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Astérisque, tome 287 (2003), p. 251-272

<http://www.numdam.org/item?id=AST_2003__287__251_0>

Astérisque
287, 2003, p. 251–272

COMPLEX SCHOTTKY GROUPS

by

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Abstract. — In this work we study a certain type of discrete groups acting on higher dimensional complex projective spaces. These generalize the classical Schottky groups acting on the Riemann sphere. We study the limit sets of these actions, which turn out to be solenoids. We also look at the compact complex manifols obtained as quotient of the region of discontinuity, divided by the action. We determine their topology and the dimension of the space of their infinitesimal deformations. We show that every such deformation arises from a deformation of the embedding of the group in question into the group of automorphisms of the corresponding complex projective space, which is a reminiscent of the classical Teichmüller theory.

Introduction

The theory of Kleinian groups introduced by Poincaré [**Po**] in the 1880's played a major role in many parts of mathematics throughout the 20th century, as for example in Riemann surfaces and Teichmüller theory, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry, 3-manifolds theory, etc. These groups are, by definition, discrete groups of holomorphic automorphisms of the complex projective line $P^1_{\mathbb{C}}$, whose limit set is not the whole $P^1_{\mathbb{C}}$. Equivalently, these can be regarded as groups of isometries of the hyperbolic 3-space, or as groups of conformal automorphisms of the sphere S^2. Much of the theory of Kleinian groups has been generalised to conformal Kleinian groups in higher dimensions (also called

2000 ***Mathematics Subject Classification.*** — Primary: 32H, 58D, 58E20, 58F23, 53C; Secondary: 20H10, 22E40, 30D05, 30F40, 30F35, 32L25, 32J18, 32G05, 32M05, 57S20, 57S25.
Key words and phrases. — Kleinian groups, Schottky groups, complex manifolds, complex solenoids or laminations, minimal sets, ergodicity, Hausdorff dimension.

Research partially supported by CONACYT (Mexico), projects E-27992 and E-27958, PAPIIT (UNAM, Mexico) IN 104 596 IM, Academia Mexicana de Ciencias and the London Royal Society.

Möbius or *hyperbolic* Kleinian groups), *i.e.*, to discrete groups of conformal automorphisms of the sphere S^n whose limit set is not the whole sphere (see, for instance, [**Ku1, Ku2, Ma1, Su1, Su2, Su3, Su4**]).

Many interesting results about the dynamics of rational maps on $P^1_{\mathbb{C}}$ in the last decades have been motivated by the dynamics of Kleinian groups, and there is an interesting "dictionary" between these two theories (see, for instance, [**Su1, Su2, Su3, Su4, Mc1, Mc2**]). The theory of rational maps has also been generalised to automorphisms of $P^2_{\mathbb{C}}$, and recently many results are being obtained about the dynamics of automorphisms and rational endomorphisms of $P^n_{\mathbb{C}}$ in general. This led us to define in [**SV**] the concept of a *higher dimensional complex Kleinian groups*. By this we meant (infinite) discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, the group of *holomorphic* automorphisms of $P^n_{\mathbb{C}}$, $n > 1$, acting with a non-empty region of discontinuity.

One of the most interesting families of (conformal) Kleinian groups is provided by the Schottky groups, and the aim of this article is to study the analogous construction for groups acting by holomorphic transformations on complex projective spaces. We call these *Complex Schottky Groups*.

We consider an arbitrary configuration $\{(L_1, M_1), \ldots, (L_r, M_r)\}$ of pairs of projective n-spaces in $P^{2n+1}_{\mathbb{C}}$, which are all of them pairwise disjoint. Given arbitrary neighbourhoods $U_1, \ldots, U_r$ of the L_i's, pairwise disjoint, we show that there exists, for each $i = 1, \ldots, r$, projective transformations T_i of $P^{2n+1}_{\mathbb{C}}$, which interchange the interior with the exterior of a compact tubular neighbourhood N_i of L_i contained in U_i, leaving invariant the boundary $E_i = \partial(N_i)$. The E_i's are *mirrors*, they play the same role in $P^{2n+1}_{\mathbb{C}}$ as circles play in S^2 to define the classical Schottky groups. Each mirror E_i is a $(2n+1)$-sphere bundle over $P^n_{\mathbb{C}}$. The group of automorphisms of $P^{2n+1}_{\mathbb{C}}$ generated by the T_i's is a complex Kleinian group Γ. The region of discontinuity $\Omega(\Gamma)$ is a fibre bundle over $P^n_{\mathbb{C}}$ with fibre S^{2n+2} minus a Cantor set $\mathcal{C}$. The limit set Λ is the complement of $\Omega(\Gamma)$ in $P^{2n+1}_{\mathbb{C}}$; it is the set of accumulation points of the Γ-orbit of the $L_i's$, and it is a product $\mathcal{C} \times P^n_{\mathbb{C}}$. The action of Γ on this set of projective lines is minimal in the sense that the Γ-orbit of every point x_o in $P^{2n+1}_{\mathbb{C}}$ accumulates to (at least a point in) each one of the projective lines in Λ. This set is transversally *projectively self-similar*, i.e., Λ corresponds to a Cantor set in the Grassmannian $G_{2n+1,n}$, which is dynamically-defined. Hence Λ is a *solenoid* (or *lamination*) by projective spaces, which is transversally Cantor and projectively self-similar. Each of these groups Γ contains a subgroup $\check{\Gamma}$ of index two, which is a free group of rank $r-1$ and acts *freely* on $\Omega(\Gamma)$. The quotient $\Omega(\Gamma)/\check{\Gamma}$ is a compact complex manifold, which is a fibre bundle over $P^n_{\mathbb{C}}$ with fibre the connected sum of $(r-1)$ copies of $S^{2n+1} \times S^1$. As mentioned above, these manifolds have a canonical projective structure [**Gu**], *i.e.*, they have an atlas $\{(\mathcal{U}_i, \phi_i)\}$ whose changes of coordinates are restrictions of complex projective transformations. However, these manifolds are never Kähler, due to cohomological reasons. When $n = 1$, the manifolds that we obtain are *Pretzel twistor spaces* in the sense of [**Pe**]; and if the configuration $\{(L_1, M_1), \ldots, (L_r, M_r)\}$ consists of twistor lines of the fibration $p\colon P^3_{\mathbb{C}} \to S^4$, then Γ and $\check{\Gamma}$ descend to conformal Schottky groups on S^4. In this case $\Omega(\Gamma)/\check{\Gamma}$ is the twistor space of the conformally flat manifold $S^4/p(\check{\Gamma})$, which is a Schottky manifold [**Ku2**]; $\Omega(\Gamma)/\check{\Gamma}$ is a *flat twistor space* [**Si**]. We also generalise our construction of Schottky groups to $P^{\infty}_{\mathbb{C}}$, the projectivization of a separable complex infinite dimensional Hilbert space.

We then compare the deformations of our Schottky groups with the deformations of the complex manifolds that one gets as quotients of the action of the group on its region of discontinuity. For this we estimate an upper bound for the Hausdorff dimension of the limit set of the complex Schottky groups. We use this to show that, with the appropriate conditions for the Schottky group $\check{\Gamma}$, the Kuranishi space $\mathfrak{K}$ of versal deformations of the complex manifold $M_{\check{\Gamma}} = \Omega(\check{\Gamma})/\check{\Gamma}$, is smooth near the reference point determined by $M_{\check{\Gamma}}$. Furthermore, we estimate the dimension of $\mathfrak{K}$ and we prove that every infinitesimal deformation of $M_{\check{\Gamma}}$ actually corresponds to an infinitesimal deformation of the group $\check{\Gamma}$ in the projective group $\mathrm{PSL}(2n+2, \mathbb{C})$, in analogy with the classical Teichmüller and moduli theory for Riemann surfaces.

While preparing this article we had very useful conversations with a number of people, and we wish to thank all of them for their support. We are most grateful to François Lescure for very useful comments regarding the algebraic aspects of this paper. Special thanks are also due to Misha Kapovich and Etienne Ghys.

1. Complex Schottky groups

We recall that (in the classical case) the Schottky groups are obtained by considering pairwise disjoint $(n-1)$-spheres $\mathbb{S}_1, \ldots, \mathbb{S}_r$ in S^n, see [**Ma2**]. Each sphere $\mathbb{S}_i$ plays the role of a mirror: it divides S^n in two diffeomorphic components, and one has an involution T_i of S^n interchanging these components, the inversion on $\mathbb{S}_i$. The Schottky group is defined to be the group of conformal transformations generated by these involutions. We are going to make a similar construction on $P^{2n+1}_{\mathbb{C}}$, $n > 0$. (For $n = 0$, if we take $P^0_{\mathbb{C}}$ to be a point, this construction gives the classical Schottky groups on $P^1_{\mathbb{C}}$.)

Consider the subspaces of $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ defined by $\widehat{L}_0 := \{(a, 0) \in \mathbb{C}^{2n+2}\}$ and $\widehat{M}_0 := \{(0, b) \in \mathbb{C}^{2n+2}\}$. Let $\widehat{S}$ be the involution of $\mathbb{C}^{2n+2}$ defined by $\widehat{S}(a, b) = (b, a)$. This interchanges $\widehat{L}_0$ and $\widehat{M}_0$.

