

Astérisque

ROBERTO FERNÁNDEZ

ANDRÉ TOOM

Non-Gibbsianness of the invariant measures of non-reversible cellular automata with totally asymmetric noise

Astérisque, tome 287 (2003), p. 71-87

<http://www.numdam.org/item?id=AST_2003_287_71_0>

© Société mathématique de France, 2003, tous droits réservés.

L'accès aux archives de la collection « Astérisque » ([http://smf4.emath.fr/
Publications/Asterisque/](http://smf4.emath.fr/Publications/Asterisque/)) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>*

NON-GIBBSIАНNESS OF THE INVARIANT MEASURES OF NON-REVERSIBLE CELLULAR AUTOMATA WITH TOTALLY ASYMMETRIC NOISE

by

Roberto Fernández & André Toom

Abstract. — We present a class of random cellular automata with multiple invariant measures which are all non-Gibbsian. The automata have configuration space $\{0, 1\}^{\mathbb{Z}^d}$, with $d > 1$, and they are noisy versions of automata with the “eroder property”. The noise is totally asymmetric in the sense that it allows random flippings of “0” into “1” but not the converse. We prove that all invariant measures assign to the event “a sphere with a large radius L is filled with ones” a probability μ_L that is too large for the measure to be Gibbsian. For example, for the NEC automaton $(-\ln \mu_L) \asymp L$ while for any Gibbs measure the corresponding value is $\asymp L^2$.

1. Introduction

Studies of cellular automata and of their continuous-time counterpart, the spin-flip dynamics, have been successful in determining how many invariant measures the automaton or dynamics have. Much less is known about properties of these measures. A natural question is whether they are Gibbsian, that is whether they could correspond to measures describing the equilibrium state of some statistical mechanical system. There are two categories of evolutions —both with local and strictly positive updating rates— for which the answer is known to be positive: (1) If the updating prescription has a high level of stochasticity —*high noise regime*—, in which case Gibbsianness comes together with uniqueness of the invariant measure [15, 19, 18]; and (2) if the updating satisfies a detailed balance condition for some Boltzmann-Gibbs weights [20]. Known cases of non-Gibbsianness, on the other hand, refer to automata where the updating rates are either non-strictly positive [16], [30, Chapter 7] or non-local [23].

2000 Mathematics Subject Classification. — 37B15, 37C40, 60A10.

Key words and phrases. — Gibbs vs. non-Gibbs measures, cellular automata, invariant measures, non-ergodicity, eroders, convex sets.

R.F.: Travel support by an agreement USP-Cofecub (projet UC/68/99, *Comportement à temps intermédiaire des systèmes aléatoires*).

A.T.: Supported by FAPESP, grant # 98/15994-0 and CNPq, grant # 300991/1998-3.

In this paper we present some examples of stochastic *non-reversible automata* — that is, automata not satisfying any form of detailed balance —, with multiple invariant measures, all of them non-Gibbsian. Our class of automata can be seen as a generalization of the North-East-Center (NEC) majority model introduced in [24] and discussed in many papers. Its non-ergodicity was first proved in [28] (see also the discussion in [15]) and later by another method in [2]. Also it was simulated more than once [1, 21, 22]. Models of this sort are obtained by superimposing stochastic errors (noise) to deterministic automata having the so-called *eroder property*: finite islands of aligned spins, within a sea of spins aligned in the opposite direction, disappear in a finite time.

We allow only *one-sided* noise or stochastic error —a “0” can stochastically be turned into a “1”, but not the reverse. Thus some of our transition rates are zeros and therefore the “dichotomy” result of [20, Corollary 1] is not applicable. Our work does not settle the long-standing issue of the Gibbsianness of the invariant measures of NEC models with non totally asymmetric noise. There are conflicting arguments and evidences for the model with symmetric noise: An interesting heuristic argument has been put forward [30, Chapter 5] pointing in the direction of Gibbsianness, and a couple of pioneer numerical studies yielded findings respectively consistent with Gibbsianness [21] and non-Gibbsianness [22]. However, we hope that the simple non-Gibbsianness mechanism clearly illustrated by our examples could be a useful guide and reference for the study of the more involved two-way-noise situation.

In our examples, non-Gibbsianness shows up in the same way as in the basic voter model [16]: Large droplets of aligned (“unanimous”) spins have too large probability for the invariant measures to be Gibbsian. More precisely, we show that once a suitable “spider” of “1” appears, the dynamics causes the alignment of the spins in a neighboring sphere. This sort of damage-spreading property (or error-correcting deficiency) implies that the presence of a sphere of “1” is penalized by the invariant measures only as a sub-volume exponential. This contradicts well known Gibbsian properties. In fact, we can be more precise. Gibbsian measures are characterized by two properties [13]: uniform non-nullness and quasilocality. As we comment in Section 3, the large probability of aligned droplets means that the invariant measures can not be uniformly non-null. More generally, such invariant measures can not be the result of block renormalizations of non-null, in particular Gibbsian, measures. Furthermore, known arguments [7] (briefly reviewed in Section 3 below), imply that if one of these measures is not a product measure, then its non-Gibbsianness is preserved by further single-site renormalization transformations.

2. Simple examples

Before plunging into the technical and notational details needed to describe our results in full generality, we would like to present some simple examples that contain the essential ideas. The examples are defined on the configuration space $\{0, 1\}^{\mathbb{Z}^2}$.

Example 1: The NEC model. — Its deterministic version is defined by a translation-invariant parallel updating defined by the rule

$$(1) \quad x_{\text{det}}^{t+1}(0,0) = \text{major}\left\{x^t(0,1), x^t(1,0), x^t(0,0)\right\},$$

where $x^t(i,j)$ denotes the configuration at site $(i,j) \in \mathbb{Z}^2$ immediately after the t -th iteration of the transformation and $\text{major} : \{0,1\}^{2k+1} \rightarrow \{0,1\}$ is the majority function, i.e. the Boolean function of any odd number of arguments, which equals “1” if and only if most of its arguments equal “1”. This prescription yields an evolution, which is symmetric with respect to the flip $0 \leftrightarrow 1$ [a function with this property is called a *self-spin-flip function* in Section 4 below]. We consider a noisy version, where in addition spins “0” flip into “1” independently with a certain probability ε , while spins “1” remain unaltered. This corresponds to stochastic updating

$$(2) \quad \text{Prob}\left(x^{t+1}(i,j) = 0 \mid x^t\right) = (1 - \varepsilon) \left[1 - x_{\text{det}}^{t+1}(i,j)\right].$$

The “all-ones” delta-measure δ_1 is invariant for this automaton. For small ε there is at least another invariant measure, as a consequence of Theorem 4.2 below.

