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# GENERATING FUNCTION ASSOCIATED WITH THE DETERMINANT FORMULA FOR THE SOLUTIONS OF THE PAINLEVÉ II EQUATION

*by*

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**Abstract.** — In this paper we consider a Hankel determinant formula for generic solutions of the Painlevé II equation. We show that the generating functions for the entries of the Hankel determinants are related to the asymptotic solution at infinity of the linear problem of which the Painlevé II equation describes the isomonodromic deformations.

**Résumé (Fonction génératrice associée à la formule déterminant pour les solutions de l'équation de Painlevé II)**

On s'intéresse à la formule déterminant de Hankel pour les solutions génériques de l'équation de Painlevé II. On établit une relation reliant les fonctions génératrices des coefficients des déterminants de Hankel aux solutions asymptotiques à l'infini du problème linéaire dont les déformations isomonodromiques sont décrites par cette équation de Painlevé II.

## 1. Introduction

The Painlevé II equation ( $P_{II}$ ),

$$(1) \quad \frac{d^2u}{dx^2} = 2u^3 - 4xu + 4\left(\alpha + \frac{1}{2}\right),$$

where  $\alpha$  is a parameter, is one of the most important equations in the theory of nonlinear integrable systems. It is well-known that  $P_{II}$  admits unique rational solution when  $\alpha$  is a half-integer, and one-parameter family of solutions expressible in terms of the solutions of the Airy equation when  $\alpha$  is an integer. Otherwise the solution is non-classical [13, 14, 17].

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The rational solutions for  $P_{II}(1)$  are expressed as logarithmic derivative of the ratio of certain special polynomials, which are called the “Yablonski-Vorob’ev polynomials”, [18, 20]. Yablonski-Vorob’ev polynomials admit two determinant formulas, namely, Jacobi-Trudi type and Hankel type. The latter is described as follows: For each positive integer  $N$ , the unique rational solution for  $\alpha = N + 1/2$  is given by

$$u = \frac{d}{dx} \log \frac{\sigma_{N+1}}{\sigma_N},$$

where  $\sigma_N$  is the Hankel determinant

$$\sigma_N = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-2} \end{vmatrix},$$

with  $a_n = a_n(x)$  being polynomials defined by the recurrence relation

$$(2) \quad \begin{aligned} a_0 &= x, & a_1 &= 1, \\ a_{n+1} &= \frac{da_n}{dx} + \sum_{k=0}^{n-1} a_k a_{n-1-k}. \end{aligned}$$

The Jacobi-Trudi type formula implies that the Yablonski-Vorob’ev polynomials are nothing but the specialization of the Schur functions [12]. Then, what does the Hankel determinant formula mean? In order to answer this question, a generating function for  $a_n$  is constructed in [6]:

**Theorem 1.1** ([6]). — Let  $\theta(x, t)$  be an entire function of two variables defined by

$$(3) \quad \theta(x, t) = \exp(2t^3/3) \operatorname{Ai}(t^2 - x),$$

where  $\operatorname{Ai}(z)$  is the Airy function. Then there exists an asymptotic expansion

$$(4) \quad \frac{\partial}{\partial t} \log \theta(x, t) \sim \sum_{n=0}^{\infty} a_n(x) (-2t)^{-n},$$

as  $t \rightarrow \infty$  in any proper subsector of the sector  $|\arg t| < \pi/2$ .

This result is quite suggestive, because it shows that the Airy functions enter twice in the theory of classical solutions of the  $P_{II}$ :

(1) in the formula [3]

$$u = \frac{d}{dx} \log \operatorname{Ai}(2^{1/3}x), \quad \alpha = 0.$$

the one parameter family of classical solutions of  $P_{II}$  for integer values of  $\alpha$  is expressed by Airy functions,

(2) in formulae (3), (4) the Airy functions generate the entries of determinant formula for the rational solutions.