1.1. Lemma. — *Let $\Phi\colon \mathbb{C}^{2n+2} \to \mathbb{R}$ be given by $\Phi(a, b) = |a|^2 - |b|^2$. Then:*

i) $\widehat{E}_{\widehat{S}} := \Phi^{-1}(0)$ *is a real algebraic hypersurface in $\mathbb{C}^{2n+2}$ with an isolated singularity at the origin 0. It is embedded in $\mathbb{C}^{2n+2}$ as a (real) cone over $S^{2n+1} \times S^{2n+1}$, with vertex at $0 \in \mathbb{C}^{2n+2}$.*

ii) $\widehat{E}_{\widehat{S}}$ *is invariant under multiplication by* $\lambda \in \mathbb{C}$, *so it is in fact a complex cone.* $\widehat{E}_{\widehat{S}}$ *separates* $\mathbb{C}^{2n+2} - \{(0,0)\}$ *in two diffeomorphic connected components* U *and* V, *which contain respectively* $\widehat{L_0} - \{(0,0)\}$ *and* $\widehat{M_0} - \{(0,0)\}$. *These two components are interchanged by the involution* $\widehat{S}$, *for which* $\widehat{E}_{\widehat{S}}$ *is an invariant set.*

iii) *Every linear subspace* $\widehat{K}$ *of* $\mathbb{C}^{2n+2}$ *of dimension* $n+2$ *containing* $\widehat{L_0}$ *meets transversally* $\widehat{E}_{\widehat{S}}$ *and* $\widehat{M_0}$. *Therefore a tubular neighbourhood* V *of* $\widehat{M_0} - \{(0,0)\}$ *in* $P_{\mathbb{C}}^{2n+1}$ *is obtained, whose normal disc fibres are of the form* $\widehat{K} \cap V$, *with* $\widehat{K}$ *as above.*

Proof. — The first statement is clear because Φ is a quadratic form with $0 \in \mathbb{C}^{2n+2}$ as unique critical point. Clearly $\widehat{E}_{\widehat{S}}$ is invariant under multiplication by complex numbers, so it is a complex cone. That $\widehat{E}_{\widehat{S}} \cap S^{4n+3} = S^{2n+1} \times S^{2n+1} \subset \mathbb{C}^{2n+2}$, is because this intersection consists of all pairs (x,y) so that $|x| = |y| = 1/\sqrt{2}$. That $\widehat{S}$ leaves $\widehat{E}_{\widehat{S}}$ invariant is obvious, and so is that $\widehat{S}$ interchanges the two components of $\mathbb{C}^{2n+2} - \{(0,0)\}$ determined by $\widehat{E}_{\widehat{S}}$, which must be diffeomorphic because $\widehat{S}$ is an automorphism. Finally, if $\widehat{K}$ is a subspace as in the statement (iii), then $\widehat{K}$ meets transversally $\widehat{E}_{\widehat{S}}$, because through every point in $\widehat{E}_{\widehat{S}}$ there exists an affine line in $\widehat{K}$ which is transverse to $\widehat{E}_{\widehat{S}}$. □

Let S be the linear projective involution of $P_{\mathbb{C}}^{2n+1}$ defined by $\widehat{S}$. Since $\widehat{E}_{\widehat{S}}$ is a complex cone, it projects to a codimension 1 real submanifold of $P_{\mathbb{C}}^{2n+1}$, that we denote by E_S.

1.2. Corollary

i) E_S *is an invariant set of* S.

ii) E_S *is a* S^{2n+1}*-bundle over* $P_{\mathbb{C}}^n$, *in fact* E_S *is the sphere bundle associated to the holomorphic bundle* $(n+1)\mathcal{O}_{P_{\mathbb{C}}^n}$, *which is the normal bundle of* $P_{\mathbb{C}}^n$ *in* $P_{\mathbb{C}}^{2n+1}$.

iii) E_S *separates* $P_{\mathbb{C}}^{2n+1}$ *in two connected components which are interchanged by* S *and each one is diffeomorphic to a tubular neighbourhood of the canonical* $P_{\mathbb{C}}^n$ *in* $P_{\mathbb{C}}^{2n+1}$.

Definition. — We call E_S *the canonical mirror* and S *the canonical involution.*

It is an exercise to show that (1.1) holds in the following more generally setting. Of course one has the equivalent of (1.2) too.

1.3. Lemma. — *Let* λ *be a positive real number and consider the involution*

$$\widehat{S}_\lambda : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1} \times \mathbb{C}^{n+1},$$

given by $\widehat{S}_\lambda(a,b) = (\lambda b, \lambda^{-1} a)$. *Then* $\widehat{S}_\lambda$ *also interchanges* $\widehat{L}_0$ *and* $\widehat{M}_0$, *and the set*

$$\widehat{E}_\lambda = \{(a,b) : |a|^2 = \lambda^2 |b|^2\}$$

satisfies, with respect to $\widehat{S}_\lambda$, *the analogous properties* (i)-(iii) *of (1.1) above.*

We notice that as λ tends to ∞, the manifold E_λ gets thiner and approaches the L_0-axes. Consider now two arbitrary disjoint projective subspaces L and M of dimension n in $P^{2n+1}_{\mathbb{C}}$, and the corresponding linear subspaces $\widehat{L}, \widehat{M}$ of $\mathbb{C}^{2n+2}$. It is clear that $\mathbb{C}^{2n+2} = \widehat{L} \oplus \widehat{M}$ and there is a linear automorphism $\widehat{H}$ of $\mathbb{C}^{2n+2}$ taking $\widehat{L}$ to $\widehat{L}_0$ and $\widehat{M}$ to $\widehat{M}_0$. For every $\lambda \in \mathbb{R}_+$, the automorphism $\widehat{H}^{-1} \circ \widehat{S}_\lambda \circ \widehat{H}$, is an involution that descends to an involution $H^{-1} \circ S_\lambda \circ H$ of $P^{2n+1}_{\mathbb{C}}$ that interchanges L and M. It is clear that one has results analogous to (1.1) and to (1.2). One also has:

1.4. Lemma. — *Let T be a linear projective involution of $P^{2n+1}_{\mathbb{C}}$ that interchanges L and M. Then T is conjugate in* $\mathrm{PSL}(2n+2, \mathbb{C})$ *to the canonical involution S.*

Proof. — Let $\widehat{L}$ and $\widehat{M}$ be linear subspaces of $\mathbb{C}^{2n+2}$ as above. Let $\{l_1, \ldots, l_{n+1}\}$ be a basis of $\widehat{L}$. Then $\{l_1, \ldots, l_{n+1}, \widehat{T}(l_1), \ldots, \widehat{T}(l_{n+1})\}$ is a basis of $\mathbb{C}^{2n+2}$. The linear transformation that sends the canonical basis of $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1}$ to this basis induces a projective transformation which realizes the required conjugation. □

In this paper, *mirrors* in $P^{2n+1}_{\mathbb{C}}$ are, by definition, the images of E_S under the action of $\mathrm{PSL}(2n+2, \mathbb{C})$. A mirror is the boundary of a tubular neighbourhood of a $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$, so it is an S^{2n+1}-bundle over $P^n_{\mathbb{C}}$.

We summarise the previous discussion in the following result.

1.5. Proposition. — *Let $L \cong M \cong P^n_{\mathbb{C}}$ be disjoint projective subspaces of $P^{2n+1}_{\mathbb{C}}$. Then:*

i) *There exist involutions of $P^{2n+1}_{\mathbb{C}}$ that interchange L and M.*

ii) *Each of these involutions has a mirror, i.e., an invariant set $E = E_T \subset P^{2n+1}_{\mathbb{C}}$ which separates $P^{2n+1}_{\mathbb{C}}$ in two connected components which are interchanged by T. Each component is diffeomorphic to a tubular neighbourhood of the canonical $P^n_{\mathbb{C}} \subset P^{2n+1}_{\mathbb{C}}$.*

iii) *Given an arbitrary tubular neighbourhood U of L, we can choose T so that the corresponding mirror E_T is contained in the interior of U.*

In fact one can obviously make stronger the last statement of (1.5):

1.6. Lemma. — *Let L and M be as above. Given an arbitrary constant λ, $0 < \lambda < 1$, we can find an involution T interchanging L and M, with a mirror E such that if U^* is the open component of $P^{2n+1}_{\mathbb{C}} - E$ which contains M and $x \in U^*$, then $d(T(x), L) < \lambda d(x, M)$, where the distance d is induced by the Fubini-Study metric.*

Proof. — The involution $T_\lambda := H^{-1} \circ S_\lambda \circ H$, with H and S_λ as above, satisfies (1.6). □

We notice that the parameter λ in (1.6) gives control upon the degree of expansion and contraction of the generators of the groups, so one can estimate bounds on the Hausdorff dimension of the limit set (see section 2 below).

The previous discussion can be summarized in the following theorem (cf. [**No**]):

1.7. Theorem. — *Let $\mathcal{L} := \{(L_1, M_1), \ldots, (L_r, M_r)\}$, $r > 1$, be a set of r pairs of projective subspaces of dimension n of $P_{\mathbb{C}}^{2n+1}$, all of them pairwise disjoint. Then:*

i) *There exist involutions $T_1, \ldots, T_r$ of $P_{\mathbb{C}}^{2n+1}$, such that each T_i, $i = 1, \ldots, r$, interchanges L_i and M_i, and the corresponding mirrors E_{T_i} are all pairwise disjoint.*

ii) *If we choose the T_i's in this way, then the subgroup of* $\mathrm{PSL}(2n+2, \mathbb{C})$ *that they generate is complex Kleinian.*

iii) *Moreover, given a constant $C > 0$, we can choose the T_i's so that if $T := T_{j_1} \cdots T_{j_k}$ is a* reduced word of length $k > 0$ *(i.e., $j_1 \neq j_2 \neq \cdots \neq j_{k-1} \neq j_k$), then $T(N_i)$ is a tubular neighbourhood of the projective subspace $T(L_i)$ which becomes very thin as k increases: $d(x, T(L_i)) < C\lambda^k$ for all $x \in T(N_i)$, where N_i is the connected component of $P_{\mathbb{C}}^{2n+1} - E_{T_i}$ that contains L_i, for all $i = 1, \ldots, r$.*

1.7.1. Definition. — A Complex Kleinian group constructed as above will be called a *Complex Schottky Group*.

1.7.2. Definition. — Given a Complex Schottky group Γ, we define its *limit set* $\Lambda := \Lambda(\Gamma)$ to be the set of accumulation points of the Γ-orbit of the union $L_1 \cup \cdots \cup L_r$. Its complement $\Omega = \Omega(\Gamma) := P_{\mathbb{C}}^{2n+1} - \Lambda$ is the *region of discontinuity*.

