Astérisque

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Appendice A, Facsimilé : Orsay - Transparents de l'exposé d'Ofer Gabber, fait le 27 juin 2005 à la conférence en l'honneur de Luc Illusie ([Gabber, 2005a])

APPENDICE A

FACSIMILÉ : ORSAY

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A finiteness theorem for non abelian H^1 of excellent schemes

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1. Statements

Theorem 1.0 Let $f: X \to Y$ be a morphism of finite type of noetherian schemes and F a constructible sheaf of sets on X_{et}.

Then $f_* F$ is constructible.

The proof is by reducing to the case f proper (SGA 4) and f an open immersion (easier when normalizations are finite)

Theorem 1.1 Let $\mathbb{L}$ be a finite set of primes, $f: X \to Y$ a morphism of finite type between quasi-excellent schemes on which $\mathbb{L}$ _{every $p \in$} is invertible. Let F be a constructible $\mathbb{L}$-torsion sheaf of groups on X.

Then $R^1 f_* F$ is constructible.

Also stack version.

Proved here using ultraproducts.

Theorem 1.2. $X, Y, \mathbb{L}, f$ as in 1.1.
Y qc.   If F is a constructible
$\mathbb{L}$ - torsion abelian sheaf on X , then
the sheaves $R^i f_x F$ are constructible
and 0 for $i \gg 0$.

(Planned for Deligne's conference,
  Using alterations)

2. Reductions.

Th.1.1 known for proper f, so enough to consider open immersions. Reduce to Y normal affine, $X = Y - Z$ (Z reduced) and F constant.

Enough to find $Y' \to Y$, normalization of Y in a finite extension of $R(Y)$ which kills $R^1 f_* F$.

This exists outside $\mathrm{Sing}(Y) \cup \mathrm{Sing}(Z)$ (Abh. Lemma). Reduce to

Th. 2.1. Let Y be a normal excellent scheme, Z a closed subscheme of $\mathrm{cod} \gtrsim 2$, $j: Y - Z \to Y$, G a finite group. Then $R^1 j_* G$ is constructible.

(The order of G not necessarily inv. on Y)

This will be reduced to the following case:

Th. 2.2. Let A be a strictly henselian excellent normal local ring of dimension 2. For every finite group G

$$H^1(\text{Spec}(A) - \{m\} , G) \quad \text{is finite.}$$

3. Lefschetz
Recall (SGA 2 XIII §2)

Th. 3.0. A cnlr $f \in m$, f nonzerodivisor, $X = \text{Spec}(A)$ $X' = X - \{m\}$, depth $O_{X',x} \geqslant 2$

for closed points of X'. Then
$$\Gamma(X', \theta) \overset{\sim}{\to} \Gamma(\hat{X}', \theta) \quad \text{so}$$
$$\pi_0(X' \cap V(f)) \overset{\sim}{\to} \pi_0(X').$$

Cor. 1. A excellent normal nlr of dim $\geqslant 3$ $f \in A$ nonzero nonunit, then the punctured spectrum of $\underset{Y}{\text{Spec}}(A/fA)$ is connected.

Y is connected in cod. 1, i.e. if $Z \subset Y$ is of cod ≥ 2 then $Y - Z$ is connected.

Cor.2. Let X be normal excellent scheme

$D \subset X$ effective Cartier divisor

$Z \subset D$ closed, $\text{cod}_X Z \geq 3$

$$
\begin{array}{ccc}
D - Z & \xrightarrow{\ j'\ } & D \\
\downarrow & & \downarrow \\
X - Z & \xrightarrow{\ i\ } & X
\end{array}
$$

If L is a locally ~~system~~ constant constr sheaf on $X - Z$

$$(j_* L)|_D \xrightarrow{\sim} j'_* (L|_{D-Z}).$$

$(R^1 j_* G)|_D$ injects into $R^1 j'_* G$

for every finite group G.

