all invariant domains in M, with respect to the relation $\prec$, the number of $\prec$-minimal domains is finite. Moreover, every invariant domain contains at least one minimal domain.*

Proof. — The proof relies in showing that Zorn's Lemma can be applied to this partially ordered set and that minimal domains are pairwise separated. See [**Ar**, Section 3]. □

Let us now fix $x \in M$ and consider

$$\mu_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} (f_x^j)_* \theta_\varepsilon^{\mathbb{N}}. \tag{17}$$

Since this is a sequence of probability measures on the compact manifold M, then it has weak* accumulation points.

Lemma 3.5. — *Every weak* accumulation point of $\big(\mu_n(x)\big)_n$ is stationary and absolutely continuous with respect to the Lebesgue measure.*

Proof. — Let μ be a weak* accumulation point of $\big(\mu_n(x)\big)_n$. We may write

$$\iint \varphi\big(f_t(x)\big)\, d\mu(x)\, d\theta_\varepsilon(t) = \int \lim_{k \to +\infty} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \int \varphi\left(f_t\big(f_{\underline{t}}^j(x)\big)\right) d\theta_\varepsilon^{\mathbb{N}}(\underline{t})\, d\theta_\varepsilon(t)$$

for each continuous $\varphi : M \to \mathbb{R}$. Moreover dominated convergence ensures that we may exchange the limit and the outer integral sign and, by definition of $f_{\underline{t}}^j(x)$, we get

$$\lim_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \int \varphi\big(f_{\underline{t}}^{j+1}(x)\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{t}) = \int \varphi\, d\mu,$$

according to the definition of μ. Thus (16) must hold and μ is stationary.

Noting that $C^0(M,\mathbb{R})$ is dense in $L^1(M,\mu)$ with the L^1 norm, we see that (16) holds for all μ-integrable functions $\varphi : M \to \mathbb{R}$. In particular, if $E \subset M$ is such that $m(E) = 0$, then

$$\begin{aligned}\int 1_E \, d\mu &= \iint 1_E\big(f_t(x)\big)\, d\mu(x)\, d\theta_\varepsilon(t)\\ &= \iint 1_E\big(f_t(x)\big)\, d\theta_\varepsilon(t)\, d\mu(x)\\ &= \iiint 1_E\big(f_t(f_s(x))\big)\, d\theta_\varepsilon(t)\, d\mu(x)\, d\theta_\varepsilon(s)\\ &= \iint 1_E\big(f_{\underline{t}}^2(x)\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{t})\, d\mu(x)\\ &= \int (f_x^2)_*\theta_\varepsilon^{\mathbb{N}}(E)\, d\mu(x).\end{aligned}$$

This process may be iterated to yield

$$\mu(E) = \int (f_x^{n_0})_*\theta_\varepsilon(E)\, d\mu(x)$$

and, since $(f_x^{n_0})_*\theta_\varepsilon \ll m$ by nondegeneracy condition 2, we must have $\mu(E) = 0$. $\square$

Clearly if $x \in M$ belongs to some set of an invariant domain $(U_0, \dots, U_{p-1})$, then $\mu_n(x)$ have supports contained in $\overline{U}_0 \cup \dots \cup \overline{U}_{p-1}$ for all $n \geqslant 1$ and any weak* accumlation point μ of $(\mu_n(x))_n$ is a stationary measure with $\mathrm{supp}(\mu) \subset \overline{U}_0 \cup \dots \cup \overline{U}_{p-1}$. We will now see these measures are physical.

Lemma 3.6. — *If $(U_0, \dots, U_{p-1})$ is a minimal invariant domain, then there is a unique absolutely continuous stationary measure ν such that* $\mathrm{supp}(\nu) \subset \overline{U}_0 \cup \dots \cup \overline{U}_{p-1}$. *Moreover, this ν is a physical measure and* $\mathrm{supp}(\nu) = \overline{U}_0 \cup \dots \cup \overline{U}_{p-1}$.

Proof. — Let us assume $n_0 = 1$ for simplicity (see [**Ar**, Section 7] for the general case) and let us consider a stationary absolutely continuous probability measure ν with $\mathrm{supp}(\nu) \subset \overline{U}_0 \cup \dots \cup \overline{U}_{p-1}$. We first show the ergodicity of ν, in the sense that $\theta_\varepsilon^{\mathbb{N}} \times \nu$ is F-ergodic. It turns out that to be F-ergodic it suffices that either $\nu(G) = 0$ or $\nu(G) = 1$ for every Borel set $G \subset M$ satisfying

$$1_G(x) = \int 1_G\left(f_t(x)\right)\, d\theta_\varepsilon(t) \tag{18}$$

for ν almost every x (cf. [**Ar**] and [**Vi2**]). So let us take G such that $\nu(G) > 0$ and G satisfies the left hand side of (18). Then it must be $m(G) > 0$ because $\nu \ll m$ and there is a closed set $J \subset G$ such that $m(G \setminus J) = 0$ and also $\nu(G \setminus J) = 0$. Hence J also satisfies the left hand side of (18) because of nondegeneracy condition (2) (with $n_0 = 1$), since

$$\int 1_E(f_t(x))\, d\theta_\varepsilon(t) = (f_x)_*\theta_\varepsilon^{\mathbb{N}}(E).$$

This means that when $x \in J$ we have $f_t(x) \in J$ for θ_ε almost all $t \in \operatorname{supp}(\theta_\varepsilon)$. Since a set of θ_ε measure 1 is dense in $\operatorname{supp}(\theta_\varepsilon)$ (we are supposing θ_ε to be positive on open sets) and $f_t(x)$ varies continuously with t, we see that $f_t(x) \in J$ for all $t \in \operatorname{supp}(\theta_\varepsilon)$ because J is closed. We then have that the interior of J is nonempty by condition (1) on random perturbations and we may apply the methods of decomposition into connected components as before (Lemma 3.2). In this manner we construct an invariant domain inside J which, in turn, is inside a minimal invariant domain. This contradicts minimality and so we conclude that J must contain $\overline{U}_0 \cup \cdots \cup \overline{U}_{p-1}$. Thus we have $\nu(G) = \nu(J) = 1$ proving $\theta_\varepsilon^{\mathbb{N}} \times \nu$ to be F-ergodic.

Now, given $\varphi : M \to \mathbb{R}$ continuous we consider the map $\psi = \varphi \circ \pi$ from $T^{\mathbb{N}} \times M$ to $\mathbb{R}$, where $\pi : T^{\mathbb{N}} \times M \to M$ is the natural projection. The Ergodic Theorem then ensures

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(F^j(\underline{t}, x)) = \int \psi \, d(\theta_\varepsilon^{\mathbb{N}} \times \nu)$$

for $\theta_\varepsilon^{\mathbb{N}} \times \nu$ almost all $(\underline{t}, x)$, which is just the same as

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_{\underline{t}}^j(x)) = \int \varphi \, d\nu \tag{19}$$

for $\theta_\varepsilon^{\mathbb{N}} \times \nu$ almost all $(\underline{t}, x)$. Finally considering the ergodic basin $B(\nu)$, defined as the set of points $x \in M$ for which

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_{\underline{t}}^j(x)) = \int \varphi \, d\nu$$

for all $\varphi \in C^0(M, \mathbb{R})$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$, it is easy to see that $B(\nu)$ satisfies (18) in the place of G and we must have as before $B(\nu) \supset \overline{U}_0 \cup \cdots \cup \overline{U}_{p-1}$.

This shows that if another stationary absolutely continuous probability measure $\widetilde{\nu}$ is such that $\operatorname{supp}(\widetilde{\nu}) \subset \overline{U}_0 \cup \cdots \cup \overline{U}_{p-1}$, then the basins of ν and $\widetilde{\nu}$ must have nonempty intersection. Thus these measures must be equal. Moreover $\nu\big(B(\nu)\big) = 1$ and so, by absolute continuity, $m\big(B(\nu)\big) > 0$ and thus ν is a physical probability. □

4. The number of physical measures

In this section we will prove that the number ℓ of physical measures is bounded by the number p of SRB measures. Moreover we will present examples of dynamical systems for which $\ell = p$ and $\ell < p$.

Let $\mu_1, \ldots, \mu_\ell$ be the physical measures supported on the minimal invariant domains in M, which exist by Lemmas 3.2 and 3.4 through 3.6. If μ is an absolutely continuous stationary measure, its restrictions to the minimal invariant domains of M, normalized when not equal to the constant zero measure, are absolutely continuous stationary measures by Lemma 3.3. After Lemma 3.6 these restrictions must be the physical measures $\mu_1, \ldots, \mu_\ell$ of the minimal domains. Hence μ must decompose into a linear combination of physical measures. Moreover, the union of $\text{supp}(\mu_1), \ldots, \text{supp}(\mu_\ell)$ must contain $\text{supp}(\mu)$, except possibly for a μ null set. In fact, if the following set function

$$\mu - \mu\big(\text{supp}(\mu_1)\big)\mu_1 - \cdots - \mu\big(\text{supp}(\mu_\ell)\big)\mu_\ell$$

were nonzero, then its normalization μ' would be an absolutely continuous stationary measure, and the above decomposition could be applied to μ', thus giving another minimal domain inside $\text{supp}(\mu)$. Clearly this cannot happen. We then have a convex linear decomposition

$$\mu = \alpha_1\mu_1 + \cdots + \alpha_\ell\mu_\ell \tag{20}$$

where $\alpha_i = \mu(\text{supp}(\mu_i)) \geqslant 0$ and $\alpha_1 + \cdots + \alpha_\ell = 1$. We will see that this decomposition is uniquely defined.