1.7.3. Remark. — We notice that this definition is not standard but it is suitable for Schottky groups.

1.8. Theorem. — *Let Γ be a complex Schottky group in $P_{\mathbb{C}}^{2n+1}$, generated by involutions $\{T_1, \ldots, T_r\}$, $n \geqslant 1$, $r > 1$, as in (1.7) above. Let $\Omega(\Gamma)$ be the region of discontinuity of Γ and let $\Lambda(\Gamma) = P_{\mathbb{C}}^{2n+1} - \Omega(\Gamma)$ be the limit set. Then, one has:*

i) *Let $W = P_{\mathbb{C}}^{2n+1} - \cup_{i=1}^{r} \mathring{N}_i$, where $\mathring{N}_i$ is the interior of the tubular neighbourhood N_i as in (1.7). Then W is a compact fundamental domain for the action of Γ on $\Omega(\Gamma)$. One has: $\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(W)$, and the action on Ω is properly discontinuous.*

ii) *$\Lambda(\Gamma)$ is an intersection of nested sets: $\Lambda(\Gamma) = \cap_{i=1}^{\infty} \gamma_i(N_{j(i)})$, where $\{\gamma_i\}_{i=1}^{\infty}$ is a sequence of distinct elements of Γ and $j \colon \mathbb{N} \to \{1, \ldots, r\}$ is a function such that $\gamma_{i+1}(N_{j(i+1)}) \subset \gamma_i(N_{j(i)})$.*

iii) *If $r = 2$, then $\Gamma \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, the infinite dihedral group, and $\Lambda(\Gamma)$ is the union of two disjoint projective subspaces L and M of dimension n. In this case we say that Γ is* elementary, *in analogy with Kleinian groups acting on $P_{\mathbb{C}}^{1}$.*

iv) *If $r > 2$, then $\Lambda(\Gamma)$ is a complex solenoid (lamination), homeomorphic to $P_{\mathbb{C}}^{n} \times \mathcal{C}$, where $\mathcal{C}$ is a Cantor set. Γ acts minimally on the set of projective subspaces in $\Lambda(\Gamma)$ considered as a closed subset of the Grassmannian $G_{2n+1,n}$.*

v) *If $r > 2$, let $\check{\Gamma} \subset \Gamma$ be the index 2 subgroup consisting of the elements which are reduced words of even length. Then $\check{\Gamma}$ is free of rank $r - 1$ and acts freely on $\Omega(\Gamma)$. The compact manifold with boundary $\check{W} = W \cup T_1(W)$ is a fundamental domain for the action of $\check{\Gamma}$ on $\Omega(\Gamma)$. We also call $\check{\Gamma}$ a* complex Schottky group.

vi) *Each element* $\gamma \in \check{\Gamma}$ *leaves invariant two copies,* P_1 *and* P_2*, of* $P^n_{\mathbb{C}}$ *in* $\Lambda(\Gamma)$*. For every* $L \subset \Lambda(\Gamma)$*,* $\gamma^i(L)$ *converges to* P_1 *(or to* P_2*) as* $i \to \infty$ *(or* $i \to -\infty$*).*

In fact we prove that if $r > 2$, then Γ acts on a graph whose vertices have all valence either 2 or r. This graph is actually a tree, which can be compactified by adding its "ends". These form a Cantor set and the action of Γ can be extended to this compactification. The limit set $\Lambda(\Gamma)$ corresponds to the uncountable set of ends of this tree. We use this to prove statement v) above.

Proof of i). — Let ∂W be the boundary of $W = P^{2n+1}_{\mathbb{C}} - \cup \mathring{N}_i$, i.e., the union $E_1 \cup \cdots \cup E_r$ of the mirrors. Set $W_0 := W$. Now define $W_1 = \bigcup_{i=0}^r T_i(W)$, where T_0 is the identity, by definition. Then W_1 is a manifold whose boundary consists of $r(r-1)$ components $E_{ij} := T_i(E_j)$, $i \neq j$, $i, j = 1, \ldots, r$, each one being a mirror. Define, by induction on $k > 1$, $W_k = \bigcup_{i=o}^r T_i(W_{k-1})$. Then W_k is a manifold whose boundary consists of $r(r-1)^k$ components, $E_{j_1,\ldots,j_k} := T_{j_1} \cdots Tj_{k-1}(E_{j_k})$, where $j_1, j_2, \ldots, j_k \in \{1, \ldots, r\}$ and $j_1 \neq j_2, \ldots, j_{k-1} \neq j_k$. Thus W_k is contained in the interior of W_{k+1}: $W_k \subset \mathring{W}_{k+1}$.

Let $U = \bigcup_{k=0}^{\infty} W_k$, so U is Γ-invariant, since $T_j(W_k) \subset W_{k+1}$ for every $j \in \{1, \ldots, r\}$. It is clear that U is open, since any $x \in U$ is contained in the interior of some W_k. Let $\gamma = T_{j_1} \cdots T_{j_k}$ be any element of Γ represented as a reduced word of length $k > 1$. Then $\gamma(W) \subset W_k - \mathring{W}_{k-1}$. Thus, for any $\gamma \neq \beta$, $\gamma(\mathring{W}) \bigcap \beta(\mathring{W}) = \varnothing$. Since $U = \bigcup_{\gamma \in \Gamma} \gamma(W)$, then U is obtained from translates of W, glued along some boundary components. Thus U is open, connected, with a properly discontinuous action of Γ. Therefore $U \subset \Omega(\Gamma)$. To finish the proof of i) we must prove $P^{2n+1}_{\mathbb{C}} - U = \Lambda(\Gamma)$. For this we consider, for each $k \geqslant 0$, the set $F_k := P^{2n+1}_{\mathbb{C}} - \mathring{W}_k$. Then $F_{k+1} \subset F_k$, hence $\bigcap_{k=0}^{\infty} F_k = P^{2n+1}_{\mathbb{C}} - U$ is a nonempty closed invariant set. For each $k \geqslant 0$, F_k is a disjoint union of closed tubular neighbourhoods of projective subspaces of dimension n of $P^{2n+1}_{\mathbb{C}}$. These are of the form $\gamma(N_i) = T_{j_1} \cdots T_{j_k}(N_i)$, for a $\gamma \in \Gamma$ which is represented in terms of the generators as the reduced word $T_{j_1} \cdots T_{j_k}$. They are closed tubular neighbourhoods of the projective subspace $T_{j_1} \cdots T_{j_k}(L_i)$. For each sequence $\{\gamma_j\}_{j=1}^{\infty}$ in Γ, such that the length of γ_{j+1} is bigger than the length of γ_j and $\gamma_{j+1}(N_i) \subset \gamma_j(N_i)$, the tubular neighbourhood becomes thinner. By (1.7), the sequence $\{\gamma_j(L_i)\}_{j=1}^{\infty}$ converges, in the Hausdorff metric, to a linear subspace of dimension n. Hence, also by (1.7), $P^{2n+1}_{\mathbb{C}} - U$ is a nowhere dense closed subset of $P^{2n+1}_{\mathbb{C}}$, which is a disjoint union of projective subspaces of dimension n. Thus U is open and dense in $P^{2n+1}_{\mathbb{C}}$; since $U \subset \Omega(\Gamma)$, it follows that $\Omega(\Gamma)$ is also connected. We have that U/Γ is compact and it is obtained from the compact fundamental domain W after identifications in each component of its boundary. If $\Omega(\Gamma) \neq U$ we arrive to a contradiction, because Ω/Γ is connected and U/Γ is open, compact and properly contained in Ω/Γ. Therefore, $\Omega(\Gamma) = U$ and $\Lambda(\Gamma) = \bigcap_{i=0}^{\infty} F_i$. This proves i).

Proof of ii). — If $x \in \Lambda(\Gamma)$ then, as shown above, $x \in \bigcap_{i=0}^{\infty} F_i$. To prove ii) it is sufficient to choose, for each i, the component of F_i which contains x. Such component is of the form $\gamma(N_j)$ for a *unique* $\gamma \in \Gamma$ (we set $\gamma = \gamma_i$) and a *unique* $j \in \{1, \ldots, r\}$. We set $j = j(i)$. This proves ii). This also shows that $\bigcap_{i=0}^{\infty} F_i$ is indeed the limit set according to Kulkarni's definition in [**Ku1**].

Proof of iii). — We have two involutions, T and S, and two neighbourhoods, N_T and N_S, whose boundaries are the mirrors of T and S, respectively. The limit set is the disjoint union $A \cup B$, where $A := \bigcap_{\gamma \in \Gamma'} \gamma(N_S)$, $B := \bigcap_{\gamma \in \Gamma''} \gamma(N_T)$, Γ' is the set of elements in Γ which are words ending in T and Γ'' is the set of elements which are words ending in S. By (1.7), A and B are each the intersection of a nested sequence of tubular neighbourhoods of projective subspaces of dimension n, whose intersection is a projective subspace of dimension n. Hence A and B are both projective subspaces of dimension n, and they are disjoint. Two reduced words ending in T and S, act differently on N_T (or N_S). Hence Γ is the free product of the groups generated T and S, proving iii).

Proof of iv). — Let $L \subset P_{\mathbb{C}}^{2n+1}$ be a subspace of dimension n and let N be a closed tubular neighbourhood of L as above. Let D be a closed disc which is an intersection of the form $\widehat{L} \cap N$, where $\widehat{L}$ is a subspace of complex dimension $n+1$, transversal to L. If M is a subspace of dimension n contained in the interior of N, then M is transverse to D, otherwise the intersection of M with $\widehat{L}$ would contain a complex line and M would not be contained in N. From the proofs of i) and ii) we know that $\Lambda(\Gamma)$ is the disjoint union of uncountable subspaces of dimension n. Let $x \in \Lambda(\Gamma)$ and let $L \subset \Lambda(\Gamma)$ be a projective subspace with $x \in L$. Let N be a tubular neighbourhood of L and D a transverse disc as above. Then $\Lambda(\Gamma) \cap D$ is obtained as the intersection of families of discs of decreasing diameters, exactly as in the construction of Cantor sets. Therefore $\Lambda(\Gamma) \cap D$ is a Cantor set and $\Lambda(\Gamma)$ is a solenoid (or lamination) by projective subspaces which is transversally Cantor. It follows that $\Lambda(\Gamma)$ is a fibre bundle over $P_{\mathbb{C}}^n$, with fibre a Cantor set $\mathcal{C}$. Since $P_{\mathbb{C}}^n$ is simply connected and $\mathcal{C}$ is totally disconnected, this fibre bundle must be trivial, hence the limit set is a product $P_{\mathbb{C}}^n \times \mathcal{C}$, as stated.

