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Fractional Sobolev inequalities : symmetrization, isoperimetry and interpolation

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To Iolanda, Quim and Vanda

FRACTIONAL SOBOLEV INEQUALITIES: SYMMETRIZATION, ISOPERIMETRY AND INTERPOLATION

Joaquim Martín, Mario Milman

Abstract. — We obtain new oscillation inequalities in metric spaces in terms of the Peetre $K-$functional and the isoperimetric profile. Applications provided include a detailed study of Fractional Sobolev inequalities and the Morrey-Sobolev embedding theorems in different contexts. In particular we include a detailed study of Gaussian measures as well as probability measures between Gaussian and exponential. We show a kind of reverse Pólya-Szegö principle that allows us to obtain continuity as a self improvement from boundedness, using symmetrization inequalities. Our methods also allow for precise estimates of growth envelopes of generalized Sobolev and Besov spaces on metric spaces. We also consider embeddings into BMO and their connection to Sobolev embeddings.

Résumé **(Inégalités de Sobolev fractionnaires : symétrisation, isopérimétrie et interpolation)**

Nous démontrons de nouvelles inégalités d'oscillations dans des espaces métriques qui s'expriment *via* la fonctionnelle K de Peetre et le profile isopérimétrique. Cela permet une étude détaillée des inégalités de Sobolev fractionnaires. Nous en déduisons aussi une démonstration du théorème de plongement de Morrey-Sobolev dans différentes situations. Nous donnons en particulier une étude détaillée de mesures gaussiennes ainsi que de mesures de probabilités entre gaussiennes et exponentielles. Nous démontrons un principe de Polya-Szego inverse qui donne un résultat de continuité à partir de certaines majorations. Nos méthodes permettent aussi d'estimer précisément la croissance d'espaces de Sobolev ou de Besov généralisés sur des espaces métriques. Enfin nous considérons des plongements dans des BMO et leur liens avec des plongements de Sobolev.

CONTENTS

PREFACE

This paper is devoted to the study of fractional Sobolev inequalities and Morrey-Sobolev type embedding theorems in metric spaces, using symmetrization. The connection with isoperimetry plays a crucial role. The aim was to provide a unified account and develop the theory in the general setting of metric measure spaces whose isoperimetric profiles satisfy suitable assumptions. In particular, the use of new pointwise inequalities for suitable defined moduli of continuity allow us to treat in a unified way Euclidean and Gaussian measures as well as a large class of different geometries. We also study the role of isoperimetry in the estimation of *BMO* oscillations. The connection with Interpolation/Approximation theory also plays a crucial role in our development and suggests further applications to optimization...

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CHAPTER 1

INTRODUCTION

In this paper we establish general versions of fractional Sobolev embeddings, including Morrey-Sobolev type embedding theorems, in the context of metric spaces, using symmetrization methods. The connection of the underlying inequalities with interpolation and isoperimetry plays a crucial role.

We shall consider connected, measure metric spaces (Ω, d, μ) equipped with a finite Borel measure μ. For measurable functions $u : \Omega \to \mathbb{R}$, the distribution function is defined by

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} \qquad (t \in \mathbb{R}).$$

The signed **decreasing rearrangement** of u, which we denote by u^*_μ, is the right-continuous non-increasing function from $[0, \mu(\Omega))$ into $\mathbb{R}$ that is equimeasurable with u; *i.e.*, u^*_μ satisfies

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} = m\left(\{s \in [0, \mu(\Omega)) : u^*_\mu(s) > t\}\right), \ \ t \in \mathbb{R}$$

(where m denotes the Lebesgue measure on $[0, \mu(\Omega))$. The maximal average of u^*_μ is defined by

$$u^{**}_\mu(t) = \frac{1}{t}\int_0^t u^*_\mu(s)\, ds, \ (t > 0).$$

For a Borel set $A \subset \Omega$, the **perimeter** or **Minkowski content** of A is defined by

$$P(A; \Omega) = \liminf_{h \to 0} \frac{\mu\left(\{x \in \Omega : d(x, A) < h\}\right) - \mu\left(A\right)}{h}.$$

The **isoperimetric profile** $I_\Omega(t), t \in (0, \mu(\Omega))$, is maximal with respect to the inequality

$$I_\Omega(\mu(A)) \leq P(A; \Omega). \tag{1.1.1}$$

From now on we only consider connected metric measure spaces whose isoperimetric profile I_Ω is zero at zero, continuous, concave and symmetric around $\mu(\Omega)/2$.

The starting point of the discussion are the rearrangement inequalities [(1)] of [**70**] and [**71**], where we showed that [(2)], under our current assumptions on the profile I_Ω, for all Lipschitz functions f on Ω (briefly $f \in Lip(\Omega)$),

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \leq \frac{t}{I(t)} |\nabla f|_\mu^{**}(t), \; 0 < t < \mu(\Omega), \tag{1.1.2}$$

where

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In fact, in [**70**] we showed that (1.1.2) is equivalent to (1.1.1).

Since the integrability properties do not change by rearrangements (*i.e.*, integrability properties are "rearrangement invariant"), rearrangement inequalities are particularly useful to prove embeddings of Sobolev spaces into rearrangement invariant spaces [(3)]. On the other hand, the use of rearrangement inequalities to study smoothness of functions is harder to implement. The main difficulty here is that while the classical Pólya-Szegö principle (cf. [**60**], [**15**]) roughly states that symmetrizations are smoothing, *i.e.*, they preserve the (up to first order) smoothness of Sobolev/Besov functions, the converse does not hold in general. In other words, it is not immediate how to deduce smoothness properties of f from inequalities on f_μ^*. From this point of view, one could describe some of the methods we develop in this paper as "suitable converses to the Pólya-Szegö principle".

As it turns out, related issues have been studied long ago, albeit in a less general context, by A. Garsia and his collaborators. The original impetus of Garsia's group was to study the path continuity of certain stochastic processes (cf. [**42**], [**41**]); a classical topic in Probability theory. This task led Garsia *et al.* to obtain rearrangement inequalities for general moduli of continuity, including L^p or even Orlicz moduli of continuity. Moreover, in [**40**], [**44**], and elsewhere (cf. [**43**]), these symmetrization inequalities were also applied to problems in Harmonic Analysis and, in particular, to study the absolute convergence of Fourier series. From our point of view, a remarkable aspect of the approach of Garsia *et al.* (cf. [**44**]) is precisely that the sought continuity can be recovered using rearrangement inequalities. In other words, one can reinterpret this part of the Garsia-Rodemich analysis as an approach to the Morrey-Sobolev embedding theorem using rearrangement inequalities.

It will be instructive to show how Garsia's analysis can be combined with (1.1.2). To fix ideas we consider the setting of Garsia-Rodemich: The metric measure space

1. For more detailed information we refer to Chapter 2 below.
2. See also the extensive list of references provided in [**70**].
3. Roughly speaking, a rearrangement invariant space is a Banach function space where the norm of a function depends only on the μ-measure of its level sets.

is $((0,1)^n, |\cdot|, dx)$ (that is $(0,1)^n$ provided with the Euclidean distance and Lebesgue measure). For functions $f \in Lip(0,1)^n$ the inequality (1.1.2) takes the following form [4]

$$|f|^{**}(t) - |f|^{*}(t) \leq c_n \frac{t}{\min(t, 1-t)^{1-1/n}} |\nabla f|^{**}(t), \quad 0 < t < 1.$$

In fact (cf. Chapter 6), the previous inequality remains true for all functions $f \in W^1_{L^p}(0,1)^n$ (where $1 \leq p < \infty$), and $W^1_{L^p}(0,1)^n$ is the Sobolev space of real-valued weakly differentiable functions on $(0,1)^n$ whose first-order derivatives belong to L^p). Moreover, as we shall see (cf. Chapter 4), the inequality also holds for (signed) rearrangements; *i.e.*, for all $f \in W^1_{L^p}(0,1)^n$, we have that,

$$f^{**}(t) - f^{*}(t) \leq c_n \frac{t}{\min(t, 1-t)^{1-1/n}} |\nabla f|^{**}(t), \quad 0 < t < 1. \tag{1.1.3}$$