4. Use of desingularization of 2 dimensional schemes.

Lemma 4.1. In the situation of Th. 2.1 if Z_i is an irreducible component of Z of codim 2 in Y then there is an open dense $U_i \subset Z_i$ s.t. the sheaf of pointed sets $F = R^1 j_* G$ on U_i has specialization maps $F_s \longrightarrow F_t$ (for $t \longrightarrow s$ map of geom. pts of U_i) "topos pts" with trivial kernel.

Proof. W.M.A. $Z = Z_i$ and that there is $p: Y' \longrightarrow Y$ proper, birational, isomorphism outside Z, Y' regular, $p^{-1}(Z)_{red}$ is a normal crossings divisor with V irr components dominating Z and $p^{-1}(Z) \longrightarrow Z$ is (univ) locally 0-acyclic.
      (connected Milnor fibres)

5. Proof of Th. 2.1 assuming Th. 2.2.

WMA $Y = \text{Spec}(A)$, A excellent normal domain and that the result is known

for (~~proper subschemes of Y and for~~)

normalizations of $Y' \underset{\neq}{\subseteq} Y$; Y' irr, in

finite extensions of $R(Y')$.

Using section 4 there is

$\text{Spec}(A') \underset{\text{finite}}{\twoheadrightarrow} \text{Spec}(A)$ which kills

$R^i j_* G$ outside a cod $\gtrless 3$ locus,

so reduce to Z of cod $\gtrless 3$ and

use section 3.

6. Discriminants.

Lemma 6.1. Let $A = (A_{ij})$ be an $n \times n$ matrix of $m \times m$ matrices over a commutative ring k, with commuting A_{ij}. Let A be the resulting block $nm \times nm$ matrix. Then $\det A = \det \det A$.

[Hint: replace A_{11} by $A_{11} + t$, invertible over $k' \supset k[t]$.]

Lemma 6.2. Let $A \subset B \subset C$ be comm. rings with C a free B-module with basis f_j $(1 \le j \le n)$ and B a free A-module with basis e_i $(1 \le i \le m)$. (so $(e_i f_j)$ is an A basis of C). We use these bases to define discriminants, e.g.

$$\text{disc}_{B/A} = \det Tr_{B/A} (e_i e_j).$$

Then $\text{disc}_{C/A} = (\text{disc}_{B/A})^n \, \text{Norm}_{B/A} (\text{disc}_{C/B}).$

7. Rigidity.

Th. 7.1. Let (A, I) be a henselian pair with I finitely generated, $U \subset \operatorname{Spec}(A)$ a quasi-compact open $U \supset \operatorname{Spec}(A) - V(I)$, $\hat{A}$ the I-adic completion of A, $\hat{U} \subset \operatorname{Spec}(\hat{A})$ the inverse image of U. Let F be a sheaf of sets (resp. an ind-finite sheaf of groups) on U. Then $H^0(U, F) \xrightarrow{\sim} H^0(\hat{U}, F)$ (resp. $H^1(U, F) \xrightarrow{\sim} H^1(\hat{U}, F)$).

version for stacks.

Can reduce to case F constant, which holds by Elkik's approximation when A is noetherian and also when I is principal and $U = \operatorname{Spec}(A) - V(I)$. In general blow up a f.g. ideal defining the complement of U and use proper base change for stacks (Giraud + extension to non noeth. case) and affine base change to reduce to the principal ideal case.

Th. 7.2. (close to SGA4 XV)

$$p: X' \longrightarrow X$$

$$j'\uparrow \qquad \uparrow j$$

$$U' \qquad U$$

p smooth, X normal excellent, $U' = p^{-1}(U)$,
U open containing all points of cod ≤ 1.

Then $p^* R^1 j_* G \overset{\sim}{\longrightarrow} R^1 j'_* G$ (G finite).

Lemma 7.3. Let Z be a nowhere dense
closed subscheme of a noetherian scheme X.
Then the following conditions are equivalent

(1) Let $p: X' \longrightarrow X_{red}$ be the normalization.
Then $p^{-1}(Z)$ is of codim ≥ 2 in X'.