In this paper we clarify the nature of this phenomenon. First, we reformulate the Hankel determinant formula for generic, namely non-classical, solutions of  $P_{II}$  already found in [10, 11]. We next construct generating functions for the entries of our Hankel determinant formula. We then show that the generating functions are related to the asymptotic solution at infinity of the isomonodromic problem introduced by Jimbo and Miwa [9]. More explicitly, the generating functions we construct are represented formally by series in powers of a variable  $t$  that does not appear in the second Painlevé equation. We show that they satisfy two Riccati equations, one in the  $x$  variable of  $P_{II}$ , the other in the auxiliary variable  $t$ . These Riccati equations simultaneously linearise to the two linear systems whose compatibility is given by  $P_{II}$ . This is the first time in the literature, to our knowledge, that the construction of the isomonodromic deformation problem has been carried out by starting directly from the Painlevé equation of interest.

This result explains the appearance of the Airy functions in Theorem 1.1. In fact, for rational solutions of  $P_{II}$ , the asymptotic solution at infinity of the isomonodromic problem is indeed constructed in terms of Airy functions [7, 8, 15].

We expect that the generic solutions of the so-called Painlevé II hierarchy [1, 2, 4] should be expressed by some Hankel determinant formula. Of course the generating functions for the entries of Hankel determinant should be related to the asymptotic solution at infinity of the isomonodromic problem for the Painlevé II hierarchy. We also expect that the similar phenomena can be seen for other Painlevé equations. We shall discuss these generalizations in future publications.

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## 2. Hankel Determinant Formula and Isomonodromic Problem

**2.1. Hankel Determinant Formula.** — We first prepare the Hankel determinant formula for generic solutions for  $P_{II}$  (1). To show the parameter dependence explicitly, we denote equation (1) as  $P_{II}[\alpha]$ . The formula is based on the fact that the  $\tau$  functions for  $P_{II}$  satisfy the Toda equation,

$$(5) \quad \sigma_n''\sigma_n - (\sigma_n')^2 = \sigma_{n+1}\sigma_{n-1}, \quad n \in \mathbb{Z}. \quad ' = d/dx.$$

Putting  $\tau_n = \sigma_n/\sigma_0$  so that the  $\tau$  function is normalized as  $\tau_0 = 1$ , equation (5) is rewritten as

$$(6) \quad \tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \varphi\psi\tau_n^2, \quad \tau_{-1} = \psi, \quad \tau_0 = 1, \quad \tau_1 = \varphi, \quad n \in \mathbb{Z}.$$

Then it is known that  $\tau_n$  can be written in terms of Hankel determinant as follows [11]:

**Proposition 2.1.** — Let  $\{a_n\}_{n \in \mathbb{Z}}$ ,  $\{b_n\}_{n \in \mathbb{Z}}$  be the sequences defined recursively as

$$(7) \quad a_n = a'_{n-1} + \psi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} a_i a_j, \quad b_n = b'_{n-1} + \varphi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} b_i b_j, \quad a_0 = \varphi, \quad b_0 = \psi.$$

For any  $N \in \mathbb{Z}$ , we define Hankel determinant  $\tau_N$  by

$$(8) \quad \tau_N = \begin{cases} \det(a_{i+j-2})_{i,j \leq N} & N > 0, \\ 1, & N = 0, \\ \det(b_{i+j-2})_{i,j \leq |N|} & N < 0. \end{cases}$$

Then  $\tau_N$  satisfies equation (6).

Since the above formula involves two arbitrary functions  $\varphi$  and  $\psi$ , it can be regarded as the determinant formula for general solution of the Toda equation. Imposing appropriate conditions on  $\varphi$  and  $\psi$ , we obtain determinant formula for the solutions of  $P_{II}$ :

**Proposition 2.2.** — Let  $\psi$  and  $\varphi$  be functions in  $x$  satisfying

$$(9) \quad \frac{\psi''}{\psi} = \frac{\varphi''}{\varphi} = -2\psi\varphi + 2x,$$

$$(10) \quad \varphi'\psi - \varphi\psi' = 2\alpha.$$

Then we have the following:

- (1)  $u_0 = (\log \varphi)'$  satisfies  $P_{II}[\alpha]$ .
- (2)  $u_{-1} = -(\log \psi)'$  satisfies  $P_{II}[\alpha - 1]$ .
- (3)  $u_N = \left(\log \frac{\tau_{N+1}}{\tau_N}\right)'$ , where  $\tau_N$  is defined by equation (8), satisfies  $P_{II}[\alpha + N]$ .