Suppose that $p > n$. Integrating, and using the fundamental theorem of calculus [5], we get

$$\begin{aligned} f^{**}(0) - f^{**}(1) &= \int_0^1 (f^{**}(t) - f^{*}(t)) \frac{dt}{t} \\ &\leq c_n \int_0^1 |\nabla f|^{**}(t) \frac{dt}{\min(t, 1-t)^{1-1/n}} \\ &\leq c_{n,p} \left\| |\nabla f| \right\|_{L^p} \left\| \frac{1}{\min(t, 1-t)^{1-1/n}} \right\|_{L^{p'}} \quad \text{(by Hölder's inequality)} \\ &= C_{n.p} \left\| |\nabla f| \right\|_{L^p}, \end{aligned}$$

where the last inequality follows from the fact that for $p > n$, $\left\| \frac{1}{\min(t,1-t)^{1-1/n}} \right\|_{L^{p'}} < \infty$. Summarizing our findings, we have

$$ess \sup_{x \in (0,1)^n} f - \int_0^1 f = f^{**}(0) - f^{**}(1) \leq C_{n.p} \left\| |\nabla f| \right\|_{L^p}. \tag{1.1.4}$$

Applying (1.1.4) to $-f$ yields

$$\int_0^1 f - ess \inf_{x \in (0,1)^n} f \leq C_{n.p} \left\| |\nabla f| \right\|_{L^p}. \tag{1.1.5}$$

Therefore, adding (1.1.4) and (1.1.5) we obtain

$$Osc(f; (0,1)^n) := ess \sup_{x \in (0,1)^n} f - ess \inf_{x \in (0,1)^n} f \leq 2C_{n.p} \left\| |\nabla f| \right\|_{L^p}. \tag{1.1.6}$$

4. The rearrangement of f with respect to the Lebesgue measure is simply denoted by f^*.
5. Recall that $\frac{d}{dt}(f^{**}(t)) = -\frac{(f^{**}(t) - f^{*}(t))}{t}$.

We have shown that (1.1.2) gives us good control of the oscillation of the original function on the whole cube $(0,1)^n$. To control the oscillation on any cube $Q \subset (0,1)^n$, we use a modification of an argument that originates [6] in the work of Garsia *et al.* (cf. [**40**]). The idea is that if an inequality scales appropriately, one can re-scale. Namely, given two fixed points $x < y \in (0,1)$, one can apply the inequality at hand to the re-scaled function [7] $\tilde{f}(t) = f(x + t(y-x)), t \in [0,1]$. This type of "change of scale argument" can be extended to the cube $(0,1)^n$, but for general domains becomes unmanageable. Therefore, we needed to reformulate the idea somewhat differently. From our point of view the idea is that if we control $|\nabla f|$ on $(0,1)^n$ then we ought to be able to control its restrictions. The issue then becomes: How do our inequalities scale under restrictions? Again for $(0,1)^n$ all goes well. In fact, if $f \in W^1_{L^p}((0,1)^n)$ then, for any open cube $Q \subset (0,1)^n$, we have $f\chi_Q \in W^1_{L^p}(Q)$. Moreover, the fundamental inequality (1.1.3) has the following scaling

$$(f\chi_Q)^{**}(t) - (f\chi_Q)^{*}(t) \leq c_n \frac{t}{\min(t, |Q|-t)^{1-1/n}} \left|\nabla (f\chi_Q)\right|^{**}(t), \; 0 < t < |Q|.$$

Using the previous argument applied to $f\chi_Q$ we thus obtain

$$Osc(f;Q) \leq c_{n,p} \left\| \frac{t}{\min(t, |Q|-t)^{1-1/n}} \right\|_{L^{p'}(0,|Q|)} \left\|\left|\nabla f\right|\right\|_{L^p(Q)}.$$

By computation, and a classical argument, it is easy to see from here that (cf. Remark 6 in Chapter 5)

$$|f(y) - f(z)| \leq c_{n,p} |y-z|^{(1-\frac{n}{p})} \left\|\left|\nabla f\right|\right\|_p, \text{ a.e. } y, z.$$

When $p = n$ this argument fails but, nevertheless, by a simple modification, it yields a result due independently to Stein [**90**] and C. P. Calderón [**23**]: namely [8], if $\left\|\left|\nabla f\right|\right\|_{L^{n,1}} < \infty$, then f is essentially continuous (cf. Remark 6 in Chapter 5).

We will show that, with suitable technical adjustments, this method can be extended to the metric setting [9]. To understand the issues involved let us note that, since our inequalities are formulated in terms of isoperimetric profiles, to achieve

6. In the original one dimensional argument (cf. [**40**], [**44**]), one controls the oscillation of f in terms of an expression that involves the modulus of continuity, rather than the gradient.

7. For example, consider the case $n = 1$. Given $0 < x < y < 1$, the inequality (1.1.6) applied to $\tilde{f}$ yields

$$\begin{aligned} ess \sup_{[x,y]} f - ess \inf_{[x,y]} f &\leq c_p |y-x| \left(\int_0^1 |f'(x + t(y-x))|^p \, dt \right)^{1/p} \\ &= c_p |y-x|^{1-1/p} \left(\int_0^1 |f'|^p \right)^{1/p}. \end{aligned}$$

8. We refer to (7.1.3), (7.1.4) for the definition of Lorentz spaces.

9. For a different approach to the Morrey-Sobolev theorem on metric spaces we refer to the work of Coulhon [**28**], [**27**].

the local control or "the change of scales (in our situation through the restrictions)" we need suitable control of the (relative) isoperimetric profiles on the (new) metric spaces obtained by restriction. More precisely, if $Q \subset \Omega$ is an open set, we shall consider the metric measure space $\left(Q, d_{|Q}, \mu_{|Q}\right)$. Then the problem we face is that, in general, the isoperimetric profile I_Q of $\left(Q, d_{|Q}, \mu_{|Q}\right)$ is different from the isoperimetric profile I_Ω of (Ω, d, μ). What we needed is to control the "relative isoperimetric inequality"[10], and make sure the corresponding inequalities scale appropriately. We say that an **isoperimetric inequality relative to** G holds, if there exists a positive constant C_G such that

$$I_G(s) \geq C_G \min(I_\Omega(s), I_\Omega(\mu(G) - s)).$$

We will say that a metric measure space (Ω, d, μ) has the **relative uniform isoperimetric property** if there is a constant C such that for any ball B in Ω, its **relative isoperimetric profile** I_B satisfies

$$I_B(s) \geq C \min(I_\Omega(s), I_\Omega(\mu(B) - s)), \quad 0 < s < \mu(B).$$

For metric spaces (Ω, d, μ) satisfying the relative uniform isoperimetric property we have the scaling that we need to apply the previous analysis. This theme is developed in detail in Chapter 4. The previous discussion implicitly shows that $(0,1)^n$ has the relative uniform isoperimetric property. We shall show in Chapter 5 that many familiar metric measure spaces also have the relative uniform isoperimetric property.

We now turn to the main objective of this paper which is to develop the corresponding theory for fractional order Besov-Sobolev spaces. This is, indeed, the original setting of Garsia's work, and our aim in this paper is to extend it to the metric setting. The first part of our program for Besov spaces was to formulate a suitable replacement of (1.1.2) for the fractional setting. To explain the peculiar form of the underlying inequalities that we need requires some preliminary background information.

Let $X = X(\mathbb{R}^n)$ be a rearrangement invariant space[11] on $\mathbb{R}^n$, and let ω_X be the modulus of continuity associated with X defined for $g \in X$ by

$$\omega_X(t, g) = \sup_{|h| \leq t} \|g(\cdot + h) - g(\cdot)\|_X .$$

10. Recall that given an open set $G \subset \Omega$, and a set $A \subset G$, the **perimeter** of A **relative** to G (cf. Chapter 2) is defined by

$$P(A; G) = \lim\inf_{h \to 0} \frac{\mu(\{x \in G : d(x, A) < h\}) - \mu(A)}{h}.$$

The corresponding **relative isoperimetric profile** of $G \subset \Omega$ is given by

$$I_G(s) = I_{(G,d,\mu)}(s) = \inf\{P(A; G) : \ A \subset G, \ \mu(A) = s\}.$$

11. See Section 2.2.

It is known (for increasing levels of generality see [**40**], [**55**], [**65**], [**63**] and the references therein), that there exists $c = c_n > 0$ such that, for all functions $f \in X(\mathbb{R}^n) + \dot{W}^1_X(\mathbb{R}^n)$,

$$|f|^{**}(t) - |f|^{*}(t) \leq c_n \frac{\omega_X\left(t^{1/n}, f\right)}{\phi_X(t)}, \ t > 0, \tag{1.1.7}$$

where $\dot{W}^1_X(\mathbb{R}^n)$ is the homogeneous Sobolev space defined by means of the seminorm $\|u\|_{\dot{W}^1_X(\mathbb{R}^n)} := \||\nabla u|\|_{X(\mathbb{R}^n)}$, $\phi_X(t)$ is the fundamental function (12) of X, and $|f|^*$ is the rearrangement of $|f|$ with respect to the Lebesgue measure (13).