Let us start with the following simple observations which are immediate consequences of the NEC rule (1) and the one-sidedness of the noise:

- (i) Horizontal lines (parallel to axis i) filled with spins “1” remain invariant under the evolution.
- (ii) The same invariance holds for vertical lines (parallel to axis j) filled with spins “1”.
- (iii) After one evolution-step (that is, after one parallel updating of all the spins), a line of slope -1 filled with spins “1” moves into the parallel line immediately to the south-west.
- (iv) If the (infinite) “spider” formed by the i -axis, the j -axis and the line $i+j=0$ is filled with “1”, then after t steps the evolution causes the whole triangle $\{(i,j) : i,j \leq 0, i+j \geq -t\}$ to be filled with “1”.

The last observation can be visualized as a displacement, at speed 1, of the “front” formed by the line $i+j=0$, with a simultaneous displacement (here a trivial one), at speed 0, of the “fronts” formed by the i - and j -axis. This combined displacement produces a growing triangle full of “1”.

The same observations hold if full lines are replaced by finite segments, except that, depending on the values of neighboring spins, in each iteration each segment can lose one or both of the “1” at its endpoints. We conclude that if at some time the spider

$$(3) \quad \text{SP}_{(0,0),L} = \left\{(i,0) \in \mathbb{Z}^2 : -8L \leq i \leq 4L\right\} \cup \left\{(0,j) \in \mathbb{Z}^2 : -8L \leq j \leq 4L\right\} \\ \cup \left\{(i,j) \in \mathbb{Z}^2 : i+j=0, -6L \leq i \leq 6L\right\}$$

is filled with “1”, then after $4L$ iterations the “1” fill a triangular region that contains the sphere of radius L centered at $(-L, L)$, to be denoted $S_{(-L,-L),L}$. Therefore, if μ

is a invariant measure,

$$(4) \quad \mu(1_{S_{(-L,-L),L}}) \geq \mu(1_{\text{SP}_{(0,0),L}}) \geq \varepsilon^{3(12L+1)}.$$

We have denoted 1_Λ , for $\Lambda \subset \mathbb{Z}^2$, the event $\{x : x(i,j) = 1, (i,j) \in \Lambda\}$. The last inequality in (4) follows from the fact that a “1” has a probability at least ε to appear at a given site because of the noise. As commented in Section 3, such a probability is too large for the invariant measure to be Gibbsian, or block-transformed Gibbsian.

Example 2: North-South maximum of minima (NSMM). — The initial deterministic prescription is defined by

$$(5) \quad x_{\text{det}}^{t+1}(0,0) = \max \left\{ \min(x^t(0,0), x^t(1,0)), \min(x^t(0,1), x^t(1,1)) \right\}$$

plus translation-invariance. The corresponding evolution is not symmetric under flipping, unlike the previous example. The stochastic version is obtained by adding one-sided noise as in (2). For small ε this automaton has more than one invariant measure (see comment after Theorem 4.2). One of them is, of course, the “all-ones” delta-measure δ_1 .

The mechanism for non-Gibbsianness for this model is even simpler to describe than for the NEC model. Indeed, it suffices to observe that whenever a horizontal line is filled with “1”, then in the next iteration these “1” survive and in addition the parallel line immediately to the South becomes also filled with “1”. The same phenomenon happens for finite horizontal segments, except that each creation of a new segment filled with “1” can be accompanied by shrinkages of up to two sites (the spins at the endpoints) of all the previously created segments. We conclude that if the “spider” (which looks more like a snake in this case)

$$(6) \quad \widetilde{\text{SP}}_{(0,0),L} = \{(i,0) \in \mathbb{Z}^2 : -3L \leq i \leq 3L\}$$

is filled with “1” at some instant, then $2L$ instants later the “1” will cover at least a square region that includes the sphere $S_{(0,-L),L}$. Arguing as for (4), we obtain for all invariant measures μ the bound

$$(7) \quad \mu(1_{S_{(0,-L),L}}) \geq \mu(1_{\widetilde{\text{SP}}_{(0,0),L}}) \geq \varepsilon^{6L+1},$$

which implies that μ is neither Gibbsian nor block-transformed Gibbsian.

A comment by A. van Enter (private communication) gives a colorful description of the mechanism acting in both preceding examples: “the spider fills his stomach faster ($\asymp L$ sites at a time) than his legs shrink ($\asymp 1$ sites at a time)”.

Example 3: A non-example. — The automata defined by the deterministic prescription

$$(8) \quad x_{\text{det}}^{t+1}(0,0) = \text{major} \left\{ \min(x^t(0,2), x^t(-1,2)), \min(x^t(2,0), x^t(2,-1)), \right. \\ \left. \min(x^t(0,-1), x^t(-1,0)) \right\}$$

followed by one-sided noise (2), also has multiple invariant measures; this follows from Theorem 4.2 (see the comment following this theorem). Nevertheless, neither the mechanism of Example 1 (travelling fronts), nor that of Example 2 (growing strips) are present, so the theory of the present paper does not apply.

3. Non-nullness and the probability of aligned spheres

We present in this section the key property used in our paper to detect non-Gibbsianness. To state it in its natural generality we introduce some definitions.

We consider a general space of the form $\Omega = \mathcal{A}^{\mathbb{Z}^d}$ where \mathcal{A} is some finite set, equipped with the usual product σ -algebra. For $\Lambda \subset \mathbb{Z}^d$ and $z \in \Omega$ we denote z_Λ the cylinder

$$(9) \quad z_\Lambda = \{x \in \Omega : x_i = z_i, i \in \Lambda\}.$$

Definition 3.1. — A measure μ in Ω is said to have the *alignment-suppression property (ASP)* if there is a positive number C such that the inequality

$$(10) \quad -\ln \mu(z_\Lambda) \geq C \cdot |\Lambda|$$

holds for every configuration $z \in \Omega$ and for every finite set $\Lambda \subset \mathbb{Z}^d$.