We remark that so far we did not use more than the continuity of the map f. For the next result we assume that $f : M \to M$ is a C^2 nonuniformly expanding map whose orbits have slow approximation to the critical $\mathcal{C}$ (possibly the emptyset) with $m(\mathcal{C}) = 0$. This result contains the assertions of the first two items of Theorem A (if we think of $\mathcal{C} = \varnothing$) and Theorem C.

Proposition 4.1. — *If $\varepsilon > 0$ is small enough, then there exist physical measures $\mu_1^\varepsilon, \ldots, \mu_\ell^\varepsilon$ (with ℓ not depending on ε) such that*

(1) *for $x \in M$ there is a $\theta_\varepsilon^{\mathbb{N}}$ mod 0 partition $T_1(x), \ldots, T_\ell(x)$ of $T^{\mathbb{N}}$ such that*

$$\mu_i^\varepsilon = w^*\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j(x)} \quad \textit{if and only if} \quad \underline{t} \in T_i(x);$$

(2) *for each $i = 1, \ldots, \ell$ we have*

$$\mu_i^\varepsilon = w^*\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \int (f_{\underline{t}}^j)_*\big(m \mid B(\mu_i^\varepsilon)\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{t}),$$

where $m \mid B(\mu_i^\varepsilon)$ is the normalized restriction of Lebesgue measure to $B(\mu_i^\varepsilon)$.

Proof. — Take $x \in M$ and let μ be a weak* accumulation point of the sequence $(\mu_n(x))_n$ defined in (17). We will prove that this is the only accumulation point of (17) by showing that the values of the $\alpha_1, \ldots, \alpha_\ell$ in decomposition (20) depend only on x and not on the subsequence that converges to μ. The definition of the average in (17) implies that there is a subset of parameter vectors $\underline{t} \in \text{supp}(\theta_\varepsilon^{\mathbb{N}})$ with positive $\theta_\varepsilon^{\mathbb{N}}$ measure for which there is $j \geqslant 1$ such that $f_{\underline{t}}^j(x) \in \text{supp}(\mu_i)$. We define for $i = 1, \ldots, \ell$

$$T_i(x) = \Big\{\underline{t} \in \text{supp}(\theta_\varepsilon^{\mathbb{N}}) : f_{\underline{t}}^j(x) \in \text{supp}(\mu_i) \quad \text{for some} \quad j \geqslant 1\Big\}.$$

We clearly have

$$T_i(x) = \bigcup_{j\geqslant 1} T_i^j(x) \quad \text{where} \quad T_i^j(x) = \{\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}}) : f_{\underline{t}}^j(x) \in \operatorname{supp}(\mu_i)\}$$

and $T_i^j(x) \subset T_i^{j+1}(x)$ for all $i, j \geqslant 1$, since the supports of stationary measures are themselves invariant. In addition, since μ is a regular (Borel) probability measure, we may find for each $\eta > 0$ an open set U and a closed set K such that $K \subset \operatorname{supp}(\mu_i) \subset U$ with $\mu(U \setminus K) < \eta$ and $\mu(\partial U) = \mu(\partial K) = 0$. In fact, there is an at most countable number of δ-neighborhoods of $\operatorname{supp}(\mu_i)$ whose boundaries have positive μ measure, and likewise for the compacts coinciding with the complement of the δ-neighborhood of $M \setminus \operatorname{supp}(\mu_i)$. Then, taking $\alpha_i = \mu(\operatorname{supp}(\mu_i))$ we have

$$\begin{aligned}\alpha_i + \eta \geqslant \mu(U) &= \lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\varepsilon^{\mathbb{N}}\{\underline{t} \in T^{\mathbb{N}} : f_{\underline{t}}^j(x) \in U\} \\ &\geqslant \limsup_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\varepsilon^{\mathbb{N}}\big(T_i^j(x)\big)\end{aligned}$$

for some sequence of integers $n_1 < n_2 < n_3 < \cdots$, and likewise for

$$\begin{aligned}\alpha_i - \eta \leqslant \mu(K) &= \lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\varepsilon^{\mathbb{N}}\{\underline{t} \in T^{\mathbb{N}} : f_{\underline{t}}^j(x) \in K\} \\ &\leqslant \liminf_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\varepsilon^{\mathbb{N}}\big(T_i^j(x)\big),\end{aligned}$$

where $\eta > 0$ is arbitrary. This shows

$$\alpha_i = \mu(\operatorname{supp}(\mu_i)) = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \theta_\varepsilon^{\mathbb{N}}\big(T_i^j(x)\big).$$

We also have

$$\theta_\varepsilon^{\mathbb{N}}\big(T_i(x)\big) = \lim_{j\to\infty} \theta_\varepsilon^{\mathbb{N}}\big(T_i^j(x)\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \theta_\varepsilon^{\mathbb{N}}\big(T_i^j(x)\big) = \alpha_i$$

which shows that the α_i depend only on the random orbits of x and not on the particular sequence $(n_k)_k$. Thus we see that the sequence of measures in (17) converges in the weak* topology. Moreover the sets $T_1(x), \ldots, T_\ell(x)$ are pairwise disjoint by definition and their total $\theta_\varepsilon^{\mathbb{N}}$ measure equals $\alpha_1 + \cdots + \alpha_\ell = 1$, thus forming a $\theta_\varepsilon^{\mathbb{N}}$ modulo zero partition of $T^{\mathbb{N}}$. We observe that if $\underline{t} \in T_i(x)$, then $f_{\underline{t}}^n(x) \in \operatorname{supp}(\mu_i) \subset B(\mu_i)$ for some $n \geqslant 1$ and $i = 1, \ldots, \ell$. This means this $\theta_\varepsilon^{\mathbb{N}}$ modulo zero partition of $T^{\mathbb{N}}$ satisfies the first item of the proposition.

Now fixing $i = 1, \ldots, \ell$, for all $x \in B(\mu_i)$ (the ergodic basin of μ_i) it holds that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f_{\underline{t}}^j(x)) = \int \varphi \, d\mu_i$$

for $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. Recall that $m(B(\mu_i)) > 0$ by the definition of physical measure. Using dominated convergence and integrating both sides of the above equality twice, first with respect to the Lebesgue measure m, and then with respect to $\theta_\varepsilon^{\mathbb{N}}$, we arrive at the statement of item 2.

Recall that up until now the noise level $\varepsilon > 0$ was kept fixed. For small enough $\varepsilon > 0$ the measures $\mu_i = \mu_i^\varepsilon$ depend on the noise level, but we will see that the number of physical measures is constant.

Fixing $i \in \{1, \ldots, \ell\}$ we let x in the interior of $\operatorname{supp}(\mu_i^\varepsilon)$ be such that the orbit $(f^j(x))_j$ has infinitely many hyperbolic times. Recall that $f \equiv f_{t^*}$ is nonuniformly expanding (possibly with criticalities). Then there is a big enough hyperbolic time n so that $V_n(\underline{t}^*, x) \subset \operatorname{supp}(\mu_i^\varepsilon)$, by Proposition 2.6, where we take $\underline{t}^* = (t^*, t^*, t^*, \ldots)$. Since $t^* \in \operatorname{supp}(\theta_\varepsilon)$ and $\operatorname{supp}(\mu_i^\varepsilon)$ is invariant under f_t for all $t \in \operatorname{supp}(\theta_\varepsilon)$, we must have

$$f_{\underline{t}^*}^n\big(V_n(\underline{t}^*, x)\big) = B\big(f_{t^*}^n(x), \delta_1\big) \subset \operatorname{supp}(\mu_i^\varepsilon), \tag{21}$$

where $\delta_1 > 0$ is the constant given by Proposition 2.6 and $B\big(f_{t^*}^n(x), \delta_1\big)$ is the ball of radius δ_1 around $f_{t^*}^n(x)$.

On the one hand, we deduce that the number $\ell = \ell(\varepsilon)$ is bounded from above by some uniform constant N since M is compact. On the other hand, since each invariant set must contain some physical measure (by Lemma 3.4), we see that for $0 < \varepsilon' < \varepsilon$ there must be some physical measure $\mu^{\varepsilon'}$ with $\operatorname{supp}(\mu^{\varepsilon'}) \subset \operatorname{supp}(\mu^\varepsilon)$. In fact $\operatorname{supp}(\mu^\varepsilon)$ is invariant under f_t for every $t \in \operatorname{supp}(\theta_{\varepsilon'}) \subset \operatorname{supp}(\theta_\varepsilon)$. This means the number $\ell(\varepsilon)$ of physical measures is a nonincreasing function of $\varepsilon > 0$. Thus we conclude that there must be $\varepsilon_0 > 0$ such that $\ell = \ell(\varepsilon)$ is constant for $0 < \varepsilon < \varepsilon_0$, ending the proof of the proposition. □

Remark 4.2. — Let us point out that from (21) one easily deduces that the Lesbesgue measure of the basin of each physical measure is uniformly bounded from below, since the support of such a measure is always contained in its basin.

Remark 4.3. — Observe that if the map $f : M \to M$ is topologically transitive, then every stationary measure must be supported on the whole of M, since the support is invariant and has nonempty interior. According to the discussion above, there must be only one such stationary measure, which must be physical.

We note that the number ℓ of physical measures for small $\varepsilon > 0$ and the number p of SRB measures for f are obtained by different existential arguments. It is natural to ask if there is any relation between ℓ and p.

Proposition 4.4. — *If $p \geqslant 1$ is the number of SRB measures of f and $\ell \geqslant 1$ is the number of physical measures of the random perturbation of f, then for $\varepsilon > 0$ small enough we have $\ell \leqslant p$.*

Proof. — We observe that $\mathrm{supp}(\mu^\varepsilon)$ is forward invariant under $f = f_{t^*}$ and, moreover, condition (3) holds for Lebesgue almost every x in $\mathrm{supp}(\mu^\varepsilon)$ because holds almost everywhere in M (by assumption) and $\mathrm{supp}(\mu^\varepsilon)$ has nonempty interior. Thus from [**ABV**, Theorem C] we assure the existence of at least one SRB measure μ with $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu^\varepsilon)$.