There is another way to describe the above construction: Γ acts, via the differential, on the Grassmannian $G_{2n+1,n}$ of projective subspaces of dimension n of $P_{\mathbb{C}}^{2n+1}$. This action also has a region of discontinuity and contains a Cantor set which is invariant. This Cantor set corresponds to the closed family of disjoint projective subspaces in $\Lambda(\Gamma)$. It is clear that the action on the Grassmannian is minimal on this Cantor set.

Proof of v). — Choose a point x_0 in the interior of W. Let Γ_{x_0} be the Γ-orbit of x_0. We construct a graph $\check{\mathcal{G}}$ as follows: to each $\gamma(x_0) \in \Gamma_{x_0}$ we assign a vertex v_γ. Two vertices $v_\gamma, v_{\gamma'}$ are joined by an edge if $\gamma(W)$ and $\gamma'(W)$ have a common boundary component, which corresponds to a mirror E_i. This means that γ' is γ followed by an involutions T_i or vice-versa. This graph can be realized geometrically by joining the corresponding points $\gamma(x_0), \gamma'(x_0) \in \Omega(\Gamma)$ by an arc $\alpha_{\gamma,\gamma'}$ in $\Omega(\Gamma)$, which is chosen to be transversal to the corresponding boundary component of $\gamma(W)$; we also choose these arcs so that no two of them intersect but at the extreme points. Clearly $\check{\mathcal{G}}$ is a tree and each vertex has valence r. To construct a graph $\mathcal{G}$ with an appropriate Γ-action we introduce more vertices in $\check{\mathcal{G}}$: we put one vertex at the middle point of each edge in $\check{\mathcal{G}}$; these new vertices correspond to the points where the above arcs intersect the boundary components of $\gamma(W)$. Then we have an obvious simplicial action of Γ on $\mathcal{G}$. Let $\check{\Gamma}$ be the index-two subgroup of Γ consisting of elements which can be written as reduced words of even length in terms of $T_1, \ldots, T_r$. A fundamental domain for $\check{\Gamma}$ in $\Omega(\Gamma)$ is $\check{W} = W \cup T_1(W)$, so this group acts freely on the vertices of $\check{\mathcal{G}}$. Hence $\check{\Gamma}$ is a free group of rank $r-1$. The tree $\check{\mathcal{G}}$ can be compactified by its ends by adding a Cantor set on which $\check{\Gamma}$ acts minimally; this corresponds to the fact that Γ acts minimally on the set of projective subspaces which constitute $\Lambda(\Gamma)$.

Proof of vi). — By (1.7), if $\gamma \in \check{\Gamma}$, then either $\gamma(N_1)$ is contained in N_1 or $\gamma^{-1}(N_1)$ is contained in N_1; say $\gamma(N_1)$ is contained in N_1. Thus $\{\gamma^i(N_1)\}$, $i > 0$, is a nested sequence of tubular neighbourhoods of projective subspaces whose intersection is a projective subspace P_1 of dimension n; $\{\gamma^i(N_1)\}$, $i < 0$, is also nested sequence of tubular neighbourhoods of projective subspaces whose intersection is a projective subspace P_2 of dimension n. For every $L \subset \Lambda(\Gamma)$, $\gamma^i(L)$ converges to P_1 and P_2 as $i \to \infty$ or $i \to -\infty$, respectively, and both P_1 and P_2 are invariant under γ, as claimed. $\square$

1.9. Remarks

i) The action of $\check{\Gamma}$ in the Cantor set of projective subspaces is analogous to the action of a classical Fuchsian group of the second kind on its Cantor limit set. We also observe that, since each involution T_i is conjugate to the canonical involution defined in lemma 1.1, the laminations obtained in theorem 1.8 are transversally *projectively self-similar*. Hence one could try to apply results analogous to the results for (conformally) self-similar sets (for instance Bowen's formula [**Bo**]) to estimate the transverse Hausdorff dimension of the laminations obtained. Here by *transverse Hausdorff dimension* we mean the Hausdorff dimension of the Cantor set $\mathcal{C}$ of projective subspaces of $G_{2n+1,n}$ which conform the limit set. If $\widetilde{T}_i$, $i = 1, \ldots, r$, denote the maps induced in the Grassmannian $G_{2n+1,n}$ by the linear projective transformations T_i, then $\mathcal{C}$ is dynamically-defined by the group generated by the set $\{\widetilde{T}_i\}$.

ii) The construction of Kleinian groups given in 1.8 actually provides families of Kleinian groups, obtained by varying the size of the mirrors that bound tubular neighbourhoods around the $L_i's$. In Section 3 below we will look at these families.

iii) The above construction of complex Kleinian groups, using involutions and mirrors, can be adapted to produce discrete groups of automorphisms of quaternionic projective spaces of odd (quaternionic) dimension. Every "quaternionic Kleinian group" on $P_{\mathcal{H}}^{2n+1}$ lifts canonically to a complex Kleinian group on $P_{\mathbb{C}}^{4n+3}$.

2. Quotient Spaces of the region of discontinuity

We now discuss the nature of the quotients $\Omega(\Gamma)/\Gamma$ and $\Omega(\Gamma)/\check{\Gamma}$, for the groups of section 1. The proof of proposition (2.1) is straightforward and is left to the reader.

2.1. Proposition. — *Let L be a copy of the projective space $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$ and let x be a point in $P^{2n+1}_{\mathbb{C}} - L$. Let $K_x \subset P^{2n+1}_{\mathbb{C}}$ be the unique copy of the projective space $P^{n+1}_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$ that contains L and x. Then K_x intersects transversally every other copy of $P^n_{\mathbb{C}}$ embedded in $P^{2n+1}_{\mathbb{C}} - L$, and this intersection consists of one single point. Thus, given two disjoint copies L and M of $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$, there is a canonical projection map*

$$\pi := \pi_L : P^{2n+1}_{\mathbb{C}} - L \longrightarrow M,$$

which is a (holomorphic) submersion. Each fibre $\pi^{-1}(x)$ is diffeomorphic to $\mathbb{R}^{2n+2}$.

2.2. Theorem. — *Let Γ be a complex Schottky group as in theorem 1.8, with $r > 2$. Then:*

i) *The fundamental domain W of Γ is (the total space of) a locally trivial differentiable fibre bundle over $P^n_{\mathbb{C}}$ with fibre $S^{2n+2} - \overset{\circ}{D}_1 \cup \cdots \cup \overset{\circ}{D}_r$, where each $\overset{\circ}{D}_i$ is the interior of a smooth closed $(2n+2)$-disc D_i in S^{2n+2} and the D_i's are pairwise disjoint.*

ii) $\Omega(\Gamma)$ *fibres differentiably over $P^n_{\mathbb{C}}$ with fibre S^{2n+2} minus a Cantor set.*

iii) *If $\check{\Gamma}$ is the subgroup of index two as in theorem 1.8, which acts freely on $\Omega(\Gamma)$, then $\Omega(\Gamma)/\check{\Gamma}$ is a compact complex manifold that fibres differentiably over $P^n_{\mathbb{C}}$ with fibre $(S^{2n+1} \times S^1)\# \cdots \#(S^{2n+1} \times S^1)$, the connected sum of $r-1$ copies of $S^{2n+1} \times S^1$.*

Proof of i). — Let $P_1, P_2 \subset \Lambda(\Gamma)$ be two disjoint projective subspaces of dimension n contained in $\Lambda(\Gamma) \subset P^{2n+1}_{\mathbb{C}}$. Since $\Omega(\Gamma)$ is open in P^{2n+1}, the restriction to $\Omega(\Gamma)$ of the map π given by 2.1, using P_1 as L and P_2 as M, is a holomorphic submersion. We know, by theorem 1.8.iv, that $\Lambda(\Gamma)$ is a compact set which is a *disjoint* union of projective subspaces of dimension n and which is a transversally Cantor lamination. By 2.1, for each $y \in P_2$, K_y meets transversally each of these projective subspaces (in other words, K_y is transverse to the lamination $\Lambda(\Gamma)$, outside P_1). Hence, by theorem 1.8, for each $y \in P_2$, K_y intersects $\Lambda(\Gamma) - P_1$ in a Cantor set minus one point (this point corresponds to P_1). The family of subspaces K_y of dimension $n+1$ are all transverse to P_2.

Let us now choose P_1 and P_2 as in 1.8.vi, so they are invariant sets for some $\gamma \in \check{\Gamma}$, and $\gamma^j(L)$ converges to P_2 as $j \to \infty$ for every projective n-subspace $L \subset \Lambda(\Gamma) - P_1$. We see that *every* mirror E_i, $i \in \{1, \ldots, r\}$ is transverse to all K_y. Hence the restriction

$$\pi_1 := \pi_{P_1}|_W : W \longrightarrow P_2 \cong P^n_{\mathbb{C}},$$

of π to W, is a submersion which restricted to each component of the boundary is also a submersion. For each $y \in P_2$ one has $\pi_1^{-1}(\{y\}) = K_y \cap W$, so $\pi_1^{-1}(\{y\})$ is compact. Thus π_1 is the projection of a locally trivial fibre bundle with fibres $K_y \cap W$, $y \in P_2$, by Ehresmann's lemma [**Eh**]. On the other hand for a fixed $y_0 \in P_2$, $K_{y_0} \cap W$ is a closed $(2n+2)$-disc with $r-1$ smooth closed $(2n+2)$-discs removed from its interior. This is true because P_1 is contained in exactly one of the $N_i's$, say N_1, the tubular neighbourhood of P_1, and K_{y_0} intersects each N_j, $j \neq 1$, in a smooth closed $(2n+2)$-disc. This proves i).