The inequalities we seek are extensions of (1.1.7) to the metric setting. Note that, in some sense, one can consider (1.1.7) as an extension, by interpolation, of (1.1.2). Therefore, it is natural to ask: How should (1.1.7) be reformulated in order to make sense for metric spaces? Not only we need a suitable substitute for the modulus of continuity ω_X, but a suitable re-interpretation of the factor "$t^{1/n}$" is required as well. We now discuss these issues in detail.

There are several known alternative, although possibly non equivalent, definitions of modulus of continuity in the general setting of metric measure spaces (Ω, d, μ) (cf. [**47**] for the interpolation properties of Besov spaces on metric spaces). Given our background on approximation theory, it was natural for us to choose the universal object that is provided by interpolation/approximation theory, namely the Peetre $K-$functional. Indeed on $\mathbb{R}^n$, the Peetre $K-$functional is defined by:

$$K(t, f; X(\mathbb{R}^n), \dot{W}^1_X(\mathbb{R}^n)) := \inf\{\|f - g\|_X + t\,\||\nabla g|\|_X : g \in \dot{W}^1_X(\mathbb{R}^n)\}.$$

Considering the $K-$functional is justified since it is well known that (14) (cf. [**13**, Chapter 5, formula (4.41)])

$$K(t, f; X(\mathbb{R}^n), \dot{W}^1_X(\mathbb{R}^n)) \simeq \omega_X(t, f).$$

In the general case of metric measure spaces (Ω, d, μ) we shall consider:

$$K(t, f; X(\Omega), S_X(\Omega)) := \inf\{\|f - g\|_{X(\Omega)} + t\,\||\nabla g|\|_{X(\Omega)} : g \in S_X(\Omega)\},$$

where $X(\Omega)$ is a r.i. space on Ω, and $S_X(\Omega) = \{f \in Lip(\Omega) : \||\nabla f|\|_{X(\Omega)} < \infty\}$. We shall thus think of $K(t, f; X(\Omega), S_X(\Omega))$ as "a modulus of continuity".

12. For the definition, see (2.2.3) below.

13. In the background of inequalities of this type lies a form of the Pólya-Szegö principle that states that symmetric rearrangements do not increase Besov norms (cf. [**3**], [**65**] and the references therein).

14. Here the symbol $f \simeq g$ indicates the existence of a universal constant $c > 0$ (independent of all parameters involved) such that $(1/c)f \leq g \leq c\,f$. Likewise the symbol $f \preceq g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq c\,g$.

Now, given our experience with the inequality (1.1.2), we were led to conjecture the following reformulation [15] of (1.1.7): There exists a universal constant $c > 0$, such that for every r.i. space $X(\Omega)$, and for all $f \in X(\Omega) + S_X(\Omega)$, we have

$$|f|_\mu^{**}(t) - |f|_\mu^*(t) \leq c \frac{K\left(\frac{t}{I_\Omega(t)}, f; X(\Omega), S_X(\Omega)\right)}{\phi_X(t)}, \ 0 < t < \mu(\Omega). \tag{1.1.8}$$

We presented this conjectural inequality when lecturing on the topic. In particular, we communicated the conjecture to M. Mastylo, who recently proved in [**73**] that indeed (1.1.8) holds for $t \in (0, \mu(\Omega)/4)$, and for all rearrangement invariant spaces X that are "far away from L^1 and from L^∞".

The result of [**73**], while in many respects satisfying, leaves some important questions open. Indeed, the restrictions placed on the range of t (*i.e.*, the measure of the sets), as well as those placed on the spaces, precludes the investigation of the isoperimetric nature of (1.1.8). In particular, while the equivalence of (1.1.2) and the isoperimetric inequality (1.1.1) is known to hold (cf. [**70**]), the possible equivalence of (1.1.8) with the isoperimetric inequality (1.1.1) apparently cannot be answered without involving the space L^1.

One of our main results in Chapter 3 shows that (1.1.8) crucially holds for all $t \in (0, \mu(\Omega)/2)$ and without restrictions on the function spaces X (cf. Theorem 7, Chapter 3 below). The possibility of including $X = L^1$ allows us to prove the following fractional Sobolev version of the celebrated Maz'ya equivalence [16] (cf. [**75**]).

Theorem 1. — *Let (Ω, d, μ) be a metric measure space that satisfies our standard assumptions. Then (cf. Theorems 7 and 11),*

(i) For all rearrangement invariant spaces $X(\Omega)$, and for all $f \in X(\Omega) + S_X(\Omega)$,

$$|f|_\mu^{**}(t) - |f|_\mu^*(t) \leq 16 \frac{K\left(\frac{t}{I_\Omega(t)}, f; X(\Omega), S_X(\Omega)\right)}{\phi_X(t)}, \ t \in (0, \mu(\Omega)/2), \tag{1.1.9}$$

$$|f - f_\Omega|_\mu^{**}(t) - |f - f_\Omega|_\mu^*(t) \leq 16 \frac{K\left(\frac{t}{I_\Omega(t)}, f; X(\Omega), S_X(\Omega)\right)}{\phi_X(t)}, \ t \in (0, \mu(\Omega)), \tag{1.1.10}$$

where

$$f_\Omega = \frac{1}{\mu(\Omega)} \int_\Omega f \, d\mu. \tag{1.1.11}$$

15. At least for the metric measure spaces (Ω, d, μ) considered in [**70**] (for which, in particular, (1.1.2) holds).

16. Which claims the equivalence between the Gagliardo-Nirenberg inequality and the isoperimetric inequality.

(ii) Conversely, suppose that $G : (0, \mu(\Omega)) \to \mathbb{R}_+$, is a continuous function, which is concave and symmetric around $\mu(\Omega)/2$, and there exists a constant $c > 0$ such that[17] *for all $f \in X(\Omega) + S_X(\Omega)$,*

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \leq c \frac{K\left(\frac{t}{G(t)}, f; X(\Omega), S_X(\Omega)\right)}{t}, \quad t \in (0, \mu(\Omega)/2).$$

Then, for all Borel sets with $\mu(A) \leq \mu(\Omega)/2$, we have the isoperimetric inequality

$$G(\mu(A)) \leq cP(A, \Omega).$$

As a consequence, there exists a constant $c > 0$ such that for all $t \in (0, \mu(\Omega))$,

$$G(t) \leq cI_\Omega(t).$$

Using (1.1.9) and (1.1.10) as a starting point we can study embeddings of Besov spaces in metric spaces and, in particular, the corresponding Morrey-Sobolev-Besov embedding.

We now focus the discussion on the fractional Morrey-Sobolev theorem. We start by describing the inequalities of [**44**], [**43**], which for L^p spaces[18] on $[0,1]$ take the following form

$$\left.\begin{array}{c} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array}\right\} \leq c \int_x^1 \frac{\omega_{L^p}(t,f)}{t^{1/p}} \frac{dt}{t}, \quad x \in \left(0, \frac{1}{2}\right]. \tag{1.1.12}$$

Letting $x \to 0$ in (1.1.12) and adding the two inequalities then yields

$$ess \sup_{[0,1]} f - ess \inf_{[0,1]} f \leq c \int_0^1 \frac{\omega_{L^p}(t,f)}{t^{1/p}} \frac{dt}{t}.$$

Using the "change of scale argument" leads to

$$|f(x) - f(y)| \leq 2c \int_0^{|x-y|} \frac{\omega_{L^p}(t,f)}{t^{1/p}} \frac{dt}{t}; \; x, y \in [0,1],$$

from which the essential continuity[19] of f is apparent if we know that $\int_0^1 \frac{\omega_{L^p}(t,f)}{t^{1/p}} \frac{dt}{t} < \infty$. To obtain the $n-$dimensional version of (1.1.12) for $[0,1]^n$, Garsia *et al.* had to develop deep combinatorial techniques. The corresponding $n-$dimensional inequality is given by (cf. [**43**, (3.6)] and the references therein)

$$\left.\begin{array}{c} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array}\right\} \leq c \int_x^1 \frac{\omega_{L^p}(t^{1/n},f)}{t^{1/p}} \frac{dt}{t}, \quad x \in x \in \left(0, \frac{1}{2}\right],$$

17. In other words we assume that (1.1.9) holds for $X = L^1(\Omega)$, and with $\frac{t}{G(t)}$ replacing $\frac{t}{I_\Omega(t)}$.