We have seen that each support of a physical measure μ^ε must contain at least the support of one SRB measure for the unperturbed map f. Since the number of SRB measures is finite we have $\ell \leqslant p$, where p is the number of those measures. □

The reverse inequality does not hold in general, as the following examples show: it is possible for two distinct SRB measures to have intersecting supports and, in this circumstance, the random perturbations will mix their basins and there will be some physical measure whose support overlaps the supports of both SRB measures.

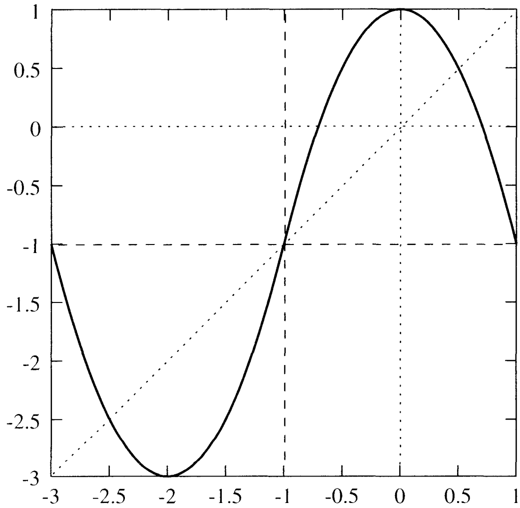

FIGURE 1. Map for which $1 = \ell < p = 2$

The first example is the map $f : [-3, 1] \to [-3, 1]$ whose graph is figure 1:

$$f(x) = \begin{cases} 1 - 2x^2 & \text{if } -1 \leqslant x \leqslant 1 \\ 2(x+2)^2 - 3 & \text{if } -3 \leqslant x \leqslant -1 \end{cases}.$$

The dynamics of f on $[-1, 1]$ and $[-3, -1]$ is conjugated to the tent map $T(x) = 1 - 2|x|$ on $[-1, 1]$. Thus understanding f as a circle map through the identification $S^1 = [-3, 1]/\{-3, 1\}$, this is a nonuniformly expanding map with a critical set satisfying conditions (S1)-(S3) and there are two ergodic absolutely continuous (thus SRB) invariant measures μ_1, μ_2 whose supports are $[-3, -1]$ and $[-1, 1]$ respectively. Moreover defining $\Phi(t) = R_t \circ f$, where $R_t : S^1 \to S^1$ is the rotation of angle t and $\theta_\varepsilon = (2\varepsilon)^{-1}(m \mid [-\varepsilon,\varepsilon])$ for small $\varepsilon > 0$, we have that $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ is a random perturbation satisfying nondegeneracy conditions (1) and (2). Since $\mathrm{supp}(\mu_1) \cap \mathrm{supp}(\mu_2) = \{-1\}$ we have that for $\varepsilon > 0$ small enough there must be a single physical measure μ^ε. Indeed, by property (P) any weak* accumulation point of a family of physical measures must have -1 in its support.

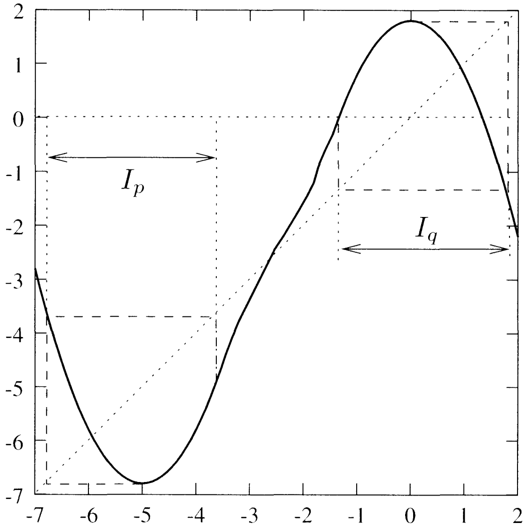

FIGURE 2. Map for which $\ell = p = 2$

The second example is defined on the interval $I = [-7, 2]$. We take the map $q_a(x) = a - x^2$ on $[-2, 2]$ for some parameter $a \in (1, 2)$ satisfying Benedicks-Carleson conditions (see [**BC1**] and [**BC2**]), and the "same" map on $[-7, -3]$ conveniently conjugated: $p_a(x) = (x+5)^2 - 5 - a$. Then the two pieces of graph are glued together in such a way that we obtain a smooth map $f : I \to I$ sending I into its interior, as figure 2 shows. The intervals $I_q = [q_a^2(0), q_a(0)]$ and $I_p = [p_a(-5), p_a^2(-5)]$ are forward invariant for f, and then we can find slightly larger intervals $I_1 \supset I_p$ and $I_2 \supset I_q$ that become trapping regions for f. So, taking $\Phi(t) = f + t$, and θ_ε as in the previous example with $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$ small enough, then $\{\Phi, (\theta_\varepsilon)_\varepsilon\}$ is a random perturbation of f leaving the intervals I_1 and I_2 invariant by each $\Phi(t)$. Moreover, Lebesgue almost every $x \in I$ eventually arrives at one of these intervals. Then by [**BC1**] and [**BY1**] the map f is nonuniformly expanding and has two SRB measures with supports contained in each trapping region. Finally f admits two distinct physical measures whose supports are contained in I_1 and I_2 respectively, for $\varepsilon_0 > 0$ small enough. Moreover, these SRB measures are stochastically stable; see [**BV**].

5. Stochastic stability

In this section we will prove the first item of Theorem B and Theorem D. The second item of Theorem B may be obtained in the same way as Theorem D, if we think of $\mathcal{C}$ as being equal to the empty set and take into account Remark 2.4.

We start by proving the first item of Theorem B. Assume that f is a stochastically stable nonuniformly expanding local diffeomorphism. We know from Proposition 4.1 that there is a finite number of physical measures $\mu_1^\varepsilon, \dots \mu_\ell^\varepsilon$ and for each $x \in M$ there is a $\theta_\varepsilon^{\mathbb{N}}$ mod 0 partition $T_1(x), \dots, T_\ell(x)$ of $T^{\mathbb{N}}$ for which

$$\mu_i^\varepsilon = w^*\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j(x)} \quad \text{for each} \quad \underline{t} \in T_i(x).$$

Furthermore, since we are taking f a local diffeomorphism, then $\log \|(Df)^{-1}\|$ is a continuous map. Thus, we have for each $x \in M$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df\big(f_{\underline{t}}^j(x)\big)^{-1}\| = \int \log \|(Df)^{-1}\| d\mu_i^\varepsilon$$

for some physical measure μ_i^ε with $1 \leqslant i \leqslant \ell$. Hence, for proving the nonuniform expansion of f on random orbits it suffices to show that there is $c_0 > 0$ such that if $\mu^\varepsilon = \mu_i^\varepsilon$ for some $1 \leqslant i \leqslant \ell$ then

$$\int \log \|(Df)^{-1}\| d\mu^\varepsilon < c_0 \quad \text{for small } \varepsilon > 0.$$

Lemma 5.1. — *Let $\varphi\colon M \to \mathbb{R}$ be a continuous map. Given $\delta > 0$ there is $\varepsilon_0 > 0$ such that if $\varepsilon \leqslant \varepsilon_0$, then*

$$\left|\int \varphi d\mu^\varepsilon - \int \varphi d\mu_\varepsilon\right| < \delta,$$

for some absolutely continuous f-invariant probability measure μ_ε.

Proof. — We will use the following auxiliary result: *Let X be a compact metric space, $K \subset X$ a closed (compact) subset and $(x_t)_{t>0}$ a curve in X (not necessarily continuous) such that all its accumulation points (as $t \to 0^+$) lie in K. Then for every open neighborhood U of K there is $t_0 > 0$ such that $x_t \in U$ for every $0 < t < t_0$.* Indeed, supposing not, there is a sequence $(t_n)_n$ with $t_n \to 0^+$ when $n \to \infty$ such that $x_{t_n} \notin U$. Since X is compact this means that $(x_t)_{t>0}$ has some accumulation point in $X \setminus U$, thus outside K, contrary to the assumption.

Now, the space $X = \mathbb{P}(M)$ of all probability measures in M is a compact metric space with the weak* topology, and the convex hull K of the (finitely many) SRB measures of f is closed. Hence, considering the curve $(\mu^\varepsilon)_\varepsilon$ in $\mathbb{P}(M)$, we are in the context of the above result, since we are supposing f to be stochastically stable.

A metric on X topologically equivalent to the weak* topology may be given by

$$\mathrm{d}_{\mathbb{P}}(\mu,\nu)=\sum_{k=1}^{\infty}\frac{1}{2^n}\left|\int\varphi_n\,d\mu-\int\varphi_n\,d\nu\right|$$

where $\mu,\nu\in\mathbb{P}(M)$ and $(\varphi_n)_{n\geqslant 1}$ is a dense sequence of functions in $C^0(M,\mathbb{R})$, see [**Ma**].