Proof of ii). — The above arguments show that for each $\overline{\gamma} \in \Gamma$, the image $\overline{\gamma}(E_i)$ of a mirror E_i is transverse to K_y for all $y \in P_2$ and $i \in \{1, \ldots, r\}$. Hence the restriction $\pi_1^k := \pi_{P_1}|_{W_k}$, where W_k is as above, is a submersion whose restriction to each boundary component of W_k is also a submersion. Thus π_1^k is a locally trivial fibration. Since $\Omega(\Gamma) = \bigcup_{k \geqslant 0} W_k$, we finish the proof of the first part of ii) by applying the slight generalisation below of Ehresmann fibration lemma [**Eh**]; we leave the proof to the reader.

Lemma. — *Let $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{N}_i$ be a smooth manifold which is the union of compact manifolds with boundary $\mathcal{N}_i$, so that each $\mathcal{N}_i$ is contained in the interior of $\mathcal{N}_{i+1}$. Let $\mathcal{L}$ be a smooth manifold and $f : \mathcal{M} \to \mathcal{L}$ a submersion whose restriction to each boundary component of $\mathcal{N}_i$, for every i, is also a submersion. Then f is a locally trivial fibration.*

Thus $\pi_{P_1} : \Omega(\Gamma) \to P_2 \cong P^n_{\mathbb{C}}$ is a holomorphic submersion which is a locally trivial differentiable fibration. To finish the proof of ii) we only need to show that the fibres of π_{P_1} are S^{2n+2} minus a Cantor set. Just as above, one shows that $K_y \cap W_k$ is diffeomorphic to the sphere S^{2n+2} minus the interior of $r(r-1)^k$ disjoint $(2n+2)$-discs. Therefore the fibre of π_{P_1} at y, which is $K_y \cap \Omega(\Gamma)$, is the intersection of S^{2n+2} minus a nested union of discs, which gives a Cantor set as claimed in ii).

Proof of iii). — We recall that by theorem 1.8.v, the fundamental domain of $\check{\Gamma}$ is the manifold $\check{W} = W \cup T_1(W)$. Then, as above, the restriction of π to $\check{W}$ is a submersion which is also a submersion in each connected component of the boundary:

$$\partial \check{W} = \Big(\bigcup_{j \neq 1} T_1(E_j)\Big) \bigcup_{j \neq 1} E_j,$$

which is the disjoint union of the $r-1$ mirrors E_j, $j \neq 1$, together with the mirrors $E_{1j} := T_1(E_j)$, $j \neq 1$. The mirror E_j is identified with E_{1j}, $j \neq 1$, by T_1, and $\Omega(\Gamma)/\check{\Gamma}$ is obtained through these identifications. Let $\check{\pi} : \check{W} \to P_2$ be the restriction of π to $\check{W}$. By the proof of i), $\check{\pi}^{-1}(y) = K_y \cap \check{W}$, $y \in P_2$, is diffeomorphic to S^{2n+2} minus the interior of $2(r-1)$ disjoint $(2n+2)$-discs. The restriction of $\check{\pi}$ to each E_j and E_{1j} determines fibrations $\check{\pi}_j : E_j \to P_2$ and $\check{\pi}_{1j} : E_{1j} \to P_2$, respectively, whose fibres are S^{2n+1}. Set $\widehat{\pi}_j := \check{\pi}_{1j} \circ (T_1|_{E_j})$. If we had that $\widehat{\pi}_j = \check{\pi}_j$ for all $j = 2, \ldots, r$, then we would have a fibration from $\check{W}/\check{\Gamma}$ to P_2, because we would have compatibility of the projections on the boundary. In fact we only need that $\widehat{\pi}_j$ and $\check{\pi}_j$

be homotopic through a smooth family of fibrations $\pi_t : E_{1j} \to P_2$, $\pi_1 = \widehat{\pi}_j$, $\pi_0 = \check{\pi}_j$, $t \in [0,1]$. Actually, to be able to glue well the fibrations at the boundary we need that $\pi_t = \check{\pi}_j$ for t in a neighbourhood of 0 and $\pi_t = \widehat{\pi}_j$ for t in a neighbourhood of 1. But this is almost trivial: $\check{\pi}_j : E_{1j} \to P_2$ is the projection of E_{1j} onto P_2 from P_1 and $\widehat{\pi} - j$ is the projection of E_{1j} from $T(P_1)$ onto P_2. The n-dimensional subspaces P_1 and $T(P_1)$ are disjoint from P_2, so there exists a smooth family of n-dimensional subspaces P_t, $t \in [0,1]$, such that the family is disjoint from P_2 and $P_t = P_1$ for t in a neighbourhood of 0 and $P_t = T(P_1)$ for t in a neighbourhood of 1. We can choose the family so that for each $t \in [0,1]$, the set of $n+1$ dimensional subspaces which contain P_t meet transversally E_{1j}. To achieve this we only need to take an appropriate curve in the Grassmannian of projective n-planes in $P^{2n+1}_{\mathbb{C}}$, consisting of a family P_t which is transverse to all K_y; this is possible by (2.1) and the fact that the set of n-dimensional subspaces which are *not* transverse to the $K'_y s$, is a proper algebraic variety of $P^{2n+1}_{\mathbb{C}}$. In this way we obtain the desired homotopy. Hence $\check{W}$ fibres over $P_2 \cong P^n_{\mathbb{C}}$; the fibre is obtained from S^{2n+2} minus the interior of $2(r-1)$ disjoint $(2n+2)$-discs whose boundaries are diffeomorphic to S^{2n+1} and are identified by pairs by diffeomorphisms which are *isotopic* to the identity (using a fixed diffeomorphism to S^{2n+1}). Hence the fibre is *diffeomorphic* to $(S^{2n+1} \times S^1)\# \cdots \#(S^{2n+1} \times S^1)$, the connected sum of $r-1$ copies of $S^{2n+1} \times S^1$. This proves iii). □

2.3. Theorem. — *Let M_Γ be the compact complex orbifold $M_\Gamma := \Omega(\Gamma)/\Gamma$, which has complex dimension $(2n+1)$. Then:*

i) *The singular set of M_Γ, $\mathrm{Sing}(M_\Gamma)$, is the disjoint union of r submanifolds analytically equivalent to $P^n_{\mathbb{C}}$, one contained in (the image in M_Γ of) each mirror E_i of Γ.*

ii) *Each component of $\mathrm{Sing}(M_\Gamma)$ has a neighbourhood homeomorphic to the normal bundle of $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$ modulo the involution $v \mapsto -v$, for v a normal vector.*

iii) *M_Γ fibres over $P^n_{\mathbb{C}}$ with fibre a real analytic orbifold with r singular points, each having a neighbourhood (in the fibre) homeomorphic to the cone over the real projective space $P^{2n+1}_{\mathbb{R}}$.*

Proof. — We notice that M_Γ is obtained from the fundamental domain W after an identification on the boundary E_j by the action of T_j. The singular set of M_Γ is the union of the images, under the canonical projection $p : \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$, of the fixed point sets of the r involutions T_j. Now, T_j is conjugate to the canonical involution S of (1.2). The lifting of S to $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ has as fixed point set the $(n+1)$-subspace $\{(a,a) : a \in \mathbb{C}^{n+1}\}$. This projectivizes to a n-dimensional projective subspace. Since we can assume, *for a fixed j*, that T_j is an isometry, we obtain the local structure of a neighbourhood of each component of the singular set. The same arguments as in theorem 2.2.iii prove that $\Omega(\Gamma)/\Gamma$ fibres over $P^n_{\mathbb{C}}$ and that the fibre has r singular points, corresponding to the r components of $\mathrm{Sing}(M_\Gamma)$, and each of these r points has a neighbourhood (in the fibre) homeomorphic to the cone over $P^{2n+1}_{\mathbb{R}}$. □

2.4. Remarks

i) The map π in (2.2.ii) is holomorphic, but the fibration is not holomorphically locally trivial, because the complex structure on the fibres may change.

ii) The Kleinian groups of 2.2 provide a method for constructing complex manifolds which is likely to produce interesting examples (cf. [**No, Ka1, Ka2, Ka3, Ka4, Pe, Si**]). These are never Kähler, because the fibration $\pi : \Omega(\Gamma)/\check{\Gamma} \to P^n_{\mathbb{C}}$ has a section, by dimensional reasons, so there can not exist a 2-cocycle with a power which is the fundamental class of $\Omega(\Gamma)/\check{\Gamma}$. The bundle $(n+1)\mathcal{O}_{P^n_{\mathbb{C}}}$ is nontrivial as a real bundle, because it has non-vanishing Pontryagin classes (except for $n = 1$), hence π is a nontrivial fibration. We notice that the fundamental group of a compact Riemann surface of genus greater than zero is never a free group; similarly, by Kodaira's classification, the only compact complex surface with non trivial free fundamental group is the Hopf surface $S^3 \times S^1$. Our examples above give compact complex manifolds with free fundamental groups (of arbitrarly high rank) in all odd dimensions greater than one. Multiplying these examples by $P^1_{\mathbb{C}}$, one obtains similar examples in all even dimensions. As pointed out by the referee, it would be interesting to know if there are other examples which are minimal, *i.e.*, they are not obtained by blowing up along a smooth subvariety of the examples above. It is natural to conjecture that our examples in odd dimensions are the only ones which have a projective structure and free fundamental group of rank greater than one.

iii) The manifolds obtained by resolving the singularities of the orbifolds in (2.3) have very interesting topology. We recall that the orbifold M_Γ is singular along r disjoint copies of $P^n_{\mathbb{C}}$: $S_1, \ldots, S_r$. The resolution $\widetilde{M_\Gamma}$ of M_Γ is obtained by a monoidal transformation along each S_i, and it replaces each point $x \in S_i$, $1 \leqslant i \leqslant r$ by a projective space $P^n_{\mathbb{C}}$. Hence, if $\mathcal{P} : \widetilde{M} \to M$ denotes the resolution map, then $\mathcal{P}^{-1}(S_i)$ is a non-singular divisor in $\widetilde{M}$, which fibres holomorphically over $P^n_{\mathbb{C}}$ with fibre $P^n_{\mathbb{C}}$, $1 \leqslant i \leqslant r$.