All Gibbs measures have the ASP property, but many non-Gibbsian measures too. We construct now a general class of measures with this property by considering renormalized measures having suitable non-nullness features. For this we consider an auxiliary configuration space $\Omega_0 = S^{\mathbb{Z}^d}$. The single-site space S can be very general, not necessarily finite or even compact. We assume that there is a σ -algebra on S and consider the usual product Borel σ -algebra on Ω_0 . A *renormalization transformation* from Ω_0 to Ω is a probability kernel $T(\cdot | \cdot)$ from Ω_0 to Ω . In words, $T(A | \omega)$ is the probability that, given a configuration $\omega \in \Omega_0$, the “renormalized” configuration is in A . This represents a general stochastic transformation while deterministic transformations are the special cases obtained via delta-like prescriptions $T(\cdot | \omega)$. A *block-renormalization transformation* is a transformation, for which probabilities factorize in the following sense: to every $i \in \mathbb{Z}^d$ there corresponds a finite set $B(i) \subset \mathbb{Z}^d$, called *block*, with the following properties:

(i) If two points are far enough from each other, the corresponding blocks are disjoint. That is, there is a positive d_0 such that if the distance between $k, \ell \in \mathbb{Z}^d$ is greater than d_0 , then $B(k) \cap B(\ell) = \emptyset$ ($d_0 = 1$ for the renormalization transformations used in statistical mechanics, while $d_0 > 1$ for common cellular-automata transformations).

(ii) If i_1, \dots, i_k are sites in \mathbb{Z}^d , and a_1, \dots, a_k are values in \mathcal{A} , then

$$(11) \quad T\left(\{x_{i_1} = a_1, \dots, x_{i_k} = a_k\} \mid \omega\right) = \prod_{j=1}^k \widehat{T}_{i_j}\left(\{x_{i_j} = a_j\} \mid \omega_{B(i_j)}\right).$$

Our notation indicates that the functions $\widehat{T}_{i_j}(\{x_{i_j} = a_j\} | \cdot)$ depend only on the values of ω_ℓ for $\ell \in B(i_j)$ (i.e., they are measurable with respect to the σ -algebra generated by the cylinders with base in $B(i_j)$). Examples of such transformations include decimation (deterministic), Kadanoff transformations (stochastic), majority rule, sign fields and transitions of cellular automata (the last three can be deterministic or stochastic, depending on the setting). These transformations are well known in physics, their precise definitions can be found, for instance, in [6, Section 3.1.2].

The kernel T naturally induces a transformation at the level of measures: each probability measure ρ on Ω_0 is mapped into a probability measure ρT on Ω —the *renormalized measure*— defined by

$$(12) \quad \int_{\Omega} f(x) (\rho T)(dx) = \int_{\Omega_0} \left[\int_{\Omega} f(x) T(dx | \omega) \right] \rho(d\omega)$$

for all suitable f (e.g. continuous or non-negative measurable). For each measure ρ on Ω_0 and each block $B(i)$ let us consider the conditional probabilities $\rho(d\omega_{B(i)} | \omega_{\mathbb{Z}^d \setminus B(i)})$. For a given transformation T we single out the set \mathcal{P}_T of measures on Ω_0 that admit conditional probabilities such that

$$(13) \quad \min_{a \in \mathcal{A}} \inf_{i \in \mathbb{Z}^d} \inf_{\omega_{\mathbb{Z}^d \setminus B(i)}} \int \widehat{T}\left(\{x_i = a\} \mid \omega_{B(i)}\right) \cdot \rho\left(d\omega_{B(i)} \mid \omega_{\mathbb{Z}^d \setminus B(i)}\right) \geq \delta,$$

for some $\delta > 0$. We denote \mathcal{P} the union of these families \mathcal{P}_T over all block-renormalization transformations T . Here is our key characterization.

Theorem 3.1. — *Every measure in \mathcal{P} has the alignment-suppression property.*

Proof. — Let T , ρ be such that $\mu = \rho T$. By property (ii) above, there exists a constant $\gamma > 0$ (proportional to d_0) such that for any $\Lambda \subset \mathbb{Z}^d$ there is a family of sites $i_1, \dots, i_k \in \Lambda$ with $k \geq \gamma|\Lambda|$, all of which are far enough from each other and therefore the blocks $B(i_1), \dots, B(i_k)$ are disjoint. We therefore have that for every $z \in \Omega$

$$(14) \quad \begin{aligned} \mu(z_\Lambda) &= \int \rho\left(\widehat{T}(\{x_{i_1} = z_{i_1}\} \mid \cdot) \mid \omega_{\mathbb{Z}^d \setminus B(i_1)}\right) \prod_{j=2}^k \widehat{T}_{i_j}(\{x_{i_j} = z_{i_j}\} \mid \omega_{B(i_j)}) \rho(d\omega) \\ &\leq (1 - \delta) \int \prod_{j=2}^k \widehat{T}_{i_j}(\{x_{i_j} = z_{i_j}\} \mid \omega_{B(i_j)}) \rho(d\omega). \end{aligned}$$

This inequality is an immediate consequence of condition (13). After k iterations of this procedure we obtain

$$(15) \quad \mu(z_\Lambda) \leq (1 - \delta)^k \leq (1 - \delta)^{\gamma|\Lambda|}. \quad \square$$

The class \mathcal{P} of measures is a very large class. It contains practically all block transformations of Gibbs measures with finite alphabet obtained via standard statistical mechanics prescriptions (decimation, Kadanoff, majority rule, etc), plus the measures generated by finite-time evolutions of usual cellular automata prescriptions. There is

by now a vast literature about such measures —see, for instance, [6, 18, 3]; for recent reviews with many references see [4, 10, 11, 8]— showing that many of them are non-Gibbsian. In fact, the family \mathcal{P}_I , where I is the identity, includes all *uniformly non-null measures*. These are measures μ that have, for each finite region $\Lambda \subset \mathbb{Z}^d$, uniformly bounded conditional probabilities $\mu(d\omega_\Lambda | \omega_{\mathbb{Z}^d \setminus \Lambda})$, that is, such that there exist $\delta_\Lambda > 0$ with

$$(16) \quad \min_{a_\Lambda \in \mathcal{A}^\Lambda} \inf_{\omega_{\mathbb{Z}^d \setminus \Lambda}} \mu\left(\{x_\Lambda = a_\Lambda\} \mid \omega_{\mathbb{Z}^d \setminus \Lambda}\right) \geq \delta_\Lambda.$$

Here we have denoted $a_\Lambda = (a_i)_{i \in \Lambda}$. Gibbs measures are uniformly non-null —and in addition quasilocal (the finite-volume conditional probabilities are continuous functions of the external conditions $\omega_{\mathbb{Z}^d \setminus \Lambda}$)— hence they also belong to \mathcal{P}_I . Property (13) seems to be more general than usual non-nullness, in particular it does not depend on the existence of a whole system of conditional probabilities.