18. Importantly, Garsia-Rodemich also can deal with $X = L^p$, or $X = L_A$ (Orlicz space), our approach covers all r.i. spaces and works for a large class of metric spaces.

19. An application of Hölder's inequality also yields *Lip* conditions.

which by the now familiar argument yields

$$|f(x) - f(y)| \leq C_{p,n} \int_0^{|x-y|} \frac{\omega_{L^p}(t,f)}{t^{n/p}} \frac{dt}{t}; \; x, y \in \left[0, \frac{1}{2}\right]^n. \tag{1.1.13}$$

However, as pointed out above, the change of scale technique is apparently not available for more general domains. Moreover, as witnessed by the difficulties already encountered by Garsia *et al.* when proving inequalities on $n-$dimensional cubes, it was not even clear at that time what form the rearrangement inequalities would take in general. In particular, Garsia *et al.* do not use isoperimetry.

For more general function spaces we need to reformulate Theorem 1 above as follows (cf. Chapter 3):

***Theorem 2* (cf. Theorem 10, Chapter 3).** — *Let (Ω, d, μ) be a metric measure space that satisfies our standard assumptions, and let X be a r.i. space on Ω. Then, there exists a constant $c > 0$ such that for all $f \in X + S_X(\Omega)$,*

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \leq c \frac{K(\psi(t), f; X, S_X(\Omega))}{\phi_X(t)}, \; 0 < t < \mu(\Omega), \tag{1.1.14}$$

where

$$\psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'},$$

and $\bar{X}'$ denotes the associated[20] *space of $\bar{X}$.*

***Remark 1*.** — *Note that when $X = L^1$ the inequalities (1.1.14) and (1.1.9) are equivalent, modulo constants.*

A second step of our program is the routine, but crucially important, reformulation of rearrangement inequalities using signed rearrangements. Once this is done, our generalized Morrey-Sobolev-Garsia-Rodemich theorem can be stated as follows

***Theorem 3* (cf. Theorem 13, Chapter 4).** — *Let (Ω, d, μ) be a metric measure space that satisfies our standard assumptions and has the relative uniform isoperimetric property. Let X be a r.i. space in Ω such that*

$$\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} < \infty.$$

If $f \in X + S_X(\Omega)$ satisfies

$$\int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X(\Omega)\right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$

then f is essentially bounded and essentially continuous.

---

20. Cf. Chapter 2 for the definition.

To see the connection [21] between our inequalities and those of Garsia *et al.* let us fix ideas and set $\mu(\Omega)=1$. Observe that on account of (1.1.9), and the fact that $K(\frac{t}{I(t)},f;X,S_X)$ is increasing and $\phi_X(t)$ is concave, we have

$$\begin{aligned} f_\mu^{**}(t)-f_\mu^*(t) &\leq 16\frac{K\left(\frac{t}{I_\Omega(t)},f;X,S_X\right)}{\phi_X(t)} \\ &\leq \frac{16}{\ln 2}2\int_t^{2t}\frac{K\left(\frac{s}{I_\Omega(s)},f;X,S_X\right)}{\phi_X(s)}\frac{ds}{s},\ t\in(0,1/2]. \end{aligned}$$

Combining the last inequality with (cf. [**9**, (4.1), p. 1222])

$$f_\mu^*\left(\frac{t}{2}\right)-f_\mu^*(t)\leq 2\left(f_\mu^{**}(t)-f_\mu^*(t)\right), \tag{1.1.15}$$

yields

$$f_\mu^*\left(\frac{t}{2}\right)-f_\mu^*(t)\leq\frac{16}{\ln 2}2\int_t^{2t}\frac{K\left(\frac{s}{I_\Omega(s)},f;X,S_X\right)}{\phi_X(s)}\frac{ds}{s},\ t\in(0,1/2].$$

Therefore, for $n=2,\dots$

$$f_\mu^*\left(\frac{1}{2^{n+1}}\right)-f_\mu^*\left(\frac{1}{2}\right)=\sum_{j=1}^n f_\mu^*\left(\frac{1}{2^{j+1}}\right)-f_\mu^*\left(\frac{1}{2^j}\right)\leq\frac{16}{\ln 2}2\int_0^1\frac{K\left(\frac{s}{I_\Omega(s)},f;X,S_X\right)}{\phi_X(s)}\frac{ds}{s},$$

and letting $n\to\infty$, we find

$$\mathit{ess}\ \sup_\Omega f-f_\mu^*\left(\frac{1}{2}\right)\leq c\int_0^1\frac{K\left(\frac{s}{I_\Omega(s)},f;X,S_X\right)}{\phi_X(s)}\frac{ds}{s}.$$

Likewise, applying the previous inequality to $-f$ and adding the two resulting inequalities, and then observing that $(-f)_\mu^*(\frac{1}{2})=-f_\mu^*(\frac{1}{2})$, yields

$$\mathit{ess}\ \sup_\Omega f-\mathit{ess}\ \inf_\Omega f\leq c\int_0^1\frac{K\left(\frac{s}{I_\Omega(s)},f;X,S_X\right)}{\phi_X(s)}\frac{ds}{s}.$$

From this point we can proceed to study the continuity or *Lip* properties of f using the arguments [22] outlined above.

21. The full metric version of Garsia's inequality is given in Chapter 10.
22. Interestingly the one dimensional case studied by Garsia et al somehow does not follow directly since the isoperimetric profile for the unit interval $(0,1)$ is 1. Therefore in this case the isoperimetric profile does not satisfy the assumptions of [**70**]. Nevertheless, our inequalities remain true and provide an alternate approach to the Garsia inequalities. See Chapter 10 below for complete details.

Next let us consider limiting cases of the Sobolev-Besov embeddings connected with these inequalities and the role of *BMO*. Note that the inequality (1.1.13) can be reformulated as the embedding of $B_p^{n/p,1}([0,1]^n)$ into the space of continuous functions $C([0,1]^n)$

$$B_p^{n/p,1}([0,1]^n) \subset C([0,1]^n), \text{ where } n/p < 1. \tag{1.1.16}$$

Moreover, since

$$\omega_{L^p}(t,f) \leq c_{p,n} t \|f\|_{W^1_{L^p}},$$

we also have

$$W^1_{L^p}([0,1]^n) \subset B_p^{n/p,1}([0,1]^n).$$

Therefore, (1.1.16) implies the (Morrey-Sobolev) continuity of Sobolev functions in W_p^1 when $p > n$. On the other hand, if we consider the Besov condition defined by the right hand side of (1.1.7), when $X = L^p$, and $n/p < 1$, we find [(23)]

$$\|f\|_{B_p^{n/p,\infty}([0,1]^n)} = \sup_{t\in[0,1]} \frac{\omega_{L^p}(t,f)}{t^{n/p}}.$$

Now, for functions in $B_p^{n/p,\infty}([0,1]^n)$ we don't expect boundedness, and in fact, apparently the best we can say directly from our rearrangement inequalities, follows from (1.1.7):

$$\sup_{[0,1]} (f^{**}(t) - f^*(t)) \leq c \|f\|_{B_p^{n/p,\infty}([0,1]^n)}. \tag{1.1.17}$$