2.5. Symmetric products of classical Kleinian groups. — Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a classical Kleinian group acting on $P^1_{\mathbb{C}}$. Let $\Lambda(\Gamma)$ and $\Omega(\Gamma) := P^1_{\mathbb{C}} - \Lambda(\Gamma)$ be, respectively, the limit set and the region of discontinuity of Γ. Since $P^n_{\mathbb{C}}$ is the n^{th} symmetric product of $P^1_{\mathbb{C}}$, $P^n_{\mathbb{C}} \cong S^n(P^1_{\mathbb{C}})$, there is a canonical diagonal action of Γ on $P^n_{\mathbb{C}}$, for all $n > 1$. The group Γ acts properly and discontinuously on $\Omega^n := P^n_{\mathbb{C}} - S^n(\Lambda(\Gamma))$. In particular, if Γ is a Schottky group of the second kind acting in $P^1_{\mathbb{C}}$ whose limit set $\Lambda(\Gamma)$ is a Cantor set, then $S^n(\Lambda(\Gamma))$ is again a Cantor set, and the action of Γ on its complement is discontinuous. Every point in $S^n(\Lambda(\Gamma))$ is an accumulation point of orbits of Γ. This provides examples of complex Klenian groups acting on $P^n_{\mathbb{C}}$ whose limit sets are Cantor sets. If in addition, the quotient of the action of Γ in $P^1_{\mathbb{C}}$ in the region of discontinuity is compact, then Ω^n/Γ is also compact.

3. Hausdorff dimension and moduli spaces

Let $\mathcal{L} := \{(L_1, M_1), \dots, (L_r, M_r)\}$ be a configuration of $P^n_{\mathbb{C}}$'s in $P^{2n+1}_{\mathbb{C}}$ as before, $r > 2$. Let Γ and Γ' be complex Schottky groups obtained from this same configuration, i.e., they are generated by sets $\{T_1, \dots, T_r\}$ and $\{T'_1, \dots, T'_r\}$ of holomorphic involutions of $P^{2n+1}_{\mathbb{C}}$ that interchange the L_i's with the M_i's. For each $i = 1, \dots, r$, the composition $T'_i \circ T_i^{-1}$ preserves the subspaces L_i, M_i. It is an exercise to see that the subgroup of $\mathrm{PSL}(n+2, \mathbb{C})$ of transformations that preserve these subspaces is the projectivization of a copy of $\mathrm{GL}(n+1, \mathbb{C}) \times \mathrm{GL}(n+1, \mathbb{C}) \subset \mathrm{GL}(2n+2, \mathbb{C})$. Therefore, we can always find an analytic family $\{\Gamma_t\}$, $0 \leqslant t \leqslant 1$, of complex Schottky groups, with configuration $\mathcal{L}$, such that $\{\Gamma_0\} = \Gamma$ and $\{\Gamma_1\} = \Gamma'$. Furthermore, let $\mathcal{L} := \{(L_1, M_1), \dots, (L_r, M_r)\}$ and $\mathcal{L}' := \{(L'_1, M'_1), \dots, (L'_r, M'_r)\}$ be two configurations of $P^n_{\mathbb{C}}$'s in $P^{2n+1}_{\mathbb{C}}$ as before. Due to dimensional reasons, we can always move these configurations to obtain a differentiable family of pairs of disjoint n-dimensional subspaces $\{(L_{1,t}, M_{1,t}), \dots, (L_{r,t}, M_{r,t})\}$, with $0 \leqslant t \leqslant 1$, providing an isotopy between $\mathcal{L}$ and $\mathcal{L}'$. Thus one has a differentiable family Γ_t of complex Kleinian groups, where $\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma'$. The same statements hold if we replace Γ and Γ' by their subgroups $\check{\Gamma}$ and $\check{\Gamma}'$, consisting of words of even length. So one has a differentiable family $\check{\Gamma}_t$ of Kleinian groups, where $\check{\Gamma}_0 = \check{\Gamma}$ and $\check{\Gamma}_1 = \check{\Gamma}'$. Hence the manifolds $\Omega(\Gamma_t)/\check{\Gamma}_t$ are all diffeomorphic. By section 2, these manifolds are (in general nontrivial) fibre bundles over $P^n_{\mathbb{C}}$ with fibre $\#^{(r-1)}(S^{2n+1} \times S^1)$, a connected sum of $(r-1)$-copies of $S^{2n+1} \times S^1$. If $n = 1$, given any configuration of r pairwise disjoint lines in $P^3_{\mathbb{C}}$, there exist an isotopy of $P^3_{\mathbb{C}}$ which carries the configuration into a family of r twistor lines. Hence $P^3_{\mathbb{C}}$ minus this configuration is diffeomorphic to the Cartesian product of S^4 minus r points with $P^1_{\mathbb{C}}$. Moreover, the attaching functions that we use to glue the boundary components of W, the fundamental domain of Γ, are all isotopic to the identity, because they live in $\mathrm{PSL}(4, \mathbb{C})$, which is connected. Thus, if $n = 1$, then $\Omega(\Gamma_t)/\check{\Gamma}_t$ is diffeomorphic to a product $P^1_{\mathbb{C}} \times \#^{(r-1)}(S^3 \times S^1)$. Hence we have:

3.1. Proposition. — *The differentiable type of the compact (complex) manifold $\Omega(\Gamma_t)/\check{\Gamma}_t$ is independent of the choice of configuration. It is a manifold of real dimension $(4n+2)$, which is a fibre bundle over $P^n_{\mathbb{C}}$ with fibre $\#^{(r-1)}(S^{2n+1} \times S^1)$; moreover, this bundle is trivial if $n = 1$. We denote the corresponding manifold by M^n_r.*

The fact that the bundle is trivial when $n = 1$ is interesting because, as pointed out in the introduction, when the configuration $\mathcal{L}$ consists of twistor lines in $P^3_{\mathbb{C}}$, the quotient $\Omega(\Gamma)/\check{\Gamma}$ is the twistor space of the conformally flat manifold $p(\Omega(\Gamma))/p(\check{\Gamma})$, which is a connected sum of the form $\#^{(r-1)}(S^3 \times S^1)$. Hence, in this case the natural fibration goes the other way round, i.e., it is a fibre bundle over $\#^{(r-1)}(S^3 \times S^1)$ with fibre $P^1_{\mathbb{C}}$.

Given a configuration $\mathcal{L}$ as above, let us denote by $[\mathcal{L}]_G$ its orbit under the action of the group $G = \mathrm{PSL}(2n+2, \mathbb{C})$. These orbits are equivalence classes of such configurations. Let us denote by $\mathcal{C}^n_r$ the set of equivalence classes of configurations consisting of r pairs of $P^n_{\mathbb{C}}$'s as above. Then $\mathcal{C}^n_r$ is a Zariski open set of the moduli space $\mathfrak{M}^n_r$, of configurations of r unordered couples of projective subspaces of dimension n in $P^{2n+1}_{\mathbb{C}}$, which is obtained as the Mumford quotient [**MFK**] of the action of G on such configurations. By [**MFK**], $\mathcal{C}^n_r$ is a complex algebraic variety: the *moduli space of configurations* of r pairs of n-planes $P^n_{\mathbb{C}}$ in $P^{2n+1}_{\mathbb{C}}$. Similarly, we denote by $\mathfrak{S}^n_r$ the equivalence classes, or moduli space, of the corresponding Schottky groups, where two such groups are equivalent if they are conjugate by an element in $\mathrm{PSL}(n+2, \mathbb{C})$. Given $\mathcal{L} := \{(L_1, M_1), \ldots, (L_r, M_r)\}$, and r-tuples of involutions $(T_1, \ldots, T_r)$ and $(S_1, \ldots, S_r)$ as above, i.e., interchanging L_i with M_i for all $i = 1, \ldots, r$ and having pairwise disjoint mirrors, we say that these r-tuples are *equivalent* if there exists $h \in G$ such that $hT_ih^{-1} = S_i$ for all i. Let $\mathfrak{T}_{\mathcal{L}}$ denote the set of equivalence classes of such r-tuples of involutions. It is clear that a conjugation h as above must leave $\mathcal{L}$ invariant. Hence, if r is big enough with respect to n, then h must be actually the identity, so the equivalence classes in fact consist of a single element.

3.2. Theorem. — *There exists a holomorphic surjective map $\pi\colon \mathfrak{S}^n_r \to \mathcal{C}^n_r$ which is a C^∞ locally trivial fibration with fibre $\mathfrak{T}_{\mathcal{L}}$. Furthermore, let Γ, Γ' be complex Schottky groups as above and let $\Omega(\Gamma), \Omega(\Gamma')$ be their regions of discontinuity. Then the complex orbifolds $M_\Gamma := \Omega(\Gamma)/\Gamma$ and $M_{\Gamma'} := \Omega(\Gamma')/\Gamma'$ are biholomorphically equivalent if and only if Γ and Γ' are projectively conjugate, i.e., they represent the same element in $\mathfrak{S}^n_r$. Similarly, if $\check{\Gamma}, \check{\Gamma}'$ are the corresponding index 2 subgroups, consisting of the elements which are words of even length, then the manifolds $M_{\check{\Gamma}} := \Omega(\Gamma)/\check{\Gamma}$ and $M_{\check{\Gamma}'} := \Omega(\Gamma')/\check{\Gamma}'$, are biholomorphically equivalent if and only if $\check{\Gamma}$ and $\check{\Gamma}'$ are projectively conjugate.*

Proof. — The first statement in (3.2) is obvious, i.e., that we have a holomorphic surjection $\pi\colon \mathfrak{S}^n_r \to \mathcal{C}^n_r$ with kernel $\mathfrak{T}_{\mathcal{L}}$. The other statements are immediate consequences of the following lemma (3.3), proved for us by Sergei Ivashkovich. Our proof below is a variation of Ivashkovich's proof.