Let $\varphi: M\to\mathbb{R}$ continuous be given and let us fix some $\delta>0$. There must be $n\in\mathbb{N}$ such that $\|\varphi-\varphi_n\|_0<\delta/3$ and, by the auxiliary result in the beginning of the proof, there exists, for some $\varepsilon_0>0$ and every $0<\varepsilon<\varepsilon_0$, a probability measure $\mu_\varepsilon\in\mathbb{P}(M)$ for which $\mathrm{d}_{\mathbb{P}}(\mu^\varepsilon,\mu_\varepsilon)<\delta(3\cdot 2^n)^{-1}$. This in particular means that

$$\frac{1}{2^n}\left|\int\varphi_n\,d\mu^\varepsilon-\int\varphi_n\,d\mu_\varepsilon\right|<\frac{\delta}{3\cdot 2^n},$$

by the definition of the distance $\mathrm{d}_{\mathbb{P}}$, which implies

$$\left|\int\varphi_n\,d\mu^\varepsilon-\int\varphi_n\,d\mu_\varepsilon\right|<\frac{\delta}{3}.$$

Hence we get

$$\begin{aligned}\left|\int\varphi\,d\mu^\varepsilon-\int\varphi\,d\mu_\varepsilon\right| &\leqslant\\ &\leqslant\left|\int\varphi\,d\mu^\varepsilon-\int\varphi_n\,d\mu^\varepsilon\right|+\left|\int\varphi_n\,d\mu^\varepsilon-\int\varphi_n\,d\mu_\varepsilon\right|+\left|\int\varphi_n\,d\mu_\varepsilon-\int\varphi\,d\mu_\varepsilon\right|\\ &<\frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}=\delta,\end{aligned}$$

which completes the proof of the lemma. □

Now we take $\varphi=\log\|(Df)^{-1}\|$ and $\delta=c/2$ in the previous lemma, where $c>0$ is the constant given by the nonuniform expansion of f (recall (3)). For each $\varepsilon\leqslant\varepsilon_0$ let μ_ε be the measure given by Lemma 5.1. Since property (P) holds, there are real numbers $w_1(\varepsilon),\dots,w_p(\varepsilon)\geqslant 0$ with $w_1(\varepsilon)+\cdots+w_p(\varepsilon)=1$ for which $\mu_\varepsilon=w_1(\varepsilon)\mu_1+\cdots+w_p(\varepsilon)\mu_p$. Since each μ_i is an SRB measure for $1\leqslant i\leqslant p$, we have for Lebesgue almost every $x\in B(\mu_i)$

$$\int\log\|(Df)^{-1}\|d\mu_i=\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f^j(x))^{-1}\|\leqslant -c<0.$$

This implies

$$\int\log\|(Df)^{-1}\|d\mu_\varepsilon\leqslant -c,$$

and so, by Lemma 5.1 and the choice of δ,

$$\int\log\|(Df)^{-1}\|d\mu^\varepsilon\leqslant -c/2.$$

This completes the proof of the first item of Theorem B.

Now we go into the proof of Theorem D. In order to prove that f is stochastically stable, and taking into account property (P), it suffices to prove that the weak* accumulation points of any family $(\mu^\varepsilon)_{\varepsilon>0}$, where each μ^ε is a physical measure of level ε, are absolutely continuous with respect to the Lebesgue measure. Let μ^ε be a physical measure of level ε for some small $\varepsilon > 0$ and define for each $n \geqslant 1$

$$\mu_n^\varepsilon = \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{m(B(\mu^\varepsilon))}\int (f_{\underline{t}}^j)_*\big(m \mid B(\mu^\varepsilon)\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{t}).$$

We know from Proposition 4.1 that each μ^ε is the weak* limit of the sequence $(\mu_n^\varepsilon)_n$. We will prove Theorem D by providing some useful estimates on the densities of the measures μ_n^ε. Define for each $\underline{t} \in T^{\mathbb{N}}$ and $n \geqslant 1$

$$H_n(\underline{t}) = \{x \in B(\mu^\varepsilon)\colon\ n \text{ is a } (\alpha,\delta)\text{-hyperbolic time for } (\underline{t}, x)\},$$

and

$$H_n^*(\underline{t}) = \{x \in B(\mu^\varepsilon)\colon\ n \text{ is the first } (\alpha,\delta)\text{-hyperbolic time for } (\underline{t}, x)\}.$$

$H_n^*(\underline{t})$ is precisely the set of those points $x \in B(\mu^\varepsilon)$ for which $h_\varepsilon(\underline{t}, x) = n$ (recall the definition of the map h_ε). For $n, k \geqslant 1$ we also define $R_{n,k}(\underline{t})$ as the set of those points $x \in M$ for which n is a (α,δ)-hyperbolic time and $n+k$ is the first (α,δ)-hyperbolic time after n, i.e.

$$R_{n,k}(\underline{t}) = \big\{x \in H_n(\underline{t})\colon\ f_{\underline{t}}^n(x) \in H_k^*(\sigma^n \underline{t})\big\},$$

where $\sigma\colon T^{\mathbb{N}} \to T^{\mathbb{N}}$ is the shift map $\sigma(t_1, t_2, \dots) = (t_2, t_3, \dots)$. Considering the measures

$$\nu_n^\varepsilon = \int (f_{\underline{t}}^n)_*\big(m \mid H_n(\underline{t})\big) d\theta_\varepsilon^{\mathbb{N}}(\underline{t})$$

and

$$\eta_n^\varepsilon = \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^{n+j})_*\big(m \mid R_{n,k}(\underline{t})\big) d\theta_\varepsilon^{\mathbb{N}}(\underline{t}),$$

we may write

$$\mu_n^\varepsilon \leqslant \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{m(B(\mu^\varepsilon))}(\nu_j^\varepsilon + \eta_j^\varepsilon).$$

Proposition 5.2. — *There is a constant $C_2 > 0$ such that for every $n \geqslant 0$ and $\underline{t} \in T^{\mathbb{N}}$*

$$\frac{d}{dm}(f_{\underline{t}}^n)_*\big(m \mid H_n(\underline{t})\big) \leqslant C_2.$$

Proof. — Take $\delta_1 > 0$ given by Proposition 2.6. It is sufficient to prove that there is some uniform constant $C > 0$ such that if A is a Borel set in M with diameter smaller than $\delta_1/2$ then

$$m\big(f_{\underline{t}}^{-n}(A) \cap H_n(\underline{t})\big) \leqslant C m(A).$$

Let A be a Borel set in M with diameter smaller than $\delta_1/2$ and B an open ball of radius $\delta_1/2$ containing A. We may write

$$f_{\underline{t}}^{-n}(B)=\bigcup_{k\geqslant 1} B_k,$$

where $(B_k)_{k\geqslant 1}$ is a (possibly finite) family of two-by-two disjoint open sets in M. Discarding those B_k that do not intersect $H_n(\underline{t})$, we choose for each $k\geqslant 1$ a point $x_k\in H_n(\underline{t})\cap B_k$. For $k\geqslant 1$ let $V_n(\underline{t},x_k)$ be the neighborhood of x_k in M given by Proposition 2.6. Since B is contained in $B\big(f_{\underline{t}}^n(x_k),\delta_1\big)$, the ball of radius δ_1 around $f_{\underline{t}}^n(x_k)$, and $f_{\underline{t}}^n$ is a diffeomorphism from $V_n(\underline{t},x_k)$ onto $B\big(f_{\underline{t}}^n(x_k),\delta_1\big)$, we must have $B_k\subset V_n(\underline{t},x_k)$ (recall that by our choice of B_k we have $f_{\underline{t}}^n(B_k)\subset B$). As a consequence of this and Corollary 2.7, we have for every k that the map $f_{\underline{t}}^n\mid B_k\colon B_k\to B$ is a diffeomorphism with bounded distortion:

$$\frac{1}{C_1}\leqslant\frac{|\det Df_{\underline{t}}^n(y)|}{|\det Df_{\underline{t}}^n(z)|}\leqslant C_1$$

for all $y,z\in B_k$. This finally gives

$$\begin{aligned}
m\big(f_{\underline{t}}^{-n}(A)\cap H_n(\underline{t})\big)&\leqslant\sum_k m\big(f_{\underline{t}}^{-n}(A\cap B)\cap B_k\big)\\
&\leqslant\sum_k C_1\frac{m(A\cap B)}{m(B)}m(B_k)\\
&\leqslant C_2 m(A),
\end{aligned}$$

where $C_2>0$ is a constant only depending on C_1, on the volume of the ball B of radius $\delta_1/2$, and on the volume of M. $\square$

It follows from Proposition 5.2 that

$$\frac{d\nu_n^\varepsilon}{dm}\leqslant C_2 \tag{22}$$

for every $n\geqslant 0$ and small $\varepsilon>0$. Our goal now is to control the density of the measures η_n^ε in such a way that we may assure the absolute continuity of the weak* accumulation points of the measures μ^ε when ε goes to zero.