*Proof.* — (i) and (ii) can be directly checked by using the relations (9) and (10). Then (iii) is the reformulation of Theorem 4.2 in [10]. □

**2.2. Riccati Equations for Generating Functions.** — Consider the generating functions for the entries as the following formal series

$$(11) \quad F_\infty(x, t) = \sum_{n=0}^\infty a_n(x) t^{-n}, \quad G_\infty(x, t) = \sum_{n=0}^\infty b_n(x) t^{-n}.$$

It follows from the recursion relations (7) that the generating functions formally satisfy the Riccati equations. In fact, multiplying the recursion relations (7) by  $t^{-n}$  and take the summation from  $n = 0$  to  $\infty$ , we have:

**Proposition 2.3.** — The generating functions  $F_\infty(x, t)$  and  $G_\infty(x, t)$  formally satisfy the Riccati equations

$$(12) \quad t \frac{\partial F}{\partial x} = -\psi F^2 + t^2 F - t^2 \varphi,$$

$$(13) \quad t \frac{\partial G}{\partial x} = -\varphi G^2 + t^2 G - t^2 \psi,$$

respectively.

Since  $F_\infty$  and  $G_\infty$  are defined as the formal power series around  $t \sim \infty$ , it is convenient to derive the differential equations with respect to  $t$ . In order to do this, the following auxiliary recursion relations are useful.

**Lemma 2.4.** — *Under the condition (9) and (10),  $a_n$  and  $b_n$  ( $n \geq 0$ ) satisfy the following recursion relations,*

$$(14) \quad (\psi a_{n+2} - \psi' a_{n+1})' = 2(n+1)\psi a_n,$$

$$(15) \quad (\varphi b_{n+2} - \varphi' b_{n+1})' = 2(n+1)\varphi b_n,$$

respectively.

We omit the details of the proof of Lemma 2.4, because it is proved by straight but tedious induction. Multiplying equations (14) and (15) by  $t^{-n}$  and taking summation over  $n = 0$  to  $\infty$ , we have the following differential equations for  $F_\infty$  and  $G_\infty$ :

**Lemma 2.5.** — *The generating functions  $F_\infty$  and  $G_\infty$  formally satisfy the following differential equations,*

$$(16) \quad 2\psi t \frac{\partial F}{\partial t} = t(\psi' - t\psi) \frac{\partial F}{\partial x} + (\psi''t - \psi't^2 + 2\psi)F + t^2(\psi\varphi' + \psi'\varphi),$$

$$(17) \quad 2\varphi t \frac{\partial G}{\partial t} = t(\varphi' - t\varphi) \frac{\partial G}{\partial x} + (\varphi''t - \varphi't^2 + 2\varphi)G + t^2(\psi\varphi' + \psi'\varphi),$$

respectively.

Eliminating  $x$ -derivatives from equations (12), (16), and equations (13), (17), respectively, we obtain the Riccati equations with respect to  $t$ :

**Proposition 2.6.** — *The generating functions  $F_\infty$  and  $G_\infty$  formally satisfy the following Riccati equations,*

$$(18) \quad 2t \frac{\partial F}{\partial t} = -(\psi' - t\psi)F^2 + \left(\frac{\psi''}{\psi}t + 2 - t^3\right)F + t^2(\varphi' + t\varphi),$$

$$(19) \quad 2t \frac{\partial G}{\partial t} = -(\varphi' - t\varphi)G^2 + \left(\frac{\varphi''}{\varphi}t + 2 - t^3\right)G + t^2(\psi' + t\psi),$$

respectively.

**2.3. Isomonodromic Problem.** — The Riccati equations for  $F_\infty$  equations (12) and (18) are linearized by standard technique, which yield isomonodromic problem for  $P_{II}$ . It is easy to derive the following theorem from the Proposition 2.3 and 2.6:

**Theorem 2.7**

(1) *It is possible to introduce the functions  $Y_1(x, t)$ ,  $Y_2(x, t)$  consistently as*

$$(20) \quad F_\infty(x, t) = \frac{t}{\psi} \left( \frac{1}{Y_1} \frac{\partial Y_1}{\partial x} + \frac{t}{2} \right) = \frac{2t}{\psi' - t\psi} \left[ \frac{1}{Y_1} \frac{\partial Y_1}{\partial t} + \frac{1}{4} \left( \frac{\psi''}{\psi} - t^2 \right) \right],$$

$$(21) \quad Y_2 = \frac{1}{\psi} \left( \frac{\partial Y_1}{\partial x} + \frac{tY_1}{2} \right).$$