The invariant measures of the automata of the present paper, on the other hand, do not have the alingment-suppression property, hence they do not belong to the class \mathcal{P} . They therefore can be neither Gibbsian nor uniformly non-null nor block-transformed Gibbsian. As further examples of measures without the ASP we mention the invariant measures of the basic voter model [16], the invariant measure of some non-local dynamics [23], and the sign-fields of massless Gaussians [14, 5], anharmonic crystals [6, Section 4.4] and solid-on-solid (SOS) models [9, 17].

For measures μ having a well defined relative entropy density $s(\cdot | \mu)$, the alignment-suppression property (10) implies that $s(\delta_z | \mu) > 0$ for every periodic configuration $z \in \Omega$. The relative entropy density is known to exist for translation-invariant Gibbs measures [12, Chapter 15]. Recent work in [25] shows that it is also well defined for most translation-invariant measures obtained through block transformations of Gibbs measures. Because of this, the non-Gibbsianness resulting from the lack of ASP has often been interpreted as “too large probabilities of large deviations”. The non-Gibbsianness (non-nullness) criterion obtained by falsifying Theorem 3.1, however, is a more general argument that needs neither translation invariance of μ nor the existence of the entropy density.

For completeness, we mention a further result obtained in [7].

Theorem 3.2. — *Suppose μ is a measure in Ω such that (i) it violates the ASP property for some periodic configuration $z \in \Omega$, and (ii) it is not a product measure. Then, for every single-site block-renormalization transformation T (i.e. a transformation defined by blocks $B(i)$ formed by only one site), the measure μT is not Gibbsian.*

This result follows from the fact that such a violation implies that $s(\delta_z | \mu) = 0$, which in its turn implies that $s(\delta_z T | \mu T) = 0$. If μT were Gibbs, then by a well known result [12, Theorem 15.37] the measure $\delta_z T$ would be Gibbs for an equivalent interaction. But this is impossible because the latter is a product measure and the former is not. Note that if T corresponds to a not-totally asymmetric noise, the

measure μT is uniformly non-null. Hence its non-Gibbsianness would correspond to lack of quasilocality.

For the automata of this paper, we suspect that many of their invariant measures are non-product.

4. General Results

We now describe a large family of automata exhibiting a general version of the non-Gibbsianness mechanism of the first two examples in Section 2. Throughout the article we consider the d -dimensional integer space \mathbb{Z}^d with $d > 1$ embedded into the d -dimensional real space \mathbb{R}^d with the same axes and Euclidean norm $\|\cdot\|$. The configuration space is $\Omega = \{0, 1\}^{\mathbb{Z}^d}$. We first need some definitions.

For any $i \in \mathbb{Z}^d$ we denote $\tau_i : \Omega \rightarrow \Omega$ the translation of Ω defined by $(\tau_i x)_j = x_{j-i}$. Any function $f : \Omega \rightarrow \{0, 1\}$ will be called a transition function. Given any transition function f , we define the corresponding operator $D_f : \Omega \rightarrow \Omega$ by the rule

$$(17) \quad \forall i \in \mathbb{Z}^d : (D_f x)_i = f(\tau_i x).$$

We call $f : \Omega \rightarrow \{0, 1\}$ *standard* if it has the following three properties:

- 1) f is *local*, i.e. there is a finite set $\Delta \subset \mathbb{Z}^d$ —the *support* of f — such that $f(x) \equiv f(x_\Delta)$. Given Δ , we denote $\|\Delta\|$ the maximum of $\|i\|$ for $i \in \Delta$.
- 2) f is *monotonic*, that is $(\forall i : x_i \leq y_i) \implies f(x) \leq f(y)$.
- 3) f is not a constant. (Otherwise our theorem is either trivially true if $f \equiv 1$ or trivially false if $f \equiv 0$.)

Since f is monotonic and non-constant,

$$(18) \quad f(\text{"all zeros"}) = 0 \text{ and } f(\text{"all ones"}) = 1.$$

For any $x \in \Omega$ we denote its indicator $\text{Ind}(x) = \{i \in \mathbb{Z}^d \mid x_i = 1\}$. Conversely, for any $S \subset \mathbb{Z}^d$ we denote $\text{Conf}(S)$ that configuration, whose indicator is S .

Let us call an element of \mathbb{R}^d a *direction* if its norm equals 1. For any direction p we call a *front* with this direction any configuration whose indicator has the form

$$(19) \quad \{i \in \mathbb{Z}^d \mid \langle i, p \rangle \leq C\},$$

where C is a real number and $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^d . It is evident that for any standard f the operator D_f transforms any front (19) into a front with the same direction, C being substituted by $C + V_p$, where V_p does not depend on C . We call V_p the *velocity* of D_f in the direction p .

Let us call a configuration $x \in \Omega$ invariant for D_f if $D_f x = x$. Given $x, y \in \Omega$, we call y a finite deviation of x if the set of those $i \in \mathbb{Z}^d$, for which $y_i \neq x_i$, is finite. We say that an invariant configuration x attracts D_f if for any its finite deviation y there is a time t such that $D_f^t y = x$.

Let \mathcal{M} denote the set of probability measures on Ω (on the σ -algebra generated by cylinder sets). For any $\varepsilon \in [0, 1]$ we define one-sided noise $N_\varepsilon : \mathcal{M} \rightarrow \mathcal{M}$ as follows: when applied to a measure δ_x concentrated in a configuration $x = (x_i)$, it produces a product measure $N_\varepsilon \delta_x$, in which the i -th component equals 1 with a probability 1 if $x_i = 1$ and with a probability ε if $x_i = 0$.

Theorem 4.1. — Take any standard f , such that “all ones” attracts D_f , and make any one of the following two assumptions:

- a) $V_p + V_{-p} \geq 0$ for all directions p .
- b) There is a direction p such that $V_p + V_{-p} > 0$.

Then for any $\varepsilon > 0$ all the invariant measures of $N_\varepsilon D_f$ satisfy

$$(20) \quad -\ln \mu(\mathbf{1}_{S_{0,L}}) \prec L^{d-1}.$$

Here and in the sequel $f \prec g$ or $g \succ f$, for f and g positive functions means that there exists a constant $C > 0$ such that $g \geq C \cdot f$. If $f \prec g$ and $f \succ g$, we write $f \asymp g$.

If $\varepsilon = 0$, our theorem may be false, for example if D is the identity. Notice also that in the case b) our assumption that “all ones” attracts D_f is redundant because it follows from b).