Proposition 5.3. — *Given $\zeta>0$, there is $C_3(\zeta)>0$ such that for every $n\geqslant 0$ and $\varepsilon>0$ we may bound η_n^ε by the sum of two non-negative measures, $\eta_n^\varepsilon\leqslant\omega^\varepsilon+\rho^\varepsilon$, with*

$$\frac{d\omega^\varepsilon}{dm}\leqslant C_3(\zeta)\quad\textit{and}\quad\rho^\varepsilon(M)<\zeta.$$

Proof. — Let A be some Borel set in M. We have for each $n \geqslant 0$

$$\begin{aligned}\eta_n^\varepsilon(A) &= \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\int m\big(f_{\underline{t}}^{-n-j}(A)\cap R_{n,k}(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t})\\ &\leqslant \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\int m\big(f_{\underline{t}}^{-n}\big(f_{\sigma^n\underline{t}}^{-j}(A)\cap H_k^*(\sigma^n\underline{t})\big)\cap H_n(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t})\\ &\leqslant \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}C_2\int m\big(f_{\underline{t}}^{-j}(A)\cap H_k^*(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t}).\end{aligned}$$

(in this last inequality we used Proposition 5.2 and the fact that $\theta_\varepsilon^{\mathbb{N}}$ is σ-invariant). Let now $\zeta > 0$ be some fixed small number. Since we are assuming $(h_\varepsilon)_\varepsilon$ with uniform L^1-tail, then there is some integer $N = N(\zeta)$ for which

$$\sum_{j=N}^{\infty} k\int m\big(H_k^*(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t}) < \frac{\zeta}{C_2}.$$

We take

$$\omega^\varepsilon = C_2\sum_{k=2}^{N-1}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^j)_*\big(m \mid H_k^*(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t})$$

and

$$\rho^\varepsilon = C_2\sum_{k=N}^{\infty}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^j)_*\big(m \mid H_k^*(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t}).$$

For this last measure we have

$$\rho^\varepsilon(M) = C_2\sum_{k=N}^{\infty}\sum_{j=1}^{k-1}\int m\big(H_k^*(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t}) \leqslant C_2\sum_{k=N}^{\infty}k\int m\big(H_k^*(\underline{t})\big)d\theta_\varepsilon^{\mathbb{N}}(\underline{t}) < \zeta.$$

On the other hand, it follows from the definition of (α,δ)-hyperbolic times that there is some constant $a = a(N) > 0$ such that $\operatorname{dist}\big(H_k(\underline{t}),\mathcal{C}\big) \geqslant a$ for $1 \leqslant k \leqslant N$. Defining $\Delta \subset M$ as the set of those points in M whose distance to $\mathcal{C}$ is greater than a, we have

$$\omega^\varepsilon \leqslant C_2\sum_{k=2}^{N-1}\sum_{j=1}^{k-1}\int (f_{\underline{t}}^j)_*(m \mid \Delta)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{t}),$$

and this last measure has density bounded by some uniform constant, as long as we take the maps f_t in a sufficiently small neighborhood of f in the C^1 topology. $\square$

It follows from Remark 4.2, Proposition 5.3 and (22) that the weak* accumulation points of μ^ε when $\varepsilon \to 0$ cannot have singular part, thus being absolutely continuous with respect to the Lebesgue measure. Moreover, the weak* accumulation points of a family of stationary measures are always f-invariant measures, cf. Remark 3.1. This together with (P) gives the stochastic stability of f.

6. Applications

In this section we will apply Theorems B and D to certain classes of nonuniformly expanding maps. Before we describe the examples we have in mind let us give a practical criterion for proving that the family of hyperbolic time maps $(h_\varepsilon)_\varepsilon$ has uniform L^1-tail.

If we look at the proof of Proposition 2.3 we see that what we did was fixing some positive number c_0 smaller than c, and then, for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$, we took a positive integer $N_\varepsilon = N_\varepsilon(\underline{t}, x)$ for which

$$\sum_{j=0}^{N_\varepsilon - 1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| \leqslant -c_0 N_\varepsilon \quad \text{and} \quad \sum_{j=0}^{N_\varepsilon - 1} -\log \operatorname{dist}_\delta(f_{\underline{t}}^j(x), \mathcal{C}) \leqslant \gamma N_\varepsilon,$$

for suitable choices of $\delta > 0$ and $\gamma > 0$. This permits us to introduce a map

$$N_\varepsilon \colon T^{\mathbb{N}} \times M \longrightarrow \mathbb{Z}^+$$

whose existence provides a first hyperbolic time map

$$h_\varepsilon \colon T^{\mathbb{N}} \times M \longrightarrow \mathbb{Z}^+ \quad \text{with} \quad h_\varepsilon \leqslant N_\varepsilon$$

(recall the proof of Proposition 2.3). Thus, the integrability of the map h_ε is implied by the integrability of the map N_ε, which is in practice easier to handle.

Remark 6.1. — In the examples we are going to study below we will show that there is a sequence of positive real numbers $(a_k^\varepsilon)_k$ for which

$$(\theta_\varepsilon^{\mathbb{N}} \times m)\left(\left\{(\underline{t}, x) \in T^{\mathbb{N}} \times M \colon N_\varepsilon(\underline{t}, x) > k\right\}\right) \leqslant a_k^\varepsilon \quad \text{and} \quad \sum_{k=1}^{\infty} k a_k^\varepsilon < \infty,$$

This gives the integrability of h_ε with respect to the measure $\theta_\varepsilon^{\mathbb{N}} \times m$. The fact the family $(h_\varepsilon)_\varepsilon$ has uniform L^1-tail can be proved by showing that the sequence $(a_k^\varepsilon)_k$ may be chosen not depending on $\varepsilon > 0$.

Now we are ready for the applications of Theorems B and D. We will describe first a class of local diffeomorphisms introduced in [**ABV**, Appendix A] that satisfies the hypotheses of Theorem B, and then a class of maps (with critical sets) introduced in [**Vi1**] satisfying the hypotheses of Theorem D.

6.1. Local diffeomorphisms. — Now we follow [**ABV**, Appendix A] and describe robust classes of maps (open in the C^2 topology) that are nonuniformly expanding local diffeomorphisms and stochastically stable. Let M be a compact Riemannian manifold and consider

$$\begin{aligned} \Phi : T &\longrightarrow C^2(M, M) \\ t &\longmapsto f_t \end{aligned}$$

a continuous family of C^2 maps, where T is a metric space. We begin with an essentially combinatorial lemma.

Lemma 6.2. — *Let $p, q \geqslant 1$ be integers and $\sigma > q$ a real number. Assume M admits a measurable cover $\{B_1, \ldots, B_p, B_{p+1}, \ldots, B_{p+q}\}$ such that for all $t \in T$ it holds*

(1) $|\det Df_t(x)| \geqslant \sigma$ *for all* $x \in B_{p+1} \cup \cdots \cup B_{p+q}$*;*

(2) $(f_t \mid B_i)$ *is injective for all* $i = 1, \ldots, p$.

Then there is $\zeta > 0$ such that for every Borel probability θ on T we have

$$\#\{0 \leqslant j < n : f_{\underline{t}}^j(x) \in B_1 \cup \cdots \cup B_p\} \geqslant \zeta n \tag{23}$$

for $\theta^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ and large enough $n \geqslant 1$. Moreover the set I_n of points $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ whose orbits do not spend a fraction ζ of the time in $B_1 \cup \cdots \cup B_p$ up to iterate n is such that $(\theta^{\mathbb{N}} \times m)(I_n) \leqslant \tau^n$ for some $0 < \tau < 1$ and for large $n \geqslant 1$.

Proof. — Let us fix $n \geqslant 1$ and $\underline{t} \in T^{\mathbb{N}}$. For a sequence

$$\underline{i} = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, p+q\}^n$$

we write

$$[\underline{i}] = B_{i_0} \cap (f_{\underline{t}}^1)^{-1}(B_{i_1}) \cap \cdots \cap (f_{\underline{t}}^{n-1})^{-1}(B_{i_{n-1}})$$

and define $g(\underline{i}) = \#\{0 \leqslant j < n : i_j \leqslant p\}$.

We start by observing that for $\zeta > 0$ the number of sequences $\underline{i}$ such that $g(\underline{i}) < \zeta n$ is bounded by

$$\sum_{k<\zeta n} \binom{n}{k} p^k q^{n-k} \leqslant \sum_{k \leqslant \zeta n} \binom{n}{k} p^{\zeta n} q^n.$$

Using Stirling's formula (cf. [**BV**, Section 6.3]) the expression on the right hand side is bounded by $(e^{\gamma} p^{\zeta} q)^n$, where $\gamma > 0$ depends only on ζ and $\gamma(\zeta) \to 0$ when $\zeta \to 0$.

Assumptions (1) and (2) ensure $m([\underline{i}]) \leqslant \sigma^{-(1-\zeta)n}$ (recall that $m(M) = 1$). Hence the measure of the union $I_n(\underline{t})$ of all the sets $[\underline{i}]$ with $g(\underline{i}) < \zeta n$ is bounded by

$$\sigma^{-(1-\zeta)n} (e^{\gamma} p^{\zeta} q)^n.$$

Since $\sigma > q$ we may choose ζ so small that $e^{\gamma} p^{\zeta} q < \sigma^{(1-\zeta)}$. Then $m(I_n(\underline{t})) \leqslant \tau^n$ with $\tau = e^{\gamma+\zeta-1} \cdot p^{\zeta} \cdot q < 1$ for big enough $n \geqslant N$. Note that τ and N do not depend on $\underline{t}$. Setting

$$I_n = \bigcup_{\underline{t} \in T^{\mathbb{N}}} (\{\underline{t}\} \times I_n(\underline{t}))$$

we also have $(\theta^{\mathbb{N}} \times m)(I_n) \leqslant \tau^n$ for all big $n \geqslant N$ and for every Borel probability θ on T, by Fubini's Theorem. Since $\sum_n (\theta^{\mathbb{N}} \times m)(I_n) < \infty$ then Borel-Cantelli's Lemma implies

$$(\theta^{\mathbb{N}} \times m) \left(\bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} I_k \right) = 0$$

and this means that $\theta^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ satisfies (23). □

Lemma 6.3. — *Let $\{B_1, \ldots, B_p, B_{p+1}, \ldots, B_{p+q}\}$ be a measurable cover of M satisfying conditions* (1) *and* (2) *of Lemma 6.2. For $0 < \lambda < 1$ there are $\eta > 0$ and $c_0 > 0$ such that, if f_t also satisfies for all $t \in T$*

(3) $\|Df_t(x)^{-1}\| \leqslant \lambda < 1$ *for* $x \in B_1, \dots, B_p$;

(4) $\|Df_t(x)^{-1}\| \leqslant 1 + \eta$ *for* $x \in B_{p+1}, \dots, B_{p+q}$;

then we have for $f \equiv f_{t^*}$, *where* t^* *is some given point in* T,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f_{\underline{t}}^j(x))^{-1}\| \leqslant -c_0 \tag{24}$$

for $\theta^{\mathbb{N}} \times m$ *almost all* $(\underline{t}, x) \in T^{\mathbb{N}} \times M$, *where* θ *is any Borel probability measure on* T. *Moreover the first hyperbolic time map* $h : T^{\mathbb{N}} \times M \to \mathbb{Z}^+$ *satisfies*

$$(\theta^{\mathbb{N}} \times m)\{(\underline{t}, x) \in T^{\mathbb{N}} \times M : h(\underline{t}, x) > k\} \leqslant a_k \quad \text{and} \quad \sum_{k=1}^{\infty} k a_k < \infty$$

with $(a_k)_k$ *independent of the choice of* θ.