In view of the celebrated result of Bennett-DeVore-Sharpley [**11**] (cf. also [**13**]) that characterizes the rearrangement invariant hull of *BMO via* the left hand side of (1.1.17), we see that (1.1.17) gives $f \in B_p^{n/p,\infty}([0,1]^n) \Rightarrow f^* \in BMO[0,1]$. In fact, a stronger result is known and readily available

$$B_p^{n/p,\infty}([0,1]^n) \subset BMO. \tag{1.1.18}$$

It turns out that our approach to the estimation of oscillations allows us to extend (1.1.18) to other geometries. Our method reflects the remarkable connections between oscillation, isoperimetry, interpolation, and rescalings. We briefly explain the ideas behind our approach to (1.1.18). Given a metric measure space (Ω, d, μ) satisfying our standard assumptions, we have the well known Poincaré inequality (cf. [**70**, p. 150] and the references therein) given by

$$\int_\Omega |f(x) - m(f)|\, d\mu \leq \frac{\mu(\Omega)}{2I_\Omega(\mu(\Omega)/2)} \int_\Omega |\nabla f(x)|\, d\mu, \text{ for all } f \in S_{L^1}(\Omega), \tag{1.1.19}$$

23. Note that we have
$$\|f\|_{B_p^{n/p,\infty}([0,1]^n)} \leq c_{p,n} \|f\|_{B_p^{n/p,1}([0,1]^n)}.$$

where $m(f)$ is a median[24] of f. Then given a r.i. space X on Ω we can extend (1.1.19) by "by interpolation" and obtain a $K-$Poincaré inequality[25] (cf. Theorem 6, Chapter 3)

$$\frac{1}{\mu(\Omega)}\int_{\Omega}|f-f_{\Omega}|\,d\mu \leq c\frac{K\left(\frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)},f;X,S_X\right)}{\phi_X(\mu(\Omega))}.$$

Now, if Ω supports the isoperimetric rescalings described above, the previous inequality self improves to (cf. Theorem 24, Chapter 7])

$$\|f\|_{BMO(\Omega)}=\sup_{B\text{ balls}}\frac{1}{\mu(B)}\int_{B}|f-f_B|\,d\mu\leq c\sup_{t<\mu(\Omega)}\frac{K\left(\frac{t}{I_{\Omega}(t)},f;X,S_X\right)}{\phi_X(t)}.$$

It is easy to see that for metric spaces with Euclidean type isoperimetric profiles, *i.e.*, $I_{\Omega}(t)\succeq t^{1-1/n}$, on $(0,1/2)$, we recover (1.1.18) (cf. Corollary 1, Chapter 7). Indeed, the result exhibits a new connection between the geometry of the ambient space and the embedding of Besov and *BMO* spaces. For further examples we refer to (7.1.14)).

In the opposite direction, we can use the insights we gained "interpolating between a r.i. space X and the corresponding space of *Lip* functions S_X" to obtain analogous results interpolating with *BMO*. This is not a coincidence; for recall the well known Euclidean interpretation of *BMO* as a limiting *Lip* condition. This can be seen by means of writing Lip_{α} conditions on a fixed Euclidean cube Q as

$$\|f\|_{Lip_{\alpha}}\simeq\sup_{\substack{Q'\subset Q\\ Q'\text{ cub}}}\frac{1}{|Q'|^{1-\alpha/n}}\int_{Q'}|f-f_{Q'}|\,dx<\infty.$$

In this fashion *BMO* appears as the limiting case of Lip_{α} conditions when $\alpha\to 0$. With this intuition at hand we were led to formulate the corresponding version of Theorem 1 for *BMO*. It reads as follows[26]

***Theorem* 4 (cf. Theorem 27, Chapter 7).** — *Suppose that (Ω,d,μ) is a metric measure space that satisfies our usual assumptions and, moreover, is such that the Bennett-DeVore-Sharpley inequality*

$$\sup_t\left(f_{\mu}^{**}(t)-f_{\mu}^{*}(t)\right)\leq c\,\|f\|_{BMO(\Omega)}, \tag{1.1.20}$$

holds. Then,

$$f_{\mu}^{**}(t)-f_{\mu}^{*}(t)\leq c\frac{K(\phi_X(t),f;X(\Omega),BMO(\Omega))}{\phi_X(t)},\ 0<t<\mu(\Omega).$$

24. For the definition of median, see Definition 1, Chapter 3.
25. See (1.1.11) above.
26. In particular, we arrive, albeit through a very different route than the original, to an extension of an inequality of Bennett-DeVore-Sharpley (cf. [**13**, combine Theorem 7.3 and Theorem 8.8]) for the space L^1.

From this point of view the Bennett-DeVore-Sharpley inequality (1.1.20) takes the role of our basic inequality (1.1.2). In this respect it is important to note that (1.1.20) has been shown to hold in great generality, for example it holds for doubling measures (cf. [**86**]). Finally, in connection with *BMO*, we considered the role of signed rearrangements. Here the import of this notion is that signed rearrangements provide a theoretical method to compute medians (cf. Theorem 25) and thus quickly lead to a version of the limiting case of the John-Stromberg-Jawerth-Torchinsky inequality (cf. (7.2.1)).

Our approach to the (Morrey-)Sobolev embedding theorem also leads to the consideration of "Lorentz spaces with negative indices" (cf. Chapter 9), providing still a very suggestive approach (27) to these results, at least in the Euclidean case.

In Chapter 5 and Chapter 6 we have considered explicit versions of our results in different classical contexts. In particular, in Chapter 6, we obtain new fractional Sobolev inequalities for Gaussian measures, as well as for probability measures that are in between Gaussian and exponential. For example, for Gaussian measure on $\mathbb{R}^n$, we have for $1 \leq q < \infty, \theta \in (0,1)$

$$\left\{ \int_0^{1/2} |f|_{\gamma_n}^* (t)^q \left(\log \frac{1}{t} \right)^{\frac{q\theta}{2}} dt \right\}^{1/q} \leq c \|f\|_{B^{\theta,q}_{L^q}(\gamma_n)},$$

where c is independent of the dimension. Likewise, the same proof yields that for probability measures on the real line of the form (28)

$$d\mu_r(x) = Z_r^{-1} \exp\left(-|x|^r\right) dx, \ r \in (1,2]$$

and their tensor products

$$\mu_{r,n} = \mu_p^{\otimes n},$$

we have

$$\left\{ \int_0^{1/2} |f|_{\mu_{r,n}}^* (t)^q \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right\}^{1/q} \leq c \|f\|_{B^{\theta,q}_{L^q}(\mu_{r,n})}.$$

We refer to Chapter 6 for the details, where the reader will also find a treatment of the case $q = \infty$, which yields the corresponding improvements on the exponential integrability:

$$\sup_{t \in (0,\frac{1}{2})} \left(|f|_{\mu_{r,n}}^{**} (t) - |f|_{\mu_{r,n}}^* (t) \right) \left(\log \frac{1}{t} \right)^{(1-\frac{1}{r})\theta} \leq c \|f\|_{\dot{B}^{\theta,\infty}_{L^\infty}(\mu_{r,n})}.$$

Applications to the computation of envelopes of function spaces in the sense of Triebel-Haroske and their school are provided in Chapter 8.

27. Although in this paper we only concentrate on the role that these spaces play on the theory of embeddings, one cannot but feel that a detailed study of these spaces could be useful for other questions connected with interpolation/approximation.

28. Where Z_r^{-1} is a normalizing constant.

The table of contents will serve to show the organization of the paper. We have tried to make the reading of the chapters in the second part of the paper as independent of each other as possible.