Let us present some further considerations that clarify the statement of the theorem. Given any non-constant affine function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ and two numbers $C_1 \leq C_2$, we call a *layer* any configuration $\text{Conf}\{i \in \mathbb{Z}^d \mid C_1 \leq \phi(i) \leq C_2\}$. We call the thickness of this layer the distance between the hyperplanes $\phi = C_1$ and $\phi = C_2$, that is $(C_2 - C_1) / \|\phi\|$, where $\|\cdot\|$ is the norm. We call a layer thick-enough if its thickness is not less than $2 \|\Delta\|$. We call the two normal unit vectors to hyperplanes $\phi = \text{const}$ the directions of this layer. If f is standard, D_f transforms any thick-enough layer into a layer with the same directions, the thickness of the layer changing by $V_p + V_{-p}$. The condition a) of our theorem means that thickness of any thick-enough layer does not decrease and the condition b) means that thickness of some layer increases under the action of D_f .

Of the examples of Section 2, the NEC automaton satisfies condition a), while the NSMM automaton satisfies condition b) for $p = (0, 1)$. For the non-example, however, $V_p + V_{-p} < 0$ for all directions p . In all the three cases f [given, respectively, by (1), (5) and (8)] is standard, and both “all zeros” and “all ones” attract D_f .

The NEC example is representative of a class of models with a further duality property. For any $x_i \in \{0, 1\}$ we denote $\neg x_i = 1 - x_i$. Accordingly, if x is a configuration, $\neg x$ is another configuration such that $(\neg x)_i \equiv \neg(x_i)$. Any transition function f has an associated spin-flip ⁽¹⁾ function denoted $\neg f$ and defined by the identity $\neg f(x) \equiv f(\neg x)$. Let us call f self-spin-flip if it coincides with its spin-flip. If f is

⁽¹⁾In the theory of Boolean functions $\neg f$ is called dual, but we have to use another term because in the theory of random processes the word “duality” is used for another purpose.

standard and self-spin-flip, then $V_p + V_{-p} \equiv 0$, so the thickness of all layers does not change under the action of D_f , which provides many examples where our results can be applied. For example, the function $\text{major}(\cdot)$, described above, is self-spin-flip.

It is evident that under the hypothesis of Theorem 4.1, the measure δ_1 is invariant for any superposition $N_\varepsilon D_f$. Hence, the theorem is not trivial only if the automata have more than one invariant measure. This is ensured by the following theorem. Given f , let us call a set $S \subset \mathbb{Z}^d$ a one-set if $f(\text{Conf}(S)) = 1$. Since one-sets belong to \mathbb{Z}^d , they belong to \mathbb{R}^d , where we can consider their convex hulls, the intersection of which is denoted σ_1 . In the analogous way we call a set $S \subset \mathbb{Z}^d$ a zero-set if $f(\text{Conf}(\mathbb{Z}^d - S)) = 0$ and denote σ_0 the intersection of their convex hulls.

Theorem 4.2. — *For any operator D_f defined by (17), where f is standard, the following four statements are equivalent:*

- 1) $N_\varepsilon D_f$ has more than one invariant measure for some positive ε .
- 2) The configuration “all zeros” attracts D_f .
- 3) σ_0 is empty.
- 4) There are a natural number $m \leq d+1$ and m affine functions $\phi_1, \dots, \phi_m : \mathbb{R}^d \rightarrow \mathbb{R}$ such that:
 - i) for every $j \in [1, m]$ the set $\{p \in \mathbb{Z}^d : \phi_j(p) \leq 0\}$ is a zero-set.
 - ii) $\phi_1 + \dots + \phi_m \equiv \text{const} > 0$.
 - iii) There is a rational point $p \in \mathbb{R}^d$ such that $\phi_j(p) > 0$ for all $j \in [1, m]$.

This theorem proves, in particular, that the three examples of Section 2 exhibit multiple invariant measures for ε small. Indeed, in the three cases the configuration “all zeros” attracts D_f .

5. Proof of Theorem 4.2

If we omit the condition iii) in 4), our Theorem 4.2 almost follows from theorems 5 and 6 and lemma 12 of [29]. However, there is some difference, so for the reader’s convenience we completely deduce 4) from 3).

Suppose that σ_0 is empty. Every zero-set can be represented as an intersection of several zero-half-spaces, i.e. half-spaces, which are zero-sets, where a half-space is a subset of \mathbb{R}^d , where some non-constant affine function does not exceed zero. Thus there are several zero-half-spaces, whose intersection is empty. Everyone of them can be represented as $\{p \in \mathbb{R}^d \mid f_i(p) \leq 0\}$, where f_i are affine functions on \mathbb{R}^d . We can choose these functions so that they have no common direction of recession (that is, no direction p such that $f_i(p) \leq f_i(0)$ for all i), which allows us to apply to them Theorem 21.3 on page 189 of [26]. Since the intersection of our zero-half-spaces is empty, the case (a) of this theorem is excluded in the present situation, whence the case (b) takes place, which amounts to our conditions i) and ii) in 4), the products

$\lambda_i f_i$ mentioned in the case (b) serving as our ϕ_i . We may assume that our m is the minimal for which there are functions satisfying i) and ii). Based on this, let us prove statement iii) using the following lemma, which is a direct consequence of Theorem 21.1 on page 186 of [26]:

Lemma 5.1. — *Let ϕ_1, \dots, ϕ_m be affine functions on R^d . Then one and only one of the following alternatives holds:*

- (a) *There exists some $x \in R^d$ such that $\phi_1(x) > 0, \dots, \phi_m(x) > 0$;*
- (b) *There exist non-negative real numbers $\lambda_1, \dots, \lambda_m$, not all zero, such that the sum $\lambda_1\phi_1(x) + \dots + \lambda_m\phi_m(x)$ is a non-positive constant.*

Let us assume that the case (b) takes place in our situation. We may assume that λ_m is the greatest of $\lambda_1, \dots, \lambda_m$, and therefore positive. From the statement ii) of 4), not all λ_i are equal to λ_m . Let us divide all terms by λ_m :

$$\frac{\lambda_1}{\lambda_m}\phi_1 + \dots + \frac{\lambda_{m-1}}{\lambda_m}\phi_1 + \phi_m = \text{const} \leq 0$$

and subtract this from the statement ii) of 4):

$$\left(1 - \frac{\lambda_1}{\lambda_m}\right)\phi_1 + \dots + \left(1 - \frac{\lambda_{m-1}}{\lambda_m}\right)\phi_1 = \text{const} \geq 0.$$