Proof. — Let $\zeta > 0$ be the constant provided by Lemma 6.2. We fix $\eta > 0$ sufficiently small so that $\lambda^{\zeta}(1+\eta) \leqslant e^{-c_0}$ holds for some $c_0 > 0$ and take $(\underline{t}, x)$ satisfying (23). Conditions (3) and (4) now imply

$$\prod_{j=0}^{n-1} \|Df(f_{\underline{t}}^j(x))^{-1}\| \leqslant \lambda^{\zeta n}(1+\eta)^{(1-\zeta)n} \leqslant e^{-c_0 n}. \tag{25}$$

for large enough n. This means (25) holds for $\theta^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$.

We observe that if $h(\underline{t}, x) = k$, then $1 \leqslant n < k$ cannot be hyperbolic times for $(\underline{t}, x)$. Hence $(\underline{t}, x) \in I_n$ for all $n = 1, \dots, k-1$. In particular

$$(\theta^{\mathbb{N}} \times m)\{(\underline{t}, x) \in T^{\mathbb{N}} \times M : h(\underline{t}, x) = k\} \leqslant (\theta^{\mathbb{N}} \times m)(I_{k-1}) \equiv a_k$$

and $\sum_k k a_k \leqslant \sum_k k \tau^{k-1} < \infty$. □

Now we will show that families of C^2 maps satisfying conditions (1) through (4) of Lemmas 6.2 and 6.3 contain open sets of families in the C^2 topology. Let M be a n-dimensional torus $\mathbb{T}^n$ and $f_0 : M \to M$ a uniformly expanding map: there exists $0 < \lambda < 1$ such that $\|Df_0(x)v\| \geqslant \lambda^{-1}\|v\|$ for all $x \in M$ and $v \in T_x M$. Let also W be some small compact domain in M where $f_0 \mid W$ is injective. Observe that f_0 is a volume expanding local diffeomorphism due to the uniform expansion.

Modifying f_0 by an isotopy inside W we may obtain a map f_1 which coincides with f_0 outside W, is volume expanding in M, i.e., $|\det Df_1(x)| > 1$ for all $x \in M$, and has bounded contraction on W near 1: $\|Df_1(x)^{-1}\| \leqslant 1 + \eta$ for every $x \in W$ and some $\eta > 0$ small. This new map f_1 may be taken C^1 close to f_0 and we may consider a C^2 map f_2 arbitrarily C^1 close to f_1.

Now any map f in a small enough C^2 neighborhood of f_2 admits $\sigma > 1$ such that $|\det Df(x)| \geqslant \sigma$ for all $x \in M$ and, for x outside W, we have $\|Df(x)^{-1}\| \leqslant \lambda$. If the C^2 neighborhood is taken sufficiently small then we maintain $\|Df(x)^{-1}\| \leqslant 1 + \eta$ for $x \in W$ and for some small $\eta > 0$. Let us take $B_1, \dots, B_p, B_{p+1} = W$ a partition of M into measurable sets where the restriction $f \mid B_i$ is injective for $i = 1, \dots, p+1$. Then any continuous family of C^2 maps $\Phi : T \to C^2(M, M)$ together with a family $(\theta_\varepsilon)_{\varepsilon>0}$ of Borel probability measures in the metric space T, satisfying $\operatorname{supp}(\theta_\varepsilon) \to \{t^*\}$ when $\varepsilon \to 0$ and $f_{t^*} \equiv f$, for some $t^* \in T$, is such that f is nonuniformly expanding for random orbits and $(h_\varepsilon)_{\varepsilon>0}$ has uniform L^1-tail — by Lemma 6.3 with $q = 1$ and $T = \operatorname{supp}(\theta_\varepsilon)$ for small enough $\varepsilon > 0$. Theorem B then shows

Corollary 6.4. — *There are open sets $\mathcal{U} \subset C^2(M, M)$ such that every $f \in \mathcal{U}$ is a stochastically stable nonuniformly expanding local diffeomorphism.*

6.2. Viana maps. — In what follows we study the class of nonuniformly expanding maps with critical sets introduced by M. Viana and prove Theorem E.

6.2.1. Nonuniform expansion. — Let $\widehat{f}$ be defined as in Subsection 1.1.2. The results in [**Vi1**] show that if the map f is sufficiently close to $\widehat{f}$ in the C^3 topology then f has two positive Lyapunov exponents almost everywhere: there is a constant $\lambda > 0$ for which

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|Df^n(s, x)v\| \geqslant \lambda$$

for Lebesgue almost every $(s, x) \in S^1 \times I$ and every non-zero $v \in T_{(s,x)}(S^1 \times I)$. As mentioned in [**ABV**], this does not necessarily imply that f is nonuniformly expanding. However a slight modification in Viana's arguments enables us to prove the nonuniform expansion of f.

For the sake of clearness, we start by assuming that f has the special form

$$f(s, x) = (g(s), q(s, x)), \quad \text{with} \quad \partial_x q(s, x) = 0 \quad \text{if and only if} \quad x = 0, \tag{26}$$

and describe how the conclusions in [**Vi1**] are obtained for each C^2 map f satisfying

$$\|f - \widehat{f}\|_{C^2} \leqslant \alpha \quad \text{on} \quad S^1 \times I. \tag{27}$$

Then we explain how these conclusions extend to the general case, using the existence of a central invariant foliation, and we show how the results in [**Vi1**] give the nonuniform expansion and slow approximation of orbits to the critical set for each map f as in (27).

The estimates on the derivative rely on a statistical analysis of the returns of orbits to the neighborhood $S^1 \times (-\sqrt{\alpha}, \sqrt{\alpha}\,)$ of the critical set $\mathcal{C} = \{(s, x) : x = 0\}$. We set

$$J(0) = I \setminus (-\sqrt{\alpha}, \sqrt{\alpha}) \quad \text{and} \quad J(r) = \{x \in I : |x| < e^{-r}\} \quad \text{for } r \geqslant 0.$$

From here on we only consider points $(s, x) \in S^1 \times I$ whose orbit does not hit the critical set $\mathcal{C}$. This constitues no restriction in our results, since the set of those points has full Lebesgue measure.

For each integer $j \geqslant 0$ we define $(s_j, x_j) = f^j(s, x)$ and

$$r_j(s, x) = \min\{r \geqslant 0 : x_j \in J(r)\}.$$

Consider, for some small constant $0 < \eta < 1/4$,

$$G = \left\{0 \leqslant j < n : r_j(s,x) \geqslant \left(\frac{1}{2} - 2\eta\right) \log \frac{1}{\alpha}\right\}.$$

Fix some integer $n \geqslant 1$ sufficiently large (only depending on $\alpha > 0$). The results in [**Vi1**] show that if we take

$$B_2(n) = \big\{(s,x) : \text{ there is } 1 \leqslant j < n \text{ with } x_j \in J([\sqrt{n}])\ \big\},$$

where $[\sqrt{n}]$ is the integer part of $\sqrt{n}$, then we have

$$m(B_2(n)) \leqslant \operatorname{const} e^{-\sqrt{n}/4} \tag{28}$$

and, for every small $c > 0$ (only depending on the quadratic map Q),

$$\log \prod_{j=0}^{n-1} |\partial_x q(s_j, x_j)| \geqslant 2cn - \sum_{j \in G} r_j(s,x) \quad \text{for } (s,x) \notin B_2(n), \tag{29}$$

see [**Vi1**, pp. 75 & 76]. Moreover, if we define for $\gamma > 0$

$$B_1(n) = \left\{(s,x) \notin B_2(n) : \sum_{j \in G} r_j(s,x) \geqslant \gamma n\right\},$$

then, for small $\gamma > 0$, there is a constant $\xi > 0$ for which

$$m\big(B_1(n)\big) \leqslant e^{-\xi n}, \tag{30}$$

see [**Vi1**, p. 77]. Taking into account the definitions of $J(r)$ and r_j, this shows that if we take $\delta = (1/2 - 2\eta)\log(1/\alpha)$, then

$$\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \leqslant \gamma n \quad \text{for } (s,x) \notin B_1(n) \cup B_2(n).$$

This in particular gives that almost all orbits have slow approximation to $\mathcal{C}$.