Acknowledgement. — *We are extremely grateful to the referees for their painstaking review that gave us the opportunity to correct, and fill-in gaps in some arguments, and thus contributed to substantially improve the quality of the paper.*

CHAPTER 2

PRELIMINARIES

2.1. Background

Our notation in the paper will be for the most part standard. In this paper we shall only consider [1] connected measure metric spaces (Ω, d, μ) equipped with a finite Borel measure μ, which we shall simply refer to, as "measure metric spaces". For measurable functions $u : \Omega \to \mathbb{R}$, the distribution function of u is given by

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} \qquad (t \in \mathbb{R}).$$

The signed **decreasing rearrangement** [2] of a function u is the right-continuous non-increasing function from $[0, \mu(\Omega))$ into $\mathbb{R}$ which is equimeasurable with u. It can be defined by the formula

$$u_\mu^*(s) = \inf\{t \geq 0 : \mu_u(t) \leq s\}, \quad s \in [0, \mu(\Omega)),$$

and satisfies

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} = m\left\{s \in [0, \mu(\Omega)) : u_\mu^*(s) > t\right\} \ , \quad t \in \mathbb{R}$$

(where m denotes the Lebesgue measure on $[0, \mu(\Omega))$. It follows from the definition that

$$(u+v)_\mu^*(s) \leq u_\mu^*(s/2) + v_\mu^*(s/2). \tag{2.1.1}$$

Moreover,

$$u_\mu^*(0^+) = ess \sup_\Omega u \quad \text{and} \quad u_\mu^*(\mu(\Omega)^-) = ess \inf_\Omega u.$$

1. See also Condition 1 below.
2. Note that this notation is somewhat unconventional. In the literature it is common to denote the decreasing rearrangement of $|u|$ by u_μ^*, while here it is denoted by $|u_\mu|^*$ since we need to distinguish between the rearrangements of u and $|u|$. In particular, the rearrangement of u can be negative. We refer the reader to [**85**] and the references quoted therein for a complete treatment.

The maximal average $u_\mu^{**}(t)$ is defined by

$$u_\mu^{**}(t) = \frac{1}{t}\int_0^t u_\mu^*(s)\,ds = \frac{1}{t}\sup\left\{\int_E u(s)\,d\mu : \mu(E) = t\right\},\ t > 0.$$

The operation $u \to u_\mu^{**}$ is sub-additive, *i.e.*,

(2.1.2) $$(u+v)_\mu^{**}(s) \leq u_\mu^{**}(s) + v_\mu^{**}(s).$$

Moreover, since u_μ^* is decreasing, u_μ^{**} is also decreasing and $u_\mu^* \leq u_\mu^{**}$.

The following lemma proved in [**40**, Lemma 2.1] will be useful in what follows.

Lemma 1. — *Let f and f_n, $n = 1,..,$ be integrable on Ω. Suppose that*

$$\lim_n \int_\Omega |f_n(x) - f(x)|\,d\mu = 0.$$

Then

$$(f_n)_\mu^{**}(t) \longrightarrow f_\mu^{**}(t),\ \textit{uniformly for } t \in [0,\mu(\Omega)],\ \textit{and}$$
$$(f_n)_\mu^{*}(t) \longrightarrow f_\mu^{*}(t)\ \textit{at all points of continuity of } f_\mu^*.$$

When the measure is clear from the context, or when we are dealing with Lebesgue measure, we may simply write u^* and u^{**}, etc.

For a Borel set $A \subset \Omega$, the **perimeter** or **Minkowski content** of A is defined by

$$P(A;\Omega) = \liminf_{h\to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x,A) < h\}$ is the open $h-$neighborhood of A.

The **isoperimetric profile** is defined by

$$I_\Omega(s) = I_{(\Omega,d,\mu)}(s) = \inf\{P(A;\Omega) :\ \mu(A) = s\},$$

i.e., $I_{(\Omega,d,\mu)} : [0,\mu(\Omega)] \to [0,\infty)$ is the pointwise maximal function such that

(2.1.3) $$P(A;\Omega) \geq I_\Omega(\mu(A)),$$

holds for all Borel sets A. A set A for which equality in (2.1.3) is attained will be called an **isoperimetric domain.** Again when no confusion arises we shall drop the subindex Ω and simply write I.

We will always assume that the metric measure spaces (Ω,d,μ) considered satisfy the following condition

Condition 1. — *We will assume throughout the paper that our metric measure spaces (Ω,d,μ) are such that the isoperimetric profile $I_{(\Omega,d,\mu)}$ is a concave continuous function, increasing on $(0,\mu(\Omega)/2)$, symmetric around the point $\mu(\Omega)/2$ that, moreover, vanishes at zero. We remark that these assumptions are fulfilled for a large class of metric measure spaces* (3).

3. These assumptions are satisfied for the classical examples (cf. [**18**], [**77**], [**10**] and the references therein).

A continuous, concave function, $J : [0, \mu(\Omega)] \to [0, \infty)$, increasing on $(0, \mu(\Omega)/2)$, symmetric around the point $\mu(\Omega)/2$, and such that

$$I_\Omega \geq J, \tag{2.1.4}$$

will be called an **isoperimetric estimator** for (Ω, d, μ). Note that (2.1.4) and the fact that $I_\Omega(0) = 0$ implies that $J(0) = 0$.

For a Lipschitz function f on Ω (briefly $f \in Lip(\Omega)$) we define the **modulus of the gradient** by [4]

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Let us recall some results that relate isoperimetry and rearrangements (see [**70**], [**65**]).

Theorem 5. — *The following statements hold*

1. *Isoperimetric inequality:* $\forall A \subset \Omega$, *Borel set,*
$$P(A;\Omega) \geq I_\Omega(\mu(A)).$$
2. *Oscillation inequality:* $\forall f \in Lip(\Omega)$,
$$(|f|_\mu^{**}(t) - |f|_\mu^{*}(t))\frac{I_\Omega(t)}{t} \leq \frac{1}{t}\int_0^t |\nabla f|_\mu^{*}(s)\,ds, \quad 0 < t < \mu(\Omega). \tag{2.1.5}$$

Lemma 2. — *Let h be a bounded Lip function on Ω. Then there exists a sequence of bounded functions $(h_n)_n \subset Lip(\Omega)$, such that*

1. (2.1.6) $$|\nabla h_n(x)| \leq (1 + \frac{1}{n})\,|\nabla h(x)|\,, \quad x \in \Omega.$$
2. (2.1.7) $$h_n \underset{n\to\infty}{\longrightarrow} h \text{ in } L^1.$$
3. (2.1.8) $$\int_0^t \left|\left(-\,|h_n|_\mu^{*}\right)'(\cdot) I_\Omega(\cdot)\right|^{*}(s)\,ds \leq \int_0^t |\nabla h_n|_\mu^{*}(s)\,ds, \quad 0 < t < \mu(\Omega).$$

(The second rearrangement on the left hand side is with respect to the Lebesgue measure on $[0, \mu(\Omega))$.)

2.2. Rearrangement invariant spaces

We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces, and refer the reader to [**13**] and [**58**] for a complete treatment.

4. In fact one can define $|\nabla f|$ for functions f that are Lipschitz on every ball in (Ω, d) (cf. [**18**, p. 2–3] for more details).

Let $X = X(\Omega)$ be a Banach function space on (Ω, d, μ), with the Fatou property [5]. We shall say that X is a **rearrangement-invariant** (r.i.) space, if $g \in X$ implies that all $\mu-$measurable functions f with $|f|_\mu^* = |g|_\mu^*$, also belong to X and moreover, $\|f\|_X = \|g\|_X$. The functional $\|\cdot\|_X$ will be called a rearrangement invariant norm. Typical examples of r.i. spaces are the L^p-spaces, Orlicz spaces, Lorentz [6] spaces, Marcinkiewicz spaces, etc.

On account of the fact that $\mu(\Omega) < \infty$, for any r.i. space $X(\Omega)$ we have

$$L^\infty(\Omega) \subset X(\Omega) \subset L^1(\Omega),$$

with continuous embeddings.