(2) for every $y \in Z$, the irreducible components
of $\operatorname{Spec}(O^h_{X,y})$ are of dim ≥ 2

(3) same for $\hat{O}_{X,y}$.

If this holds we say that Z is
$\subset 2$ in X.

Prop. 7.4. Let $A \longrightarrow B$ be a local homomorphism of nlr , formally smooth for m-adic topologies. Then There is a direct system of local essentially smooth A-algebras B_i with

$$\hat{B} \simeq (\varinjlim B_i)^{\wedge}$$

$B_i \longrightarrow B_j$ Aat $m_i B_j = m_j$.

<u>Th. 7.5.</u> Let Z be c2 in X

$p: X' \longrightarrow X$ a flat morphism of noetherian schemes, the fibres of p above points of Z are geometrically regular.

Conclusion as in 7.2.

8. Separable projection.

Lemma 8.1. Let R be a complete noetherian local domain. Then R has a regular subring R_0, R finite over R_0 with $\mathrm{Frac}(R)$ separable over $\mathrm{Frac}(R_0)$.

Only problem in equal characteristic $p > 0$. Let k be the residue field of R and $\{b_i\}_{i \in I}$ a p-basis of k. There is a bijection

$$\{\text{coefficient fields of } R\} \longleftrightarrow \{\text{liftings of } b_i\}.$$

Fix a coefficient field.

By the proof of Nagata's Jacobian criterion there is a finite subset $J \subset I$ s.t. if $k' = k^p(b_i, i \notin J)$ then

$$\Omega = \Omega^1_R \big/ \overline{k' R^p} \qquad \text{has generic rank}$$

$$\dim(R) + \mathrm{card}(J).$$

Then change the liftings of b_i $i \in J$ s.t. their differentials are linearly independent in Ω. With this coefficient field

one can take $k' = k$.

Let $t_1, \ldots, t_d$ be a system of parameters

for R and let $f_1, \ldots, f_d \in R$ be s.t.

df_i form a basis of $\Omega \underset{R}{\otimes} frac(R)$.

WMA $f_i \in m$.

Let $t_i' = t_i^p (1 + f_i)$.

Can take

$$R_0 = k [[t_1', \ldots, t_d']].$$

9. Ultraproducts

Let I be a set. There is a bijection

$$\{ultrafilters \ on \ I\} \longleftrightarrow Spec((\mathbb{Z}/2\mathbb{Z})^I)$$

$$F \longmapsto p = \{ \chi_A \mid A \notin F\}.$$

If R_i $(i \in I)$ are rings then

the ultraproduct $\prod_{/F} (R_i)$ is

$\prod_i (R_i) / \sim$ where

$$(r_i) \sim (s_i) \iff \{i \mid r_i = s_i\} \in F.$$

In the commutative case there is a
map of topological spaces

$$\hat{f} : \text{Spec} \left(\prod_i R_i \right) \longrightarrow \text{Spec} \left((\mathbb{Z}/2)^I \right)$$

defined by "$f^* \chi_A = \chi_A$".

The fibres of f (with the restriction
of the structure sheaf) are Spec of
the ultraproduct.

R_i all fields, domains, local rings,
$\Longrightarrow$ same for $\prod_{/F} (R_i) = R_\infty$

(✳) If R_i are local rings whose maximal
ideals are generated by n elements
then R_∞ has the same property, so
its completion is a̲ ̲c̲o̲m̲p̲l̲e̲t̲e̲ noetherian local ring.

• If furthermore F is not ω complete
 R_∞ maps onto $\hat{R}_\infty$

• If F is ω complete and R_i are noetherian
 (✳),

then R_∞ is noetherian.