Then  $Y_1$  and  $Y_2$  satisfy the following linear system for  $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ :

$$(22) \quad \frac{\partial}{\partial t} Y = AY, \quad A = \begin{pmatrix} \frac{t^2}{4} - \frac{z}{2} - \frac{x}{2} & -\frac{\psi}{2}(t + u_{-1}) \\ \frac{1}{\psi} \left\{ (u_{-1} - t) \frac{z}{2} + \alpha \right\} & -\frac{t^2}{4} + \frac{z}{2} + \frac{x}{2} \end{pmatrix},$$

$$(23) \quad \frac{\partial}{\partial x} Y = BY, \quad B = \begin{pmatrix} -\frac{t}{2} \psi \\ z & t \\ \frac{z}{\psi} & \frac{t}{2} \end{pmatrix},$$

where  $z = -\psi\varphi$ .

(2) Similarly, it is possible to introduce the functions  $Z_1(x, t)$ ,  $Z_2(x, t)$  consistently as

$$(24) \quad G_\infty(x, t) = \frac{t}{\varphi} \left( \frac{1}{Z_1} \frac{\partial Z_1}{\partial x} + \frac{t}{2} \right) = \frac{2t}{\varphi' - t\varphi} \left[ \frac{1}{Z_1} \frac{\partial Z_1}{\partial t} + \frac{1}{4} \left( \frac{\varphi''}{\varphi} - t^2 \right) \right],$$

$$(25) \quad Z_2 = \frac{1}{\varphi} \left( \frac{\partial Y_1}{\partial x} + \frac{tY_1}{2} \right).$$

Then  $Z_1$  and  $Z_2$  satisfy the following linear system for  $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ :

$$(26) \quad \frac{\partial}{\partial t} Z = CZ, \quad C = \begin{pmatrix} \frac{t^2}{4} - \frac{z}{2} - \frac{x}{2} & -\frac{\varphi}{2}(t - u_0) \\ -\frac{1}{\varphi} \left\{ (u_0 + t) \frac{z}{2} + \alpha \right\} & -\frac{t^2}{4} + \frac{z}{2} + \frac{x}{2} \end{pmatrix},$$

$$(27) \quad \frac{\partial}{\partial x} Z = DY, \quad D = \begin{pmatrix} -\frac{t}{2} \varphi \\ z & t \\ \frac{z}{\varphi} & \frac{t}{2} \end{pmatrix}.$$

**Remark 2.8.** — The linear systems (22), (23) and (26), (27) are the isomonodromic problems for  $P_{\Pi}[\alpha - 1]$  and  $P_{\Pi}[\alpha]$ , respectively [9]. For example, compatibility condition for equations (22) and (23), namely,

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial t} + [A, B] = 0,$$

gives

$$(28) \quad \begin{cases} \frac{dz}{dx} = -2u_{-1}z - 2\alpha, \\ \frac{du_{-1}}{dx} = u_{-1}^2 - 2z - 2x, \\ u_{-1} = -\frac{1}{\psi} \frac{d\psi}{dx}, \end{cases}$$

which yields  $P_{\Pi}[\alpha - 1]$  for  $u_{-1}$ . This fact also guarantees the consistency of two expressions for  $F_\infty$  in terms of  $Y_1$  in equation (20). Similar remark holds true for  $G_\infty$  and  $Z_1$ .

**Remark 2.9.** —  $F_\infty$  and  $G_\infty$  are also expressed as,

$$(29) \quad F_\infty = t \frac{Y_2}{Y_1}, \quad G_\infty = t \frac{Z_2}{Z_1},$$

respectively. Conversely, it is obvious that for any solution  $Y_1$  and  $Y_2$  for the linear system (22) and (23),  $F = tY_2/Y_1$  satisfies the Riccati equations (12) and (18) (Similar for  $G$ ).