Here all coefficients are non-negative and not all are zero. Therefore the functions $(1 - \lambda_i/\lambda_m)\phi_i$ for $i = 1, \dots, m-1$ also satisfy the conditions i) and ii) of 4) with a smaller value of m , which contradicts our assumption. Thus case (b) is excluded, so case (a) takes place, whence there is a point $p \in R^d$ where all $\phi_j(p) > 0$. Since all ϕ_j are continuous, there is a rational point with this property also, whence condition iii) of 4) follows, \square

6. Proof of Theorem 4.1

6.1. Proof of (20) in case a) of the theorem. — Rewording Theorem 4.2 for the case when 0 and 1 are permuted, we see that whenever f is standard and “all ones” attracts D_f , there exist a natural number $m \leq d+1$ and m affine functions $\phi_1, \dots, \phi_m : R^d \rightarrow R$ such that:

- $$(21) \quad \begin{cases} \text{i) for every } j \in [1, m] \text{ the set } \{i \in \mathbb{Z}^d : \phi_j(i) \leq 0\} \text{ is a one-set.} \\ \text{ii) } \phi_1 + \dots + \phi_m \equiv \text{const} > 0. \\ \text{iii) There is a rational point } p \in R^d \text{ such that } \phi_j(p) > 0 \text{ for all } j \in [1, m]. \end{cases}$$

For instance, for the NEC example there are $m = 3$ such affine functions, whose level lines are horizontal, vertical and lines of slope -1 respectively.

For every j let us denote $\bar{\phi}_j = \phi_j - \phi_j(0)$, whence $\phi_j = \bar{\phi}_j + \phi_j(0)$, where $\bar{\phi}_j$ is the linear part. Notice that $|\phi_j(0)| \leq \|\phi_j\| \cdot \|\Delta\|$ and that $\phi_1(0) + \dots + \phi_m(0) > 0$. Notice

also that if f is standard, “all ones” attracts D_f and $V_p + V_{-p} \geq 0$ for all directions p , then for any $j \in [1, m]$ and any thick-enough layer

$$y = \text{Conf}\{i \in \mathbb{Z}^d \mid C_1 \leq \phi_j(i) \leq C_2\},$$

$$(22) \quad \text{Ind}(D_f y) \supseteq \left\{ i \in \mathbb{Z}^d \mid C_1 + \phi_j(0) \leq \phi_j(i) \leq C_2 + \phi_j(0) \right\}.$$

Lemma 6.1. — Take any standard f and assume that “all ones” attracts D_f and that $V_p + V_{-p} \geq 0$ for all directions p . Take x^* defined by

$$(23) \quad \text{Ind}(x^*) = \bigcup_{1 \leq j \leq m} \left\{ i \in \mathbb{Z}^d \mid |\phi_j(i)| \leq 2\|\Delta\| \cdot \|\phi_j\| \right\}.$$

Then for $t = 0, 1, 2, 3, \dots$ the indicator of $D^t x^*$ includes the union $A_t \cup B_t$, where

$$(24) \quad A_t = \bigcup_{1 \leq j \leq m} \left\{ i \in \mathbb{Z}^d \mid |\phi_j(i) - t \cdot \phi_j(0)| \leq 2\|\Delta\| \cdot \|\phi_j\| \right\}$$

and

$$(25) \quad B_t = \bigcap_{1 \leq j \leq m} \left\{ i \in \mathbb{Z}^d \mid \bar{\phi}_j(i) - t \cdot \phi_j(0) \leq 0 \right\}.$$

[For the NEC example of Section 2, this lemma corresponds to observation (iv).]

Let us prove this lemma by induction. Base of induction: Since A_0 coincides with $\text{Ind}(x^*)$ and $B_0 \subset A_0$, our statement is true for $t = 0$.

Induction step. — Let us suppose that $\text{Ind}(D^t x^*) \supseteq A_t \cup B_t$, take any $i \in A_{t+1} \cup B_{t+1}$ and prove that $i \in \text{Ind}(D^{t+1} x^*)$. Let us consider two cases.

Case 1. — Let i belong to A_{t+1} . Then our statement follows from (22).

Case 2. — Let i belong to B_{t+1} , but not to A_{t+1} . Then

$$\bar{\phi}_j(i) - (t+1) \cdot \phi_j(0) \leq -2\|\Delta\| \cdot \|\phi_j\|$$

for all $j \in [1, m]$. Notice that

$$\bar{\phi}_j(i + v_k) \leq \bar{\phi}_j(i) + \|\phi_j\| \cdot |v_k| \leq \bar{\phi}_j(i) + \|\phi_j\| \cdot \|\Delta\|.$$

Therefore

$$\begin{aligned} \bar{\phi}_j(i + v_k) - t \cdot \phi_j(0) &\leq \bar{\phi}_j(i) + \|\phi_j\| \cdot \|\Delta\| - (t+1)\phi_j(0) + \phi_j(0) \\ &\leq -2\|\Delta\| \cdot \|\phi_j\| + \|\phi_j\| \cdot \|\Delta\| + \phi_j(0) \leq 0. \end{aligned}$$

Thus

$$i + \Delta \subset B_t \subset \text{Ind}(D^t x^*).$$

Hence from (18) $i \in \text{Ind}(D^{t+1} x^*)$. Lemma 6.1 is proved.

Lemma 6.2. — Under the conditions of Lemma 6.1, there is a positive constant $\alpha > 0$ such that for all $t = 0, 1, 2, \dots$ the set B_t defined by (25) contains a sphere in \mathbb{Z}^d with the radius $\alpha \cdot t$.

Proof. — In fact we shall prove that

$$\forall i \in \mathbb{Z}^d, t = 0, 1, 2, \dots : |i + t \cdot p| \leq \alpha \cdot t \implies i \in B_t,$$

where p is that rational point where all $\phi_j(p) > 0$, whose existence is provided by iii). Let us denote $\kappa_j = \phi_j(p) > 0$ and $\alpha = \min_j(\kappa_j / \|\phi_j\|)$, that is the minimal distance from p to the hyperplanes $\phi_j = 0$. Let us consider three cases.

Case 1. — Let $p = 0$. Then $\phi_j(0) = \kappa_j > 0$ for all j . Now let us take any point i in the sphere with the radius $\alpha \cdot t$ and center 0. This means that

$$|i| \leq \alpha \cdot t = \min_j(\phi_j(0) / \|\phi_j\|) \cdot t.$$

Then

$$\bar{\phi}_j(i) \leq |i| \cdot \|\phi_j\| \leq \phi_j(0) / \|\phi_j\| \cdot t \cdot \|\phi_j\| = t \cdot \phi_j(0)$$

for all j , whence $i \in B_t$.