For rearrangement invariant norms (or seminorms) $\|.\|_X$, we can compare the size of elements by comparing their averages, as expressed by a majorization principle, sometimes referred to as the Calderón-Hardy Lemma:

$$\text{(2.2.1)} \quad \text{Suppose that } \int_0^t |f|_\mu^*(s)\,ds \le \int_0^t |g|_\mu^*(s)\,ds \text{ holds for all } 0 < t < \mu(\Omega) \implies \|f\|_X \le \|g\|_X.$$

The associated space $X'(\Omega)$ is defined using the r.i. norm given by

$$\|h\|_{X'(\Omega)} = \sup_{g \neq 0} \frac{\int_\Omega |g(x)h(x)|\,d\mu}{\|g\|_{X(\Omega)}} = \sup_{g \neq 0} \frac{\int_0^{\mu(\Omega)} |h|_\mu^*(s)\,|g|_\mu^*(s)\,ds}{\|g\|_{X(\Omega)}}.$$

In particular, the following **generalized Hölder's inequality** holds

$$\text{(2.2.2)} \qquad \int_\Omega |g(x)h(x)|\,d\mu \le \|g\|_{X(\Omega)}\,\|h\|_{X'(\Omega)}.$$

The **fundamental function** of $X(\Omega)$ is defined by

$$\text{(2.2.3)} \qquad \phi_X(s) = \|\chi_E\|_X, \quad 0 \le s \le \mu(\Omega),$$

where E is any measurable subset of Ω with $\mu(E) = s$. We can assume without loss of generality that ϕ_X is concave (cf. [**13**]). Moreover,

$$\text{(2.2.4)} \qquad \phi_{X'}(s)\phi_X(s) = s.$$

For example, if $X = L^p$ or $X = L^{p,q}$ (a Lorentz space), then $\phi_{L^p}(t) = \phi_{L^{p,q}}(t) = t^{1/p}$, if $1 \le p < \infty$, while for $p = \infty$, $\phi_{L^\infty}(t) \equiv 1$. If N is a Young's function, then the fundamental function of the Orlicz space $X = L_N$ is given by $\phi_{L_N}(t) = 1/N^{-1}(1/t)$.

5. This means that if $f_n \ge 0$, and $f_n \uparrow f$, then $\|f_n\|_X \uparrow \|f\|_X$ (*i.e.*, the monotone convergence theorem holds in the X norm). The nomenclature is somewhat justified by the fact that this property is equivalent to the validity of the usual Fatou Lemma in the X norm (cf. [**13**]).

6. See (7.1.3), (7.1.4), for the definition of Lorentz spaces.

The Lorentz $\Lambda(X)$ space and the Marcinkiewicz space $M(X)$ associated with X are defined by the quasi-norms

$$\|f\|_{M(X)} = \sup_t f_\mu^*(t)\phi_X(t), \quad \|f\|_{\Lambda(X)} = \int_0^{\mu(\Omega)} f_\mu^*(t)\, d\phi_X(t). \tag{2.2.5}$$

Notice that

$$\phi_{M(X)}(t) = \phi_{\Lambda(X)}(t) = \phi_X(t),$$

and, moreover,

$$\Lambda(X) \subset X \subset M(X). \tag{2.2.6}$$

Let $X(\Omega)$ be a r.i. space, then there exists a **unique** r.i. space $\bar{X} = \bar{X}(0, \mu(\Omega))$ on $((0, \mu(\Omega)), m)$, (m denotes the Lebesgue measure on the interval $(0, \mu(\Omega))$) such that

$$\|f\|_{X(\Omega)} = \|\, |f|_\mu^* \,\|_{\bar{X}(0,\mu(\Omega))}. \tag{2.2.7}$$

$\bar{X}$ is called the **representation space** of $X(\Omega)$. The explicit norm of $\bar{X}(0, \mu(\Omega))$ is given by (see [**13**, Theorem 4.10 and subsequent remarks])

$$\|h\|_{\bar{X}(0,\mu(\Omega))} = \sup\left\{ \int_0^{\mu(\Omega)} |h|^*(s)\, |g|_\mu^*(s)\, ds : \|g\|_{X'(\Omega)} \leq 1 \right\} \tag{2.2.8}$$

(the first rearrangement is with respect to the Lebesgue measure on $[0, \mu(\Omega))$).

Classically conditions on r.i. spaces can be formulated in terms of the boundedness of the Hardy operators defined by

$$Pf(t) = \frac{1}{t}\int_0^t f(s)\, ds; \qquad Qf(t) = \int_t^{\mu(\Omega)} f(s)\frac{ds}{s}.$$

The boundedness of these operators on r.i. spaces can be best described in terms of the so called **Boyd indices** [7] defined by

$$\bar{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where $h_X(s) = \sup_{\|f\|_{\bar{X}} \leq 1} \|E_s f\|_{\bar{X}}$ denotes the norm of the compression/dilation operator E_s on $\bar{X}$, defined for $s > 0$, by

$$E_s f(t) = \begin{cases} f^*(\frac{t}{s}) & 0 < t < s, \\ 0 & s < t < \mu(\Omega). \end{cases}$$

7. Introduced by D.W. Boyd in [**21**].

The operator E_s is bounded on $\bar{X}$ on every r.i. space $X(\Omega)$, and moreover,

$$h_X(s) \leq \max(1, s), \text{ for all } s > 0. \tag{2.2.9}$$

For example, if $X = L^p$, then $\overline{\alpha}_{L^p} = \underline{\alpha}_{L^p} = \frac{1}{p}$. It is well known that

$$\begin{aligned} P \text{ is bounded on } \bar{X} &\Leftrightarrow \overline{\alpha}_X < 1, \\ Q \text{ is bounded on } \bar{X} &\Leftrightarrow \underline{\alpha}_X > 0. \end{aligned} \tag{2.2.10}$$

We shall also need to consider the restriction of functions of the r.i. space $X(\Omega)$ to measurable subsets $G \subset \Omega$ with $\mu(G) \neq 0$. We can then consider G as a metric measure space $(G, d_{|G}, \mu_{|G})$ where the corresponding distance and the measure are obtained by the restrictions of the distance d and the measure μ to G. We shall denote the r.i. space $X(G, d_{|G}, \mu_{|G})$ by $X_r(G)$. Given $u : G \to \mathbb{R}$, $u \in X_r(G)$, we let $\tilde{u} : \Omega \to \mathbb{R}$, be its extension to Ω defined by

$$\tilde{u}(x) = \begin{cases} u(x) & x \in G, \\ 0 & x \in \Omega \setminus G. \end{cases} \tag{2.2.11}$$

Then,

$$\|u\|_{X_r(G)} = \|\tilde{u}\|_{X(\Omega)} .$$

Proposition 1. — *Let $X(\Omega)$ be a r.i. space on Ω, and let G be a measurable subset of Ω with $\mu(G) \neq 0$. Then,*

1. *If $u \in X(\Omega)$, then $u\chi_G \in X_r(G)$ and*

$$\|u\chi_G\|_{X_r(G)} \leq \|u\|_{X(\Omega)} .$$

2. *Let $\bar{X}_r$ be the representation space of $X_r(G)$ and let $\bar{X}$ be the representation space of $X(\Omega)$. Let $u \in X_r(G)$. Then*

$$\|u\|_{X_r(G)} = \left\| \widetilde{\left(|u|^*_{\mu|G} \right)} \right\|_{\bar{X}} ,$$

where given $h : (0, \mu(G)) \to (0, \infty)$, $\tilde{h}$ denotes its continuation by 0 outside $(0, \mu(G))$. Thus by the uniqueness of the representation space, if $h \in \bar{X}_r$, then

$$\|h\|_{\bar{X}_r} = \left\| \tilde{h} \right\|_{\bar{X}} .$$

3. *The fundamental function of $X_r(G)$ is given by*

$$\phi_{X_r(G)}(s) = \phi_{X(\Omega)}(s) \quad (0 \leq s \leq \mu(G)). \tag{2.2.12}$$

4. *Let $(X_r(G))'$ be the associated space of $X_r(G)$. Then*

$$(X_r(G))' = (X(\Omega)')_r (G). \tag{2.2.13}$$

Proof. — Part 1 and 4 are elementary. For Part 2, note that if $u \in X_r(G)$, then

$$\begin{aligned}\|u\|_{X_r(G)} &= \|\tilde{u}\|_{X(\Omega)} \\ &= \left\|(|\tilde{u}|)^*_\mu\right\|_{\bar{X}} \quad \text{(by (2.2.7))}.\end{aligned}$$

Since $\mu_{|G} = \mu$ on G, it follows from the definition of $\tilde{u}$ that

$$(|\tilde{u}|)^*_\mu = \begin{cases} (|u|)^*_{\mu_{|G}}(s) & s \in (0, \mu(G)), \\ 0 & s \in (\mu(G), \mu(\Omega)). \end{cases}$$

Thus

$$\left\|(|\tilde{u}|)^*_\mu\right\|_{\bar{X}} = \left\|(|u|)^*_{\mu_{|G}} \chi_{(0,\mu(G))}\right\|_{\bar{X}} = \left\|\widetilde{\left(|u|^*_{\mu_{|G}}\right)}\right\|_{\bar{X}}.$$

Now Part 3 follows from Part 2 taking in account that

$$\phi_{X(\Omega)}(s) = \phi_{\bar{X}}(s). \qquad \square$$

In what follows, when $G \subset \Omega$ is clear from the context, and u is a function defined on G, we shall use the notation $\tilde{u}$ to denote its extension (by zero) defined by (2.2.11) above.