3.3. Lemma. — *Let U be a connected open set in $P^{2n+1}_{\mathbb{C}}$ that contains a subspace $L \subset P^{2n+1}_{\mathbb{C}}$ of dimension n, and let $h\colon U \to V$ be a biholomorphism onto an open set $V \subset P^{2n+1}_{\mathbb{C}}$. Suppose that V also contains a subspace M of dimension n. Then h extends uniquely to an element in* $\mathrm{PSL}(2n+2, \mathbb{C})$.

Proof. — Let $f : U \to P^n_{\mathbb{C}}$ be a holomorphic map. Then f is defined by n meromorphic functions $f_1, \ldots, f_n$ from U to $P^1_{\mathbb{C}}$ (see [**Iva**]), *i.e.*, holomorphic functions which are defined outside of an analytic subset of U (the indeterminacy set).

Consider the set of all subspaces of $P^{2n+1}_{\mathbb{C}}$ of dimension $n+1$ which contain L. Then, if N is such subspace, one has a neighbourhood U_N of L in N which is the complement of a round ball in the affine part, $\mathbb{C}^{n+1}$, of N. Since the boundary of such a ball is a round sphere S_N and, hence, it is pseudo-convex, it follows from E. Levi extension theorem, applied to each f_i, that the restriction, f_N, of f to $U \cap N$ extends to all of N as a meromorphic function. The union of all subspaces N is $P^{2n+1}_{\mathbb{C}}$ and they all meet in L. The functions f_N depend holomorphically on N as is shown in [**Iva**]. One direct way to prove this is by considering the Henkin-Ramirez reproducing kernel defined on each round sphere S_N, [**He, Ram**]. One can choose the spheres S_N in such a way that the kernel depends holomorphically on N by considering a tubular neighbourhood of L in N whose radius is independent of N. Hence the extended functions to all $N's$ define a meromorphic function in all of $P^{2n+1}_{\mathbb{C}}$, which extends f. Now let h be as in the statement lemma 3.3 and let $\widetilde{h}$ be its meromorphic extension. Then, since by hypothesis h is a biholomorphism from the open set $U \subset P^n_{\mathbb{C}}$ onto the the open set $V := h(U) \subset P^n_{\mathbb{C}}$, one can apply the above arguments to $h^{-1} : V \to U$. Let $g : P^n_{\mathbb{C}} \to P^n_{\mathbb{C}}$ be the meromorphic extension of h^{-1}. Then, outside of their sets of indeterminacy, one has $\widetilde{h}g = g\widetilde{h} = Id$. Hence the indeterminacy sets are empty and both $\widetilde{h}$ and g are biholomorphisms of $P^n_{\mathbb{C}}$. In fact, in [**Iva**] it is shown that if f is as in the statement of lemma 3.3 and if f is required only to be locally injective, then f extends as a holomorphic function. $\square$

Notice that if $n = 1$, then (3.3) becomes Lemma 3.2 in [**Ka1**].

3.4. Corollary. — *For $r > 2$ sufficiently large, the manifold $\Omega(\Gamma)/\check{\Gamma}$ has non-trivial moduli.*

In fact, if the manifolds $\Omega(\Gamma)/\check{\Gamma}$ and $\Omega(\Gamma')/\check{\Gamma}'$ are complex analytically equivalent, then $\check{\Gamma}$ is conjugate to $\check{\Gamma}'$ in $\mathrm{PSL}(2n+2, \mathbb{C})$, by (3.2), and the corresponding configurations $\mathcal{L}$ and $\mathcal{L}'$ are projectively equivalent. Now it is sufficient to choose r big enough to have two such configurations which are not projectively equivalent. This is possible because the action induced by the projective linear group G on the Grassmannian $G_{2n+1,n}$ is obtained from the projectivization of the action of $\mathrm{SL}(2n+2, \mathbb{C})$ acting on the Grassmann algebra Λ^{n+1}, of $(n+1)$-vectors of $\mathbb{C}^{2n+2}$, restricted to the set of decomposable $(n+1)$-vectors $\mathcal{D}^{n+1}$. The set $\mathcal{D}^{n+1}$ generates the Grassmann algebra and $G_{2n+1,n} = (\mathcal{D}^{n+1} - \{0\})/\sim$, where $\sim$ is the equivalence relation of projectivization.

If r is small with respect to n, then $\mathcal{C}^n_r$ consists of one point, because any two such configurations are in the same $\mathrm{PSL}(2n+2, \mathbb{C})$-orbit. Therefore, in this case $\mathfrak{T}_{\mathcal{L}}$ coincides with $\mathfrak{G}^n_r$. That is, to change the complex structure of M^n_r we need to change the corresponding involutions into a family of involutions, with the same configuration (up to conjugation), which is not conjugate to the given one.

The following result is a generalization of Theorem 1.2 in [**Ka1**]. This can be regarded as a restriction for a complex orbifold (or manifold) to be of the form $\Omega(\Gamma)/\Gamma$ (or $\Omega(\Gamma)/\check{\Gamma}$).

3.5. Proposition. — *If $r > 2$, then the compact complex manifolds and orbifolds $\Omega(\Gamma)/\check{\Gamma}$ and $\Omega(\Gamma)/\Gamma$, obtained in theorem 2.2, have no non-constant meromorphic functions.*

Proof. — Let f be a meromorphic function on one of these manifolds (or orbifolds). Then f lifts to a meromorphic function $\widetilde{f}$ on $\Omega(\Gamma) \subset P_{\mathbb{C}}^{2n+1}$, which is $\check{\Gamma}$-invariant. By lemma (3.6) below, f extends to a meromorphic function on all of $P_{\mathbb{C}}^{2n+1}$. Hence $\widetilde{f}$ must be constant, because $\check{\Gamma}$ is an infinite group. □

3.6. Lemma **([Iva])**. — *Let $U \subset P_{\mathbb{C}}^{2n+1}$, $n \geqslant 1$, be an open set that contains a projective subspace $P_{\mathbb{C}}^{n}$. Let $f : U \to P_{\mathbb{C}}^{1}$ be a meromorphic function. Then f can be extended to a meromorphic function $\widetilde{f} : U \to P_{\mathbb{C}}^{1}$.*

We refer to [**Iva**] for the proof of (3.6). In the following proposition we estimate an upper bound for the Hausdorff dimension of the limit set of some Schottky groups.

3.7. Proposition. — *Let $r > 2$, $0 < \lambda < (r-1)^{-1}$ and let Γ and $\check{\Gamma}$ be as in (1.7). Then, for every $\delta > 0$, the Hausdorff dimension of $\Lambda(\Gamma) = \Lambda(\check{\Gamma})$ is less than $2n+1+\delta$, i.e., the* transverse *Hausdorff dimension of $\Lambda(\Gamma) = \Lambda(\check{\Gamma})$ is less than $1+\delta$.*

Proof. — We recall that $\Lambda(\Gamma) = \cap_{k=0}^{\infty} F_k$, by the proof of theorem 1.8.i), where F_k is the disjoint union of the $r(r-1)^k$ closed tubular neighbourhoods $\gamma(N_i)$, $i \in \{1, \ldots, r\}$, where $\gamma \in \Gamma$ is an element which can be represented as a reduced word of length k in terms of the generators. $\gamma(N_i)$ is a closed tubular neighbourhood of $\gamma(L_i)$, as in theorem 1.7, and the "width" of each $\gamma(N_i)$, $w_{(\gamma,i)} := d(\gamma(E_i), L_i)$, satisfies $w_{(\gamma,i)} \leqslant C\lambda^k$, as was shown in lemma 1.6 and corollary 1.7. Hence,

$$w(k) := \sum_{\substack{l(\gamma)=k \\ i \in \{1,\ldots,r\}}} w_{(\gamma,i)}^{1+\delta} \leqslant Cr(r-1)^k \lambda^{k(1+\delta)} < Cr(r-1)^{-\delta k}.$$

Thus, $\lim_{k\to\infty} w(k) = 0$. Hence, just as in the proof of the theorem of Marstrand [**Mr**], the Hausdorff dimension of $\Lambda(\Gamma)$ can not exceed $2n+1+\delta$. □

Next we will apply the previous estimates to compute the versal deformations of manifolds obtained from complex Schottky groups as in (3.7), whose limit sets have small Hausdorff dimension.

We first recall [**Kod**] that given a compact complex manifold X, a deformation of X consists of a triple $(\mathcal{X}, \mathcal{B}, \omega)$, where $\mathcal{X}$ and $\mathcal{B}$ are complex analytic spaces and $\omega : \mathcal{X} \to \mathcal{B}$ is a surjective holomorphic map such that $\omega^{-1}(t)$ is a complex manifold for all $t \in \mathcal{B}$ and $\omega^{-1}(t_0) = X$ for some t_0, which is called the reference point. It is known [**Kur**] that given X, there is always a deformation $(\mathcal{X}, \mathfrak{K}_X, \omega)$ which is *universal*, in the sense that every other deformation is induced from it (see also [**KNS, Kod**]). The space $\mathfrak{K}_X$ is the Kuranishi space of *versal* deformations of X [**Kur**]. If we let $\Theta := \Theta_X$ be the sheaf of germs of local holomorphic vector fields on X, then every deformation of X determines, via differentiation, an element in $H^1(X, \Theta)$, so $H^1(X, \Theta)$ is called the *space of infinitesimal deformations* of X ([**Kod**], Ch. 4). Furthermore ([**KNS**] or [**Kod**, Th. 5.6]), if $H^2(X, \Theta) = 0$, then the Kuranishi space $\mathfrak{K}_X$ is smooth at the reference point t_0 and its tangent space at t_0 is canonically identified with $H^1(X, \Theta)$. In particular, in this case *every infinitesimal deformation* of X comes from an actual deformation, and vice-versa, every deformation of the complex structure on X, which is near the original complex structure, comes from an infinitesimal deformation.

The following lemma is an immediate application of (3.7) and Harvey's Theorem 1 in [**Ha**], which generalises the results of Scheja [**Schj**].