Case 2. — Let $p \in \mathbb{Z}^d$. Then along with our operator D_f we consider another operator D_g , where $g(x) \equiv f(\tau_p x)$. The function g is also standard, D_g is also attracted by “all ones” and the affine functions provided for D_g by iii) of (21) can be obtained from those for D_f by the same translation, so their values at 0 are $\kappa_1, \dots, \kappa_m > 0$, whence D_g fits our case 1. So the set B_t for D_g contains a sphere with the center 0 and radius $\alpha \cdot t$. Since D_f commutes with all translations, the set B^t for D_g results from the set B_t for D_f by a translation at $t \cdot p$. Thus the set B_t for D_f results from B_t for D_g by the opposite translation, whence it contains a sphere with the center $-t \cdot p$ and the same radius.

Case 3. — Let p be any rational point. Let us denote q the least common denominator of all the coordinates of p and immerse our \mathbb{Z}^d into the set \mathbb{Z}_q^d , where $\mathbb{Z}_q = \{n/q \mid n \in \mathbb{Z}\}$. Let us denote $\Omega_q = \{0, 1\}^{\mathbb{Z}_q^d}$. Now f can be considered as a function g from Ω_q to $\{0, 1\}$. Now let us “stretch” \mathbb{Z}_q^d to turn it into \mathbb{Z}^d . Under this transformation the function g remains standard and “all ones” still attracts D_g . In addition to that, the affine functions for D_g with the properties (21) now can be obtained from those for D_f by a homothety with coefficient q . Therefore their values at the integer point $q \cdot p$ are $\kappa_1, \dots, \kappa_m > 0$. So D_g fits our case 2, whence the set B_t for D_g contains a sphere with the center $-t \cdot q \cdot p$ and radius $\alpha \cdot q \cdot t$, whence the set B_t for D_f contains a sphere with the center $-t \cdot p$ and radius $\alpha \cdot t$. Lemma 6.2 is proved. \square

Now let us prove (20). From monotonicity it is sufficient to prove this inequality for $\mu = (N_\varepsilon D_f)^t \delta_0$ for some t . Let us choose t_1 such that $\alpha \cdot t_1 \geq R + d$. Then, taking x^* defined by (23) as the initial configuration, after t_1 time-steps we obtain a configuration, whose indicator contains a sphere with the radius $R + d$ and therefore contains a sphere with the radius R and center at some integer point p . However, what we actually need is a finite deviation from “all zeros”, which coincides with x^* only

within a sphere with the radius $R + t_1 \cdot \|\Delta\|$ and has zeros outside it. The cardinality of its indicator does not exceed $C(R^{d-1} + 1)$ with an appropriate C . Translating this configuration at the vector $-p$, we obtain another configuration, which fills with ones a sphere with radius R and center at the origin after t_1 time-steps. The probability that the actual configuration's indicator contains this configuration is not less than $\varepsilon^{C(R^{d-1} + 1)}$, whence (20) follows. \square

6.2. Proof of (20) in case b) of the theorem. — This time we define x^* as follows:

$$\text{Ind}(x^*) = \left\{ i \in \mathbb{Z}^d \mid |\langle i, p \rangle| \leq \|\Delta\| \right\}.$$

Then for all $t = 0, 1, 2, \dots$

$$\text{Ind}(D_f^t x^*) \supseteq \left\{ i \in \mathbb{Z}^d \mid -\|\Delta\| + t \cdot V_{-p} \leq |\langle i, p \rangle| \leq \|\Delta\| + t \cdot V_p \right\}.$$

Here the right side is a layer with the thickness $2\|\Delta\| + t(V_p + V_{-p})$. Given any $R \geq 0$, let us choose the minimal integer t_1 for which $2\|\Delta\| + t_1(V_p + V_{-p}) \geq R + d$. Then indicator of $D_f^{t_1} x^*$ contains a sphere with an integer center and radius R . If we take an initial condition which coincides with x^* within a sphere with the center at the origin and radius $R + d + t_1 \cdot \|\Delta\|$, we shall obtain the same result. This configuration has $C(R^{d-1} + 1)$ components that equal 1, where C is an appropriate constant. Further we argue like in case a). \square

7. Final notes

Note 1. — Using minoration arguments, it is easy to expand our theorem to some random cellular automata, which cannot be represented as $N_\varepsilon D_f$. Using the same Δ as before and choosing transition probabilities $\theta(x \mid y_\Delta)$ for all $x \in \{0, 1\}$ and $y \in \{0, 1\}^\Delta$, we can define a random cellular automaton as an operator $P : \mathcal{M} \rightarrow \mathcal{M}$ which transforms any δ_y , where $y \in \Omega$, into a product-measure in which the probability that the i -th component equals x is $\theta(x \mid y_{i+\Delta})$. This operator majorates $N_\varepsilon D_f$ if

$$\theta(x \mid y_\Delta) \begin{cases} = 1 & \text{if } f(y_\Delta) = 1, \\ \geq \varepsilon & \text{if } f(y_\Delta) = 0. \end{cases}$$

As soon as this condition holds and D_f satisfies conditions of our theorem, all invariant measures of P also satisfy (20) and therefore are non-Gibbs.

Note 2. — In some cases it is possible to obtain a stronger estimation than (20). Let $d > a > 0$ and $f(x)$ equal

$$\min_{i_1, \dots, i_a \in \{0, 1\}} \max_{i_{a+1}, \dots, i_d \in \{0, 1\}} x(i_1, \dots, i_d)$$

where i_1, \dots, i_d are the coordinates of \mathbb{Z}^d . In this case

$$-\ln \mu(\mathbf{1}(S_{0,L})) \prec L^a,$$

where μ is any invariant measure of $N_\varepsilon D_f$. If $a < d - 1$, this estimation is stronger than (20). This estimation can be proved in the same manner as in the case b), only x^* now is defined by the condition:

$$x_i^* = 1 \text{ if } \max(|i_{a+1}|, \dots, |i_d|) \leq \text{const.}$$

Note 3. — Given a standard f , let us assume that “all zeros” attracts D_f . Then we hope to estimate $-\ln \mu(\mathbf{1}(S_{0,L}))$ from below as follows:

$$-\ln \mu(\mathbf{1}(S_{0,L})) \succ L.$$

If we succeed, this will settle the question of asymptotics of $-\ln \mu(\mathbf{1}(S_{0,L}))$ in some cases, e.g. in our examples 1 and 2.