Lemma. If $R_i \longrightarrow S_i$ $(i \in I)$ are finite maps
of noetherian local rings, $\exists n$ s.t. $\forall i$
the maximal ideal of R_i and S_i have
n generators as an R_i-module, F is
an ultrafilter on I, $R_\infty \longrightarrow S_\infty$ the
corresponding map on ultraproducts then
$R_\infty \longrightarrow S_\infty$ ~~transmitte~~ is finite and
$\ker(R_\infty \longrightarrow \hat{R}_\infty)$ generates $\ker(S_\infty \longrightarrow \hat{S}_\infty)$.

The proof uses that in all cases
$R_{\infty, sep} = \operatorname{Im}(R_\infty \longrightarrow \hat{R}_\infty)$ is noetherian.

We will use this only for complete
local rings, in which case it is easily
seen that R_∞ maps onto $\hat{R}_\infty$,

10. Proof of Th. 2.2.

We may assume A is complete.
View A as a finite generically étale
extension of a 2 dimensional regular complete
local ring R. A/R finite free rk m.

If the assertion is false there is $n > 0$
and connected pairwise non isomorphic finite
étale maps $E_i \longrightarrow \mathrm{Spec}(A) - \{m_A\}$ of
degree n. $(i \in N)$ Let B_i be the normalization
of A in E_i. B_i is complete normal local
ring and E_i is the punctured spectrum
of B_i. B_i is finite free rk nm as
an R-module.

$$\mathrm{disc}_R B_i = (\mathrm{disc}_R A)^n \text{ up to unit.}$$

Let F be an ultrafilter on N. Consider
the ultraproducts

$$R_\infty \longrightarrow A_\infty \longrightarrow B_\infty$$

$R_\infty \to A_\infty$ free rk m

$R_\infty \to B_\infty$ free rk nm

B_∞ a finitely presented A_∞-module

information on $\mathrm{disc}_R B$ retained

Same for $R_\infty^\wedge \longrightarrow A_\infty^\wedge \longrightarrow B_\infty^\wedge$.

Note $A \to A_\infty^\wedge$ is flat : mod m_A^ℓ this reduces to the fact that over a coherent ring infinite products of flat modules are flat. The residue field of $A_\infty^\wedge$ is a regular (in particular separable) extension of the residue field of A. By " localization de la lissité formelle" the fibre rings of $A \to A_\infty^\wedge$ are geometrically regular. Hence $A_\infty^\wedge$ is normal. $B_\infty^\wedge$ is torsion free over $R_\infty^\wedge$, hence over $A_\infty^\wedge$. Hence $B_\infty^\wedge / A_\infty^\wedge$ is finite flat over the punctured spectrum, necessarily of rank n. By the discriminant information we get that

$\hat{B}_\infty/\hat{A}_\infty$ is finite étale on the punctured spectrum. Hence $\forall \xi \in \mathrm{Spec}(\hat{A}_\infty) - V(m_A)$, the criterion of flatness by fibres gives that $(B_\infty)_\xi$ is flat over $(A_\infty)_\xi$, hence finite étale. But every point of $\mathrm{Spec}(A_\infty) - V(m_A)$ is a generization of a point of $\mathrm{Spec}(\hat{A}_\infty) - V(m_A)$.

[If $I \subset \mathrm{Rad}(A)$ is a f.g. ideal in a ring A and $\hat{A}$ the I-adic completion of A then all closed points of $\mathrm{Spec}(A) - V(I)$ are in the image of $\mathrm{Spec}(\hat{A}) \to \mathrm{Spec}(A)$.]

This proves

__Theorem.__ $\rho : \mathrm{Spec}(\prod B_i) \to \mathrm{Spec}(\prod_i A)$ is finite étale of deg n on $V(m_A)^c$.

Let F be a non principal ultrafilter on $\mathbb{N}$. By the rigidity facts The restriction of ρ to $\mathrm{spec}(A_\infty) - V(m_A)$ comes from a finite étale cover of $\mathrm{Spec}(A) - V(m_A)$. By passage to the limit $\exists T \in F$ s.t. same holds for $\rho|_{\mathrm{Spec}(\prod_{i \in T} A) - V(m_A)}$.

Contradiction.