**Remark 2.10.** — Theorem 1.1 is recovered by putting  $\psi = 1, \varphi = x$ .

**Remark 2.11.** —  $Y_1$  can be formally expressed in terms of  $a_n$  by using equation (20) as

$$(30) \quad Y_1 = \text{const.} \times \exp\left(\frac{1}{12}t^3 - \frac{x}{2}t\right) t^{-\alpha} \exp\left[\frac{1}{2} \sum_{n=1}^{\infty} \frac{\psi a_{n+1} - \psi' a_n}{n} t^{-n}\right].$$

which coincides with known asymptotic behavior of  $Y_1$  around  $t \sim \infty$  [9].

### 3. Solutions of Isomonodromic Problems and Determinant Formula

In the previous section we have shown that the generating functions  $F_\infty$  and  $G_\infty$  formally satisfy the Riccati equations (12,18) and (13,19), and that their linearization yield isomonodromic problems (22, 23) and (26,27) for  $P_{II}$ . Now let us consider the converse. We start from the linear system (22) and (23). We have two linearly independent solutions around  $t \sim \infty$ , one of which is related with  $F_\infty(x, t)$ . Then, what is another solution? In fact, it is well-known that linear system (22) and (23) admit the formal solutions around  $t \sim \infty$  of the form [9],

$$(31) \quad Y^{(1)} = \begin{pmatrix} Y_1^{(1)} \\ Y_2^{(1)} \end{pmatrix} = \exp\left(\frac{t^3}{12} - \frac{xt}{2}\right) t^{-\alpha} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} y_{11}^{(1)} \\ y_{21}^{(1)} \end{pmatrix} t^{-1} + \begin{pmatrix} y_{12}^{(1)} \\ y_{22}^{(1)} \end{pmatrix} t^{-2} + \dots \right].$$

$$(32) \quad Y^{(2)} = \begin{pmatrix} Y_1^{(2)} \\ Y_2^{(2)} \end{pmatrix} = \exp\left(-\frac{t^3}{12} + \frac{xt}{2}\right) t^\alpha \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} y_{11}^{(2)} \\ y_{21}^{(2)} \end{pmatrix} t^{-1} + \begin{pmatrix} y_{12}^{(2)} \\ y_{22}^{(2)} \end{pmatrix} t^{-2} + \dots \right].$$

From Remark 2.9 we see that there are two possible power-series solutions for the Riccati equation (18) of the form,

$$(33) \quad Y^{(1)} \rightarrow F^{(1)} = t \frac{Y_2^{(1)}}{Y_1^{(1)}} = t \frac{y_{21}^{(1)} t^{-1} + \dots}{1 + y_{11}^{(1)} t^{-1} + \dots} = c_0 + c_1 t^{-1} + \dots,$$

$$(34) \quad Y^{(2)} \rightarrow F^{(2)} = t \frac{Y_2^{(2)}}{Y_1^{(2)}} = t \frac{1 + y_{21}^{(2)} t^{-1} + \dots}{y_{11}^{(2)} t^{-1} + \dots} = t^2(d_0 + d_1 t^{-1} + \dots).$$

The above two possibilities of power-series solutions for the Riccati equations are verified directly as follows:



**Proposition 3.1.** — *The Riccati equation (18) admits only the following two kinds of power-series solutions around  $t \sim \infty$ :*

$$(35) \quad F^{(1)} = \sum_{n=0}^{\infty} c_n t^{-n}, \quad F^{(2)} = t^2 \sum_{n=0}^{\infty} d_n t^{-n}.$$

*Proof.* — We substitute the expression,

$$(36) \quad F = t^\rho \sum_{n=0}^{\infty} c_n t^{-n}.$$

for some integer  $\rho$  to be determined, into the Riccati equation (18). We have

$$\begin{aligned} \sum_{n=0}^{\infty} 2(\rho - n)c_n t^{\rho+1-n} &= \sum_{n=0}^{\infty} \psi' \left( \sum_{k=0}^n c_k c_{n-k} \right) t^{2\rho-2n} - \sum_{n=0}^{\infty} \psi' \left( \sum_{k=0}^n c_k c_{n-k} \right) t^{2\rho+1-2n} \\ &+ \sum_{n=0}^{\infty} \left( \frac{\psi''}{\psi} + 2 \right) c_n t^{\rho-n} - \sum_{n=0}^{\infty} c_n t^{\rho+3-n} + t^2(\varphi' + t\varphi) \end{aligned}$$