On the other hand, we have for $(s,x) \in S^1 \times I$

$$\big(Df(s,x)\big)^{-1} = \frac{1}{\partial_x q(s,x)\partial_s g(s)} \begin{pmatrix} \partial_x q(s,x) & 0 \\ -\partial_s q(s,x) & \partial_s g(s) \end{pmatrix}. \tag{31}$$

Since all the norms are equivalent in finite dimensional Banach spaces, it is no restriction for our purposes to take the norm of $\big(Df(s,x)\big)^{-1}$ as the maximum of the absolute values of its entries. From (26) and (27) we deduce that for small α

$$|\partial_s g| \geqslant d - \alpha, \quad |\partial_s q| \leqslant \alpha|b'| + \alpha \leqslant 8\alpha \quad \text{and} \quad |\partial_x q| \leqslant |2x| + \alpha \leqslant 4,$$

which together with (31) gives

$$\big\|\big(Df(s,x)\big)^{-1}\big\| = |\partial_x q(s,x)|^{-1},$$

as long as $\alpha > 0$ is taken sufficiently small. This implies

$$\sum_{j=0}^{n-1} \log \|Df(s_j,x_j))^{-1}\| = -\sum_{j=0}^{n-1} \log |\partial_x q(s_j, x_j)| \tag{32}$$

for every $(s,x)\in S^1\times I$. If we choose $\gamma<c$, then we have

$$\sum_{j=0}^{n-1}\log|\partial_x q(s_j,x_j)| = \log\prod_{j=0}^{n-1}|\partial_x q(s_j,x_j)| \geqslant cn \tag{33}$$

for every $(s,x)\notin B_1(n)\cup B_2(n)$ (recall (29) and the definition of $B_1(n)$). We conclude from (32) and (33) that

$$\sum_{j=0}^{n-1}\log\|Df(s_j,x_j))^{-1}\| \leqslant -cn \quad \text{for } (s,x)\notin B_1(n)\cup B_2(n),$$

which, in view of the estimates on the Lebesgue measure of $B_1(n)$ and $B_2(n)$, proves that f is a nonuniformly expanding map.

Now we describe how in [**Vi1**] the same conclusions are obtained without assuming (26). Since $\widehat{f}$ is strongly expanding in the horizontal direction, it follows from the methods of [**HPS**] that any map f sufficiently close to $\widehat{f}$ admits a unique invariant central foliation $\mathcal{F}^c$ of $S^1\times I$ by smooth curves uniformly close to vertical segments, see [**Vi1**, Section 2.5]. Actually, $\mathcal{F}^c$ is obtained as the set of integral curves of a vector field $(\xi^c,1)$ in $S^1\times I$ with ξ^c uniformly close to zero. The previous analysis can then be carried out in terms of the expansion of f along this central foliation $\mathcal{F}^c$. More precisely, $|\partial_x q(s,x)|$ is replaced by

$$|\partial_c q(s,x)| \equiv |Df(s,x)v_c(s,x)|,$$

where $v_c(s,x)$ is a unit vector tangent to the foliation at (s,x). The previous observations imply that v_c is uniformly close to $(0,1)$ if f is close to $\widehat{f}$. Moreover, cf. [**Vi1**, Section 2.5], it is no restriction to suppose $|\partial_c q(s,0)|\equiv 0$, so that $\partial_c q(s,x)\approx |x|$, as in the unperturbed case. Indeed, if we define the *critical set* of f by

$$\mathcal{C}=\{(s,x)\in S^1\times I : \partial_c q(s,x)=0\}.$$

by an easy implicit function argument it is shown in [**Vi1**, Section 2.5] that $\mathcal{C}$ is the graph of some C^2 map $\eta: S^1\to I$ arbitrarily C^2-close to zero if α is small. This means that up to a change of coordinates C^2-close to the identity we may suppose that $\eta\equiv 0$ and, hence, write for $\alpha>0$ small

$$\partial_c q(s,x)=x\psi(s,x) \quad \text{with } |\psi+2| \text{ close to zero.}$$

This provides an analog to the second part of assumption (26). At this point, the arguments apply with $\partial_x q(s,x)$ replaced by $\partial_c q(s,x)$, to show that orbits have slow approximation to the critical set $\mathcal{C}$ and $\prod_{i=0}^{n-1}|\partial_c q(s_i,x_i)|$ grows exponentially fast for Lebesgue almost every $(s,x)\in S^1\times I$. A matrix formula for $\big(Df^n(s,x)\big)^{-1}$ similar to that in (31) can be obtained if we replace the vector $(0,1)$ in the canonical basis of the space tangent to $S^1\times I$ at (s,x) by $v_c(s,x)$, and consider the matrix of $\big(Df^n(s,x)\big)^{-1}$ with respect to the new basis.

For future reference, let us make some considerations on the way the sets $B_1(n)$ and $B_2(n)$ are obtained. Let $X : S^1 \to I$ be a smooth map whose graph in $S^1 \times I$ is nearly horizontal (see the notion of admissible curve in [**Vi1**, Section 2] for a precise definition). Denote $\widehat{X}_n(s) = f^n\big(s, X(s)\big)$ for $n \geqslant 0$ and $s \in S^1$. Take some leaf L_0 of the foliation $\mathcal{F}^c$. Letting $L_n = f^n(L_0)$ for $n \geqslant 1$, we define a sequence of Markov partitions $(\mathcal{P}_n)_n$ of S^1 in the following way:

$$\mathcal{P}_n = \Big\{[s', s''):\, (s', s'') \text{ is a connected component of } \widehat{X}_n^{-1}\big((S^1 \times I) \setminus L_n\big)\Big\}.$$

It is easy to check that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \geqslant 1$ and

$$(d + \operatorname{const}\alpha)^{-n} \leqslant |\omega| \leqslant (d - \operatorname{const}\alpha)^{-n}$$

for each $\omega \in \mathcal{P}_n$. Due to the large expansion of f in the horizontal direction, we have that if $J \subset I$ is an interval with $|J| \leqslant \alpha$, then for each $\omega \in \mathcal{P}_n$

$$m\big(\{s \in \omega\colon \widehat{X}_j(s) \in S^1 \times J\}\,\big) \leqslant \operatorname{const} \sqrt{|J|}\, m(\omega) \tag{34}$$

see [**Vi1**, Corollary 2.3]. The estimate (28) on the Lebesgue measure of $B_2(n)$ is now an easy consequence of (34). For that we only have to compute the Lebesgue measure of $B_2(n)$ on each horizontal line of $S^1 \times I$ and integrate. The estimate (28) on the Lebesgue measure of $B_1(n)$ is obtained by means of a large deviations argument applied to the horizontal curves in $S^1 \times I$; see [**Vi1**, pp. 76 & 77].

Remark 6.5. — The choice of the constants c, ξ, γ and δ only depends on the quadratic map Q and $\alpha > 0$. In particular the decay estimates on the Lebesgue measure of $B_1(n)$ and $B_2(n)$ only depend on the quadratic map Q and $\alpha > 0$.

6.2.2. Random perturbations. — Let f be close to $\widehat{f}$ in the C^3 topology. As we have seen before, it is no restriction to assume that $\mathcal{C} = \{(s, x) \in S^1 \times I\colon x = 0\}$ is the critical set of f. Fix $\{\Phi, (\theta_\varepsilon)_\varepsilon\}$ a random perturbation of f for which (8) holds. Our goal now is to prove that any such f satisfies the hypotheses of Theorems C and D for $\varepsilon > 0$ sufficiently small, and thus conclude that f is stochastically stable. So, we want to show that if $\varepsilon > 0$ is small enough then

- f is nonuniformly expanding for random orbits;
- random orbits have slow approximation to the critical set $\mathcal{C}$;
- the family of hyperbolic time maps $(h_\varepsilon)_\varepsilon$ has uniform L^1-tail.

We remark that in the estimates we have obtained for $\log\|(Df(s_j, x_j))^{-1}\|$ and $\log\operatorname{dist}_\delta(x_j, \mathcal{C})$ over the orbit of a given point $(s, x) \in S^1 \times I$, we can easily replace the iterates (s_j, x_j) by random iterates $(s^j_{\underline{t}}, x^j_{\underline{t}}) = f^j_{\underline{t}}(s, x)$. Actually, the methods used for obtaining estimate (29) rely on a delicate decomposition of the orbit of a given point (s, x) from time 0 until time n into finite pieces according to its returns to the neighborhood $S^1 \times (-\sqrt{\alpha}, \sqrt{\alpha})$ of the critical set. The main tools are [**Vi1**, Lemma 2.4] and [**Vi1**, Lemma 2.5] whose proofs may easily be mimicked for random orbits. Indeed, the important fact in the proof of the referred lemmas is that orbits of points in the central direction stay close to orbits of the quadratic map Q for long periods, as long as $\alpha > 0$ is taken sufficiently small. Hence, such results can easily be obtained for random orbits as long as we take $\varepsilon > 0$ with $\varepsilon \ll \alpha$ and perturbation vectors $\underline{t} \in \operatorname{supp}(\theta_\varepsilon)$.

Thus, the procedure of [**Vi1**] described in Subsection 6.2.1 applies to this situation, and we are able to prove that there is $c > 0$, and for $\gamma > 0$ there is $\delta > 0$, such that

$$\sum_{j=0}^{n-1} \log \|Df(s^j_{\underline{t}}, x^j_{\underline{t}}))^{-1}\| \leqslant -cn \quad \text{and} \quad \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(x^j_{\underline{t}}, \mathcal{C}) \leqslant \gamma n$$

for $(s, x) \notin B_1(n) \cup B_2(n)$, where $B_1(n)$ and $B_2(n)$ are subsets $S^1 \times I$ with

$$m\big(B_1(n)\big) \leqslant e^{-\xi n} \quad \text{and} \quad m(B_2(n)) \leqslant \operatorname{const} e^{-\sqrt{n}/4}$$

for some constant $\xi > 0$ only depending on γ. This gives the nonuniform expansion and slow approximation to the critical set for random orbits. Moreover, the arguments show that we may take the map N_ε with

$$(\theta_\varepsilon^{\mathbb{N}} \times m)\left(\left\{(\underline{t}, x) \in T^{\mathbb{N}} \times M \colon N_\varepsilon(\underline{t}, x) > n\right\}\right) \leqslant \operatorname{const} e^{-\sqrt{n}/4},$$

thus giving that the family of first hyperbolic time maps has uniform L^1-tail; cf. Remark 6.1.