Note 4. — Those conditions under which our theorem holds and is non-trivial can be satisfied only for $d > 1$. However, a statement similar to our theorem for the one-dimensional case was proved in [27]. Namely, it was proved that all non-trivial invariant measures of a class of one-dimensional random cellular automata did not belong to a class, which included all Markov measures.

Acknowledgments. — We thank Aernout van Enter for very useful comments and criticism. We also thank the organizers and funding agencies of the IV Brazilian School of Probability where the final discussions for this paper took place.

References

- [1] C. Bennett and G. Grinstein. Role of Irreversibility in Stabilizing Complex and Nonergodic Behavior in Locally Interacting Discrete Systems. *Phys. Rev. Letters*, v. 55 (1985), n. 7, pp. 657–660.
- [2] M. Bramson and L. Gray. A useful renormalization argument. In *Random walks, Brownian motion, and interacting particle systems*, 113–152, Birkhäuser Boston, Boston, MA, 1991.
- [3] J. Bricmont, A. Kupiainen, and R. Lefevere. Renormalization group pathologies and the definition of Gibbs states. *Comm. Math. Phys.*, 194(2):359–388, 1998.
- [4] A.C.D. van Enter. The renormalization-group peculiarities of Griffiths and Pearce: What have we learned? In *Mathematical Results in Statistical Mechanics*, S. Miracle-Sole, J. Ruiz and V. Zagrebnov eds., World Scientific, Singapore, 1999, pp. 509–26.
- [5] T.C. Dorlas and A.C.D. van Enter. Non-Gibbsian limit for large-block majority-spin transformations. *J. Stat. Phys.*, 55:171–181, 1989.
- [6] A.C.D. van Enter, R. Fernández, and A.D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. *J. Stat. Phys.*, 72:879–1167, 1993.
- [7] A.C.D. van Enter and J. Lörinczi. Robustness of the non-Gibbsian property: some examples. *J. Phys. A*, 29:2465–73, 1996.

- [8] A.C.D. van Enter, C. Maes, and S. Shlosman. Dobrushin's program on Gibbsianity restoration: weakly Gibbs and almost Gibbs random fields. In *On Dobrushin's way. From probability theory to statistical physics*, Amer. Math. Soc., Providence, RI, 2000, pp. 59–70.
- [9] A.C.D. van Enter and S.B. Shlosman. (Almost) Gibbsian description of the sign fields of SOS fields. *J. Statist. Phys.*, 92:353–368, 1998.
- [10] R. Fernández. Random fields in lattices. The Gibbsianess issue. *Resenhas*, 3(4):391–421, 1998.
- [11] R. Fernández. Measures for lattice systems. *Phys. A*, 263(1-4):117–130, 1999. STAT-PHYS 20 (Paris, 1998).
- [12] H.-O. Georgii. *Gibbs Measures and Phase Transitions*. Walter de Gruyter (de Gruyter Studies in Mathematics, Vol. 9), Berlin–New York, 1988.
- [13] O.K. Kozlov. Gibbs description of a system of random variables. *Probl. Inform. Transmission*, 10:258–65, 1974.
- [14] J. L. Lebowitz and C. Maes, The effect of an external field on an interface, entropic repulsion, *J. Stat. Phys.*, v. 46, pp. 39–49, 1987.
- [15] J.L. Lebowitz, C. Maes, and E.R. Speer. Statistical mechanics of probabilistic cellular automata. *J. Stat. Phys.*, 59:117–70, 1990.
- [16] J.L. Lebowitz and R.H. Schonmann. Pseudo-free energies and large deviations for non-Gibbsian FKG measures. *Prob. Th. Rel. Fields*, 77:49–64, 1988.
- [17] J. Lörinczi, Non-Gibbsianess of the reduced SOS-measure, *Stoch. Proc. Appl.*, v. 74, pp. 83–88, 1998.
- [18] J. Lörinczi and C. Maes. Weakly Gibbsian measures for lattice spin systems. *J. Statist. Phys.*, 89(3-4):561–579, 1997.
- [19] C. Maes and K. Vande Velde. The interaction potential of a stationary measure of a high-noise spinflip process. *J. Math. Phys.*, 34:3030–1, 1993.
- [20] C. Maes and K. Vande Velde. The (non-)Gibbsian nature of states invariant under stochastic transformations. *Physica A*, 206:587–603, 1994.
- [21] D. Makowiec. Gibbsian versus non-Gibbsian nature of stationary states for Toom probabilistic cellular automata via simulations. *Phys. Rev. E*, 55:6582–8, 1997.
- [22] D. Makowiec. Stationary states of Toom cellular automata in simulations. *Phys. Rev. E*, 60:3787–95, 1999.
- [23] F. Martinelli and E. Scoppola. A simple stochastic cluster dynamics: rigorous results. *J. Phys. A*, 24:3135–57, 1991.
- [24] M. B. Petrovskaya, I. I. Piatetski-Shapiro and N. B. Vasilyev. Modelling of voting with random errors. *Automatics and telemechanics*, v.10, pp. 103–107, 1969 (in Russian).
- [25] C.-E. Pfister. Thermodynamical aspects of classical lattice systems. Preprint. Notes for a minicourse at the IV-th Brazilian School of Probability, Mambucaba, Brazil, August 2000.
- [26] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.
- [27] A.L. Toom. On invariant measures in non-ergodic random media. In *Probabilistic Methods of Investigation*, A. Kolmogorov, editor, Moscow University Press, Moscow, 1972, pp. 43–51. (In Russian).
- [28] A.L. Toom. Nonergodic multidimensional systems of automata. *Probl. Inform. Transmission*, 10(3):70–79, 1974.
- [29] A.L. Toom. Stable and attractive trajectories in multicomponent systems. In *Multicomponent Random Systems. Advances in Probability and Related Topics*, R. Dobrushin and Ya. Sinai eds., Dekker, New York, 1980, v. 6, pp. 549–576.

- [30] K. Vande Velde. On the question of quasilocality in large systems of locally interacting components. K. U. Leuven thesis, 1995.

R. FERNÁNDEZ, Mathématiques, site Colbert, Faculté des Sciences Université de Rouen F-76821
Mont Saint-Aignan, France • *E-mail* : Roberto.Fernandez@univ-rouen.fr

A. TOOM, Departamento de Estatística, Universidade Federal de Pernambuco, Recife/PE 50740-
540, Brazil • *E-mail* : toom@de.ufpe.br or toom@member.amd.org