The leading order should be one of  $t^{2\rho+1}$ ,  $t^{\rho+3}$  and  $t^3$ . Investigating the balance of these terms, we have  $\rho = 0$  or  $\rho = 2$ .  $\square$

We also have the similar result for the solution of the Riccati equation (19):

**Proposition 3.2.** — *The Riccati equation (19) admits only the following two kinds of power-series solutions around  $t \sim \infty$ :*

$$(37) \quad G^{(1)} = \sum_{n=0}^{\infty} e_n t^{-n}, \quad G^{(2)} = t^2 \sum_{n=0}^{\infty} f_n t^{-n}.$$

It is obvious that  $F^{(1)}$  and  $G^{(1)}$  are nothing but our generating functions  $F_\infty$  and  $G_\infty$ , respectively. Then, what are  $F^{(2)}$  and  $G^{(2)}$ ? In the following, we present two observations regarding this point. First, there are unexpectedly simple relations among those functions:

**Proposition 3.3.** — *The following relations holds.*

$$(38) \quad F^{(2)}(x, t) = \frac{t^2}{G^{(1)}(x, -t)}, \quad G^{(2)}(x, t) = \frac{t^2}{F^{(1)}(x, -t)}.$$

*Proof.* — Substitute  $F(x, t) = t^2/g(x, t)$  into equation (18). This gives equation (19) for  $G(x, t) = g(x, -t)$  by using the relation (9). Choosing  $g(x, t) = G^{(1)}(x, t)$ ,  $F(x, t)$  must be  $F^{(2)}(x, t)$ , since its leading order is  $t^2$ . We obtain the second equation by the similar argument.  $\square$

Second,  $F^{(2)}(x, t)$  and  $G^{(2)}(x, t)$  are also interpreted as generating functions of entries of Hankel determinant formula for  $P_{\text{II}}$ . Recall that the determinant formula in Proposition 2.1 is for the  $\tau$  sequence  $\tau_n = \sigma_n/\sigma_0$  so that it is normalized as  $\tau_0 = 1$ . We show that  $F^{(2)}(x, t)$  and  $G^{(2)}(x, t)$  correspond to different normalizations of  $\tau$  sequence:

**Proposition 3.4.** — *Let*

$$(39) \quad F^{(2)}(x, t) = -\frac{t^2}{\psi^2} \sum_{n=0}^{\infty} d_n (-t)^{-n},$$

$$(40) \quad G^{(2)}(x, t) = -\frac{t^2}{\varphi^2} \sum_{n=0}^{\infty} f_n (-t)^{-n}.$$

be formal solutions of the Riccati equations (12),(18) and (13), (19), respectively. We put

$$(41) \quad \kappa_{-n-1} = \det(d_{i+j})_{i,j=1,\dots,n} \quad (n > 0), \quad \kappa_{-1} = 1.$$

$$(42) \quad \theta_{n+1} = \det(f_{i+j})_{i,j=1,\dots,n} \quad (n > 0), \quad \theta_1 = 1.$$

Then  $\kappa_n$  and  $\theta_n$  are related to  $\tau_n$  as

$$(43) \quad \kappa_n = \frac{\tau_n}{\psi} = \frac{\tau_n}{\tau_{-1}} \quad (n < 0),$$

$$(44) \quad \theta_n = \frac{\tau_n}{\varphi} = \frac{\tau_n}{\tau_1} \quad (n > 0).$$

To prove Proposition 3.4, we first derive recurrence relations that characterize  $d_n$  and  $f_n$ . By substituting equations (39) and (40) into the Riccati equations (12) and (13), respectively, we easily obtain the following lemma:

**Lemma 3.5**

(1)  $d_0$  and  $d_1$  are given by  $d_0 = -\psi$  and  $d_1 = \psi'$ , respectively. For  $n \geq 2$ ,  $d_n$  are characterized by the recursion relation,

$$(45) \quad d_n = d'_{n-1} + \frac{1}{\psi} \sum_{k=2}^{n-2} d_k d_{n-k}, \quad d_2 = \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi}.$$

(2)  $f_0$  and  $f_1$  are given by  $d_0 = -\varphi$  and  $d_1 = \varphi'$ , respectively. For  $n \geq 2$ ,  $f_n$  are characterized by the recursion relation,

$$(46) \quad f_n = f'_{n-1} + \frac{1}{\varphi} \sum_{k=2}^{n-2} f_k f_{n-k}, \quad f_2 = \frac{\varphi''\varphi - (\varphi')^2 + \varphi^3\psi}{\varphi}.$$