For the sake of completeness, an explanation is required on the way the Markov partitions $\mathcal{P}_n$ of S^1 can be defined in this case, in order to obtain the estimates on the Lebesgue measure of $B_1(n)$ and $B_2(n)$. We consider $M = S^1 \times I$ and define the skew-product map

$$\begin{aligned} F : T^{\mathbb{N}} \times M &\longrightarrow \quad T^{\mathbb{N}} \times M, \\ (\underline{t}, z) \quad &\longmapsto \big(\sigma(\underline{t}), f_{t_1}(z)\big) \end{aligned}$$

where σ is the left shift map. Writing $f_t(z) = \big(g_t(z), q_t(z)\big)$ for $z = (s, x) \in S^1 \times I$, we have that $q_t(s, \cdot)$ is a unimodal map close to $\widehat{q}$ for all $s \in S^1$ and $t \in \operatorname{supp}(\theta_\varepsilon)$ with $\varepsilon > 0$ small.

Proposition 6.6. — *Given $\underline{t} \in T^{\mathbb{N}}$ there is a C^1 foliation $\mathcal{F}^c_{\underline{t}}$ of M such that if $L_{\underline{t}}(z)$ is the leaf of $\mathcal{F}^c_{\underline{t}}$ through a point $z \in M$, then*

(1) *$L_{\underline{t}}(z)$ is a C^1 submanifold of M close to a vertical line in the C^1 topology;*
(2) *$f_{t_1}\big(L_{\underline{t}}(z)\big)$ is contained in $L_{\sigma\underline{t}}\big(f_{t_1}(z)\big)$.*

Proof. — This will be obtained as a consequence of the fact that the set of vertical lines constitutes a normally expanding invariant foliation for $\widehat{f}$. Let $\mathcal{H}$ be the space of continuous maps $\xi : T^{\mathbb{N}} \times M \to [-1, 1]$ endowed with the sup norm, and define the map $A : \mathcal{H} \to \mathcal{H}$ by

$$A\xi(\underline{t}, z) = \frac{\partial_x q_{t_1}(z)\xi(F(\underline{t}, z)) - \partial_x g_{t_1}(z)}{-\partial_s q_{t_1}(z)\xi(F(\underline{t}, z)) + \partial_s g_{t_1}(z)}, \quad \underline{t} = (t_1, t_2, \dots) \in T^{\mathbb{N}} \quad \text{and} \quad z \in M.$$

Note that A is well-defined, since

$$|A\xi(\underline{t},z)| \leqslant \frac{(4+\alpha+\varepsilon)+\alpha+\varepsilon}{-(\text{const }\alpha+\varepsilon)+(d-\alpha-\varepsilon)} < 1$$

for small $\alpha > 0$ and $\varepsilon > 0$. Moreover, A is a contraction on $\mathcal{H}$: given $\xi,\zeta \in \mathcal{H}$ and $(\underline{t},z) \in T^{\mathbb{N}} \times M$ then

$$\begin{aligned}
&|A\xi(\underline{t},z) - A\zeta(\underline{t},z)| \\
&\qquad \leqslant \frac{|\det Df_{t_1}(z)| \cdot |\xi(\underline{t},z) - \zeta(\underline{t},z)|}{\big|\big(-\partial_s q_{t_1}(z)\xi(F(\underline{t},z)) + \partial_s g_{t_1}(z)\big)\cdot\big(-\partial_s q_{t_1}(z)\zeta(F(\underline{t},z)) + \partial_s g_{t_1}(z)\big)\big|} \\
&\qquad \leqslant \frac{\big((d+\alpha+\varepsilon)(4+\alpha+\varepsilon)+\alpha+\varepsilon\big)\cdot\big|\xi(\underline{t},z)-\zeta(\underline{t},z)\big|}{(d-\text{const}\alpha-\varepsilon)^2}.
\end{aligned}$$

This last quantity can be made smaller than $|\xi(\underline{t},z) - \eta(\underline{t},z)|/2$, as long as α and ε are chosen sufficiently small. This shows that A is a contraction on the Banach space $\mathcal{H}$, and so it has a unique fixed point $\xi^c \in \mathcal{H}$.

It is no restriction for our purposes if we think of T as being equal to $\text{supp}(\theta_\varepsilon)$ for some small ε. Note that the map A depends continuously on F and for $\varepsilon > 0$ small enough the fixed point of A is close to the zero constant map. This holds because we are choosing $\text{supp}(\theta_\varepsilon)$ close to $\{t^*\}$, $f_{t^*} = f$ and f close to $\widehat{f}$. Then, for $\varepsilon > 0$ small enough, we have $\xi^c(\underline{t},\cdot)$ uniformly close to $\xi^c(\underline{t}^*,\cdot)$ and it is not hard to check that $\xi^c_0 = \xi^c(\underline{t}^*,\cdot)$ is precisely the map whose integral leaves of the vector field $(\xi^c_0, 1)$ give the invariant foliation $\mathcal{F}^c$ associated to $f_{t^*} = f$. Since this foliation depends continuously on the dynamics and for $f = \widehat{f}$ we have $\xi^c_0 \equiv 0$ (see [**Vi1**, Section 2.5]), we finally deduce that $\xi^c(\underline{t},\cdot)$ is uniformly close to zero for small $\varepsilon > 0$.

We have defined A in such a way that if we take $E^c(\underline{t},z) = \text{span}\{(\xi^c(\underline{t},z),1)\}$, then for every $\underline{t} \in T^{\mathbb{N}}$ and $z \in S^1 \times I$

$$Df_{t_1}(z)E^c(\underline{t},z) \subset E^c(F(\underline{t},z)). \tag{35}$$

Now, for fixed $\underline{t} \in T^{\mathbb{N}}$, we take $\mathcal{F}^c_{\underline{t}}$ to be the set of integral curves of the vector field $z \mapsto (\xi^c(\underline{t},z),1)$ defined on $S^1 \times I$. Since the vector field is taken of class C^0, it does not follow immediately that through each point in $S^1 \times I$ passes only one integral curve. We will prove uniqueness of solutions by using the fact that the map f has a big expansion in the horizontal direction.

Assume, by contradiction, that there are two distinct integral curves $Y, Z \in \mathcal{F}^c_{\underline{t}}$ with a common point. So we may take three distinct nearby points $z_0, z_1, z_2 \in S^1 \times I$ such that $z_0 \in Y \cap Z$, $z_1 \in Y$, $z_2 \in Z$ and z_1, z_2 have the same x-coordinate. Let X be the horizontal curve joining z_1 to z_2. If we consider $X_n = \pi_2 \circ F^n(\underline{t}, X)$ for $n \geqslant 1$, where π_2 is the projection from $T^{\mathbb{N}} \times S^1 \times I$ onto $S^1 \times I$, we have that the curves X_n are nearly horizontal and grow in the horizontal direction (when n increases) by a factor close to d for small α and ε, see [**Vi1**, Section 2.1]. Hence, for large n, X_n wraps many times around the cylinder $S^1 \times I$. On the other hand, since $Y_n = \pi_2 \circ F^n(\underline{t}, Y)$ and $Z_n = \pi_2 \circ F^n(\underline{t}, Z)$ are always tangent to the vector field $z \mapsto \big(\xi^c(\sigma^n\underline{t},z),1\big)$

on $S^1 \times I$, it follows that all the iterates of Y_n and Z_n have small amplitude in the s-direction. This gives a contradiction, since the closed curve made by Y, Z and X is homotopic to zero in $S^1 \times I$ and the closed curve made by Y_n, Z_n and X_n cannot be homotopic to zero for large n. Thus, for fixed $\underline{t} \in T^{\mathbb{N}}$ we have uniqueness of solutions of the vector field $z \to (\xi^c(\underline{t}, z), 1)$, and from (35) it follows that $\mathcal{F}^c_{\underline{t}}$ is an F-invariant foliation of M by nearly vertical leaves. □

Now, using the foliations given by the previous proposition we are also able to define the Markov partitions of S^1 in this setting. Given any smooth map $X : S^1 \to I$ whose graph is nearly horizontal, denote $\widehat{X}^n_{\underline{t}}(s) = f^n_{\underline{t}}\big(s, X(s)\big)$ for $n \geqslant 0$ and $s \in S^1$. Take some leaf $L^0_{\underline{t}}$ of the foliation $\mathcal{F}^c_{\underline{t}}$. Letting $L^n_{\underline{t}} = f^n_{\underline{t}}(L_{\underline{t}})$ for $n \geqslant 1$, we define the sequence of Markov partitions $(\mathcal{P}^n_{\underline{t}})_n$ of S^1 as

$$\mathcal{P}^n_{\underline{t}} = \Big\{[s', s'')\colon (s', s'') \text{ is a connected component of } (\widehat{X}^n_{\underline{t}})^{-1}\big((S^1 \times I) \smallsetminus L^n_{\underline{t}}\big)\Big\}.$$

It is easy to check that $\mathcal{P}^{n+1}_{\underline{t}}$ refines $\mathcal{P}^n_{\underline{t}}$ for each $n \geqslant 1$ and, taking $\varepsilon \ll \alpha$,

$$(d + \operatorname{const} \alpha)^{-n} \leqslant |\omega| \leqslant (d - \operatorname{const} \alpha)^{-n}$$

for each $\omega \in \mathcal{P}^n_{\underline{t}}$. This permits to obtain estimates (28) and (30) for the Lebesgue measure of the sets $B_1(n)$ and $B_2(n)$ exactly in the same way as in Subsection 6.2.1, also with the constants only depending on the quadratic map Q (cf. Remark 6.5).

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