3.8. Lemma. — *Let $r > 2$, $0 < \lambda < (r-1)^{-1}$, let $\check{\Gamma}$ be as in proposition 3.7 and let $\Omega := \Omega(\Gamma) \subset P_{\mathbb{C}}^{2n+1}$ be its region of discontinuity. Then one has:*

$$H^j(\Omega, i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})) \cong H^j(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}), \quad \textit{for } 0 \leqslant j < n,$$

where i is the inclusion of Ω in $P_{\mathbb{C}}^{2n+1}$. Hence, if $n > 1$, then one has:

$$H^0(\Omega, i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})) \cong \mathfrak{sl}(2n+2, \mathbb{C}) \quad \textit{and} \quad H^j(\Omega, i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})) \cong 0,$$

for all $0 < j < n$, where $\mathfrak{sl}(2n+2, \mathbb{C})$ is the Lie algebra of $\mathrm{PSL}(2n+2, \mathbb{C})$, *and it is being considered throughout this section as an additive group.*

Proof. — By (3.7) we have that the Hausdorff dimension d of the limit set $\Lambda(\check{\Gamma})$ satisfies $d < 2n + 1 + \delta$ for every $\delta > 0$. Therefore the Hausdorff measure of $\Lambda(\Gamma)$ of dimension s, $\mathcal{H}_s(\Lambda(\Gamma))$, is zero for every $s > 2n+1$. Hence the first isomorphism in (3.8) follows from Theorem 1.ii) in [**Ha**], because the sheaf Θ is locally free. The second statement in (3.8) is now immediate, because

$$H^0(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) \cong \mathfrak{sl}(2n+2, \mathbb{C}) \quad \text{and} \quad H^j(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) \cong 0 \text{ for } j > 0,$$

a fact which follows immediately by applying the long exact sequence in cohomology derived from the short exact sequence:

$$0 \longrightarrow \mathcal{O} \longrightarrow [\mathcal{O}(1)]^{n+1} \longrightarrow \Theta_{P_{\mathbb{C}}^{2n+1}} \longrightarrow 0,$$

where $\mathcal{O}$ is the structural sheaf of $P_{\mathbb{C}}^{2n+1}$ and $[\mathcal{O}(1)]^{n+1}$ is the direct sum of $n+1$ copies of $\mathcal{O}_{P_{\mathbb{C}}^{2n+1}}(1)$, the sheaf of germs of holomorphic sections of the holomorphic line bundle over $P_{\mathbb{C}}^{2n+1}$ with Chern class 1. See Hartshorne [**Ht**], Example 8.20.1, page 182. $\square$

We let $M := \Omega/\check{\Gamma}$, where $\check{\Gamma}$ is as above. We notice that Ω is simply connected when $n > 0$, so that Ω is the universal covering $\widetilde{M}$ of M. Let $p : \widetilde{M} \to M$ be the covering projection; since $\check{\Gamma}$ acts freely on Ω, this projection is actually given by the group action. Let Θ_M be the sheaf of germs of local holomorphic vector fields on M and let $\widetilde{\Theta}$ be the pull-back of Θ to $\widetilde{M}$ under the covering p; $\widetilde{\Theta}$ is the sheaf $i^*(\Theta_{P_{\mathbb{C}}^{2n+1}})$ on $\widetilde{M} = \Omega$.

3.9. Lemma. — *If $n > 2$, then for $0 \leqslant j \leqslant 2$ we have:*

$$H^j(M, \Theta_M) \cong H^j_\rho(\check{\Gamma}, \mathfrak{sl}(2n+2, \mathbb{C})),$$

where $\mathfrak{sl}(2n+2, \mathbb{C}))$ is considered as a $\check{\Gamma}$-left module via the representation

$$\rho : \check{\Gamma} \longrightarrow \mathrm{Aut}(\mathfrak{sl}(2n+2, \mathbb{C})))$$

given by:

$$\rho(\gamma)(v) = dT_g \circ v \circ T_g{}^{-1}, \quad v \in \mathfrak{sl}(2n+2, \mathbb{C}),$$

where T_g is the action of $g \in \check{\Gamma}$ on $P_{\mathbb{C}}^{2n+1}$.

Proof. — If $n > 2$, then (3.8) and Mumford's formula (c) in [**Mu**], pag 23, (see also Grothendieck [**Gr**], Chapter V) imply that there exists an isomorphism

$$\phi : H^j_\rho(\check{\Gamma}, H^0(\Omega, \widetilde{\Theta})) \longrightarrow H^j(M, \Theta_M),$$

for $0 \leqslant j \leqslant 2$, where $H^0(\Omega, \widetilde{\Theta})$ is the vector space of holomorphic vector fields on the universal covering $\widetilde{M} = \Omega \subset P_{\mathbb{C}}^{2n+1}$ of M.

Now, by [**Ha**], Theorem 1.i), every holomorphic vector field in $\Omega(\Gamma)$, extends to a holomorphic vector field defined in all of $P_{\mathbb{C}}^{2n+1}$. Therefore,

$$H^0(\Omega, \widetilde{\Theta}) = H^0(P_{\mathbb{C}}^{2n+1}, \Theta_{P_{\mathbb{C}}^{2n+1}}) = \mathfrak{sl}(2n+2, \mathbb{C}). \quad \square$$

We recall that $\check{\Gamma}$ is a free group of rank $r-1$; let $g_1, \ldots, g_{r-1}$ be generators of $\check{\Gamma}$. By [**HS**], page 195 Corollary 5.2, applied to $\check{\Gamma}$, we obtain:

$$H^1_\rho(\check{\Gamma}, \mathfrak{sl}(2n+2, \mathbb{C})) \cong \mathfrak{sl}(2n+2, \mathbb{C}) \times \cdots \times \mathfrak{sl}(2n+2, \mathbb{C})/Im(\psi),$$

where

$$\psi : \mathfrak{sl}(2n+2, \mathbb{C}) \longrightarrow \mathfrak{sl}(2n+2, \mathbb{C}) \times \cdots \times \mathfrak{sl}(2n+2, \mathbb{C})$$

is given by $\psi(v) = (g_1(v) - v, \ldots, g_{r-1}(v) - v)$. We claim that ψ is injective. Indeed, if v is a linear vector field in $P_{\mathbb{C}}^{2n+1}$ which is invariant by $g_1, \ldots, g_{r-1}$, then, by Jordan's theorem, this vector field is tangent to a hyperplane Π which is $\check{\Gamma}$-invariant. This can not happen. In fact, if L is a n dimensional projective subspace contained in $\Lambda(\check{\Gamma})$, then L must intersect Π transversally in a subspace of dimension $n-2$, for otherwise Π would contain the whole limit set $\Lambda(\check{\Gamma})$, which is a disjoint union of projective subspaces of dimension n. Hence, there exists $L \subset \Pi$, a projective n-subspace such that $L \cap \Lambda(\check{\Gamma}) = \varnothing$. Then, as we have shown in section 1, there exists a sequence $\{\gamma_i\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty}(\gamma_i(L)) = L_1$, where $L_1 \subset \Lambda(\check{\Gamma})$, where L_1 is not contained in Π. This is a contradiction to the invariance of Π. $\square$

Therefore,

$$\dim_{\mathbb{C}} H^1(\Omega, \widetilde{\Theta}) = \dim_{\mathbb{C}}[\mathfrak{sl}(2n+2,\mathbb{C})^{r-2}] = (r-2)\left((2n+2)^2-1\right).$$

By [**HS**], page 197, Corollary 5.6 we have $H^2_\rho(\check{\Gamma}, \mathfrak{sl}(2n+2,\mathbb{C})) = 0$. Hence, by 3.9 above, one obtains,

$$H^2(M, \Theta_M) \cong H^2_\rho(\check{\Gamma}, \mathfrak{sl}(2n+2,\mathbb{C})) = 0.$$

Thus we arrive to the following theorem:

3.10. Theorem. — *Let $n, r > 2$ and let λ be an arbitrary scalar such that $0 < \lambda < (r-1)^{-1}$. Let Γ be a Schottky group as in (1.7.iii), so that the (Fubini-Study) distance from $\gamma(x)$ to the limit set Λ decreases faster that $C\lambda^k$ for every point $x \in P^{2n+1}_{\mathbb{C}}$ and any $\gamma \in \Gamma$ of word-length k (where C is some positive constant). Let $\check{\Gamma}$ be the index-two subgroup of Γ consisting of words of even length. Let Ω be the region of discontinuity of Γ, $M := \Omega/\check{\Gamma}$, and let $\mathfrak{K}^n_r$ denote the Kuranishi space of* versal *deformations of M, with reference point $t_0 \in \mathfrak{K}^n_r$ corresponding to M. Then, we have:*

$$H^1(M, \Theta_M) \cong H^1_\rho(\check{\Gamma}, \mathfrak{sl}(2n+2,\mathbb{C})) \cong \mathbb{C}^{(r-2)((2n+2)^2-1)},$$

and

$$H^2(M, \Theta_M) = 0.$$

Hence $\mathfrak{K}^n_r$ is non-singular at t_0, of complex dimension $(r-2)((2n+2)^2-1)$, and every small deformation of M is obtained by a small deformation of $\check{\Gamma}$ as a subgroup of $\mathrm{PSL}(2n+2,\mathbb{C})$*, unique up to conjugation.*

Although we only considered $n > 2$ above, the last theorem remains valid for $n = 0, 1$. In fact, if $n = 0$ and $r > 2$, we have the classical Schottky groups. The manifold $\Omega/\check{\Gamma}$ is a compact Riemann surface of genus $r-1$. It is well known that in this case the moduli space has dimension $3(r-1)-3 = 3(r-2)$, which, of course, coincides with the formula above. When $n = 1$ and $r > 2$ the manifolds $\Omega/\check{\Gamma}$ are Pretzel twistor spaces of genus $g = r-1$, in the sense of Penrose [**Pe**]. The theorem above gives that the dimension of the moduli space of this manifold is $15g-15$, which coincides with Penrose's calculations in page 251 of [**Pe**].

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