*Proof of Proposition 3.4.* — Consider the Toda equations (5) and (6). Let us put

$$(47) \quad \tilde{\tau}_n = \frac{\sigma_n}{\sigma_{-1}} = \frac{\tau_n}{\tau_{-1}}$$

so that  $\tilde{\tau}_{-1} = 1$ . Then it is easy to derive the Toda equation for  $\tilde{\tau}_n$ :

$$(48) \quad \tilde{\tau}_n''\tilde{\tau}_n - (\tilde{\tau}_n')^2 = \tilde{\tau}_{n+1}\tilde{\tau}_{n-1} - \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi^2} \tilde{\tau}_n^2,$$

$$(49) \quad \tilde{\tau}_{-2} = \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi}, \quad \tilde{\tau}_{-1} = 1, \quad \tilde{\tau}_0 = \frac{1}{\psi}.$$

We have the determinant formula for  $\tilde{\tau}_n$  as.

$$(50) \quad \tilde{\tau}_n = \begin{cases} \det(\tilde{a}_{i+j-2})_{i,j \leq n+1} & n > 0, \\ 1, & n = 0, \\ \det(\tilde{b}_{i+j-2})_{i,j \leq |n|-1} & n < 0, \end{cases}$$

$$(51) \quad \tilde{a}_n = \tilde{a}'_{n-1} + \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi} \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} \tilde{a}_i \tilde{a}_j, \quad \tilde{a}_0 = \frac{1}{\psi},$$

$$(52) \quad \tilde{b}_n = \tilde{b}'_{n-1} + \frac{1}{\psi} \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} \tilde{b}_i \tilde{b}_j, \quad \tilde{b}_0 = \frac{\psi''\psi - (\psi')^2 + \varphi\psi^3}{\psi}.$$

Now it is obvious from Lemma 3.5 that

$$(53) \quad d_j = \tilde{b}_{j-2} \quad (j \geq 2), \quad \kappa_n = \tilde{\tau}_n \quad (n < 0),$$

which proves equation (41). Equation (42) can be proved in similar manner. □

We remark that the mysterious relations among the  $\tau$  functions and the solutions of isomonodromic problem in Proposition 3.3 and 3.4 should eventually originate from the symmetry of  $P_{II}$ , but their meaning is not sufficiently understood yet.

### 4. Summability of the generating function

To study the growth as  $n \rightarrow \infty$  of the coefficients  $a_n(x)$  (or  $b_n(x)$ ) in (11) we use a theorem proved in [5], based on a result by Ramis [16].

**Theorem 4.1.** — *Consider the following nonlinear differential equation in the variable  $s$*

$$(54) \quad s^{k+1} \frac{dH}{ds} = c(s)H + s b(s, H),$$

where  $k$  is a positive integer,  $c(s)$  is holomorphic in the neighbourhood of  $s = 0$  and  $c(0) \neq 0$ , and  $b(s, H)$  is holomorphic in the neighbourhood of  $(s, H) = (0, 0)$ . Then equation (54) admits one and only one formal solution  $H_f(s)$  of the form  $H_f(s) = \sum_{n=1}^{\infty} a_n s^n$ . Moreover  $H_f$  is  $k$ -summable in any direction  $\arg(s) = \vartheta$  except a finite number of values  $\vartheta$ . Furthermore the sum of  $H_f(s)$  in the direction  $\arg(s) = \vartheta$  is a solution of equation (54).

Equation (18) can be put into the form (54) by changing the variable  $t = 1/s$  and taking  $H = F - a_0$ . We obtain equation (54) with  $k = 3$  and

$$c(s) = \frac{1}{2} \left( 1 - \frac{\psi''}{\psi} s^2 - 2s^3 \right),$$

$$b(s, H) = -\frac{1}{2} \left( \varphi(\psi''/\psi s + 2s^2) \right. \\ \left. + \varphi' + s(\psi - \psi' s)\varphi^2 + 2s\varphi(\psi - \psi' s)H + s(\psi - \psi' s)H^2. \right)$$

Applying theorem 4.1, we obtain that equation (18) admits one and only one formal solution  $F_\infty(t)$  of the form  $F_\infty(t) = \sum_{n=0}^{\infty} a_n t^{-n}$ . This formal solution is 3-summable in any direction  $\arg(t) = \vartheta$  except a finite number of values  $\vartheta$  and its sum in the direction  $\arg(s) = \vartheta$  is a solution of equation (18). The definition of  $k$ -summability implies that  $F_\infty(t)$  is Gevrey of order 3, namely, for each  $x$  there exist positive numbers  $C(x)$  and  $K(x)$  such that

$$|a_n(x)| < C(x)(n!)^{1/3} K(x)^n, \quad \text{for all } n \geq 1.$$

Clearly, one can prove a similar result for the coefficients  $d_n$  of the second formal solution  $F^{(2)}$  of equation (18). One has to apply theorem 4.1 to a new series  $H$  defined as  $H(s) = s^2 F^{(2)} - d_0$ .

### References

- [1] M.J. ABLOWITZ & H. SEGUR – “Exact linearization of a Painlevé transcendent”, *Phys. Rev. Lett.* **38** (1977), p. 1103–1106.
- [2] H. AIRAULT – “Rational solutions of Painlevé equations”, *Stud. Appl. Math.* **61** (1979), p. 31–53.
- [3] N.P. ERUGIN – “On the second transcendent of Painlevé”, *Dokl. Akad. Nauk BSSR* **2** (1958), p. 139–142.
- [4] H. FLASCHKA & A.C. NEWELL – “Monodromy and Spectrum Preserving Deformations I”, *Comm. Math. Phys.* **76** (1980), p. 65–116.
- [5] P.F. HSIEH & Y. SIBUYA – *Basic theory of ordinary differential equations*, Universitext, Springer, New York, 1999.
- [6] K. IWASAKI, K. KAJIWARA & T. NAKAMURA – “Generating function associated with the rational solutions of the Painlevé II equation”, *J. Phys. A: Math. Gen.* **35** (2002), p. L207–L211.
- [7] M. JIMBO – “Monodromy problem and the boundary condition for some Painlevé equations”, *Publ. RIMS, Kyoto Univ.* **18** (1982), p. 1137–1161.
- [8] ———, Unpublished work.
- [9] M. JIMBO & T. MIWA – “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II”, *Physica: 2D* (1981), p. 407–448.
- [10] K. KAJIWARA & T. MASUDA – “A generalization of the determinant formulae for the solutions of the Painlevé II equation”, *J. Phys. A: Math. Gen.* **32** (1999), p. 3763–3778.
- [11] K. KAJIWARA, T. MASUDA, M. NOUMI, Y. OHTA & Y. YAMADA – “Determinant formulas for the Toda and discrete Toda equations”, *Funkcial. Ekvac.* **44** (2001), p. 291–307.
- [12] K. KAJIWARA & Y. OHTA – “Determinant structure of the rational solutions for the Painlevé II equation”, *J. Math. Phys.* **37** (1996), p. 4693–4704.
- [13] Y. MURATA – “Rational solutions of the second and the fourth Painlevé equations”, *Funkcial. Ekvac.* **28** (1985), p. 1–32.
- [14] M. NOUMI & K. OKAMOTO – “Irreducibility of the second and the fourth Painlevé equations”, *Funkcial. Ekvac.* **40** (1997), p. 139–163.
- [15] K. OKAMOTO – Private communication.
- [16] J.-P. RAMIS – *Séries divergentes et théories asymptotiques*, Panoramas & Synthèses, Société Mathématique de France, 1993.

- [17] H. UMEMURA & H. WATANABE – “Solutions of the second and fourth Painlevé equations I”, *Nagoya Math. J.* **148** (1997), p. 151–198.
- [18] A.P. VOROB'EV – “On the rational solutions of the second Painlevé equation”, *Differencial'nye Uravnenija* **1** (1965), p. 79–81.
- [19] W. WASOW – *Asymptotic expansions for ordinary differential equations*. Reprint of the 1965 edition, Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.
- [20] A.I. YABLONSKII – *Vesti Akad. Navuk. BSSR Ser. Fiz. Tkh. Nauk.* **3** (1959), p. 30–35.

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