2.3. Some remarks about Sobolev spaces

Let (Ω, d, μ) be a connected metric measure space with finite measure, and let X be a r.i. space on Ω. We let $S_X = S_X(\Omega) = \{f \in Lip(\Omega) : |\nabla f| \in X(\Omega)\}$, equipped with the seminorm

$$\|f\|_{S_X} = \||\nabla f|\|_X \,.$$

At some point in our development we also need to consider restrictions of Sobolev functions. Let $G \subset \Omega$ be an open subset, then if $f \in S_X(\Omega)$ we have $f\chi_G \in S_{X_r}(G)$, and

$$\begin{aligned}\|f\chi_G\|_{S_{X_r}(G)} &\leq \||\nabla f|\,\chi_G\|_{X(\Omega)} \\ &\leq \||\nabla f|\|_{X(\Omega)} \\ &= \|f\|_{S_X(\Omega)} \,.\end{aligned}$$

$K-$functionals play an important role in this paper. The $K-$functional for the pair $(X(\Omega), S_X(\Omega))$ for is defined by

$$K(t, f; X(\Omega), S_X(\Omega)) = \inf_{g \in S_X(\Omega)} \{\|f - g\|_{X(\Omega)} + t\,\||\nabla g|\|_{X(\Omega)}\}. \tag{2.3.1}$$

If G is an open subset of Ω, each competing decomposition for the calculation of the $K-$functional of f, $K(t, f; X(\Omega), S_X(\Omega))$, produces by restriction a decomposition for the calculation of the $K-$functional of $f\chi_G$, and we have

$$K(t, f\chi_G; X_r(G), S_{X_r}(G)) \leq K(t, f; X(\Omega), S_X(\Omega)).$$

Notice that from our definition of $S_X(\Omega)$ it does not follow that $h \in S_X$ implies that $h \in X$. However, under mild conditions on X, one can guarantee that $h \in X$. Indeed, using the isoperimetric profile $I = I_{(\Omega,d,\mu)}$, let us define the associated **isoperimetric Hardy operator** by

$$Q_I f(t) = \int_t^{\mu(\Omega)} f(s) \frac{ds}{I(s)} \quad (f \geq 0).$$

Suppose that there exists an absolute constant $c > 0$ such that, for all $f \in \bar{X}$, such that $f \geq 0$, and with $\operatorname{supp}(f) \subset (0, \mu(\Omega)/2)$, we have

$$\|Q_I f\|_{\bar{X}} \leq c \|f\|_{\bar{X}} . \tag{2.3.2}$$

Then, it was shown in [**70**] that for all $h \in S_X$,

$$\left\| h - \frac{1}{\mu(\Omega)} \int_\Omega h \right\|_X \preceq \| |\nabla h| \|_X .$$

Therefore, since constant functions belong to X we can then conclude that indeed $h \in X$. It is easy to see that if $\underline{\alpha}_X > 0$, condition (2.3.2) is satisfied. Indeed, from the concavity of I, it follows that $\frac{I(s)}{s}$ is decreasing, therefore

$$\frac{I(\mu(\Omega)/2)}{\mu(\Omega)/2} \leq \frac{I(s)}{s}, \ s \in (0, \mu(\Omega)/2).$$

It follows that if $s \in (0, \mu(\Omega)/2)$, then

$$s \leq \frac{\mu(\Omega)/2}{I(\mu(\Omega)/2)} I(s) = cI(s).$$

Consequently, for all $f \geq 0$, with $\operatorname{supp}(f) \subset (0, \mu(\Omega)/2)$,

$$Q_I f(t) = \int_t^{\mu(\Omega)/2} f(s) \frac{ds}{I(s)} \leq c \int_t^{\mu(\Omega)/2} f(s) \frac{ds}{s} = Qf(t).$$

Therefore,

$$\|Q_I f\|_{\bar{X}} \leq c \|Qf\|_{\bar{X}} \leq c_X \|f\|_{\bar{X}} ,$$

where the last inequality follows from the fact that $\underline{\alpha}_X > 0$. We can avoid placing restrictions on X if instead we impose more conditions on the isoperimetric profile. For example, suppose that the following condition [8] on I holds:

$$\int_0^{\mu(\Omega)/2} \frac{ds}{I(s)} = c < \infty. \tag{2.3.3}$$

8. A typical example is $I(t) \simeq t^{1-1/n}$, near zero.

Then, for $f \in L^\infty$ we have

$$\begin{aligned} Q_I f(t) &\leq \|f\|_{L^\infty} \int_t^{\mu(\Omega)} \frac{ds}{I(s)} \\ &\leq c \|f\|_{L^\infty} . \end{aligned}$$

Consequently, Q_I is bounded on L^∞. Since, as we have already seen $Q_I \leq Q$, it follows that Q_I is also bounded on L^1, and therefore, by Calderón's interpolation theorem, Q_I is bounded on any r.i. space X. In particular, (2.3.2) is satisfied.

CHAPTER 3

OSCILLATIONS, K-FUNCTIONALS AND ISOPERIMETRY

3.1. Summary

Let (Ω, d, μ) be a metric measure space satisfying the usual assumptions, and let X be a r.i. space on Ω. In this chapter we show (cf. Theorem 7 in Section 3.2) that for all $f \in X + S_X$,

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq 16 \frac{K\left(\frac{t}{I_{\Omega}(t)}, f; X, S_X\right)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega)/2. \tag{3.1.1}$$

Moreover, if $f_{\Omega} = \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu$, then

$$|f - f_{\Omega}|_{\mu}^{**}(t) - |f - f_{\Omega}|_{\mu}^{*}(t) \leq 16 \frac{K\left(\frac{t}{I_{\Omega}(t)}, f; X, S_X\right)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega). \tag{3.1.2}$$

This extends one of the main results of [**73**]. In Section 3.3 we prove a variant of inequality (3.1.1) that will play an important role in Chapter 5, where embeddings into the space of continuous functions will be analyzed. In this variant we replace $\frac{t}{I_{\Omega}(t)}$ by a smaller function that depends on the space X, more specifically we show that

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq 4 \frac{K(\psi(t), f; X, S_X)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega)/2,$$

where

$$\psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{I_{\Omega}(s)} \chi_{(0,2t)}(s) \right\|_{\bar{X}'}.$$

In Section 3.4 we show that (3.1.1) for $X = L^1$ implies an isoperimetric inequality.

Underlying these results is the following estimation of the oscillation (without rearrangements): there exists a constant $c > 0$ such that

$$\frac{1}{\mu(\Omega)} \int_{\Omega} |f - f_{\Omega}| \, d\mu \leq c \frac{K\left(\frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)}, f; X, S_X\right)}{\phi_X(\mu(\Omega))}.$$

3.2. Estimation of the oscillation in terms of K−functionals

The leitmotif of this chapter are the remarkable connections between oscillations, optimization and isoperimetry.

Definition 1. — *Let $f : \Omega \to \mathbb{R}$ be an integrable function. We say that $m(f)$ is a median of f if*

$$\mu\{f > m(f)\} \geq \mu(\Omega)/2; \text{ and } \mu\{f < m(f)\} \geq \mu(\Omega)/2.$$

Recall the following basic property of medians (cf. [**93**], [**70**, p. 134], etc.)

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{\mu(\Omega)}\int_\Omega |f - f_\Omega|\, d\mu\right) &\leq \inf_c \frac{1}{\mu(\Omega)}\int_\Omega |f - c|\, d\mu \\ &\leq \frac{1}{\mu(\Omega)}\int_\Omega |f - m(f)|\, d\mu \\ &\leq 3\left(\frac{1}{\mu(\Omega)}\int_\Omega |f - f_\Omega|\, d\mu\right). \end{aligned} \tag{3.2.1}$$

The starting point of our analysis is the well known Poincaré inequality (cf. [**18**], [**70**, p. 150]):

$$\int_\Omega |f - m(f)|\, d\mu \leq \frac{\mu(\Omega)}{2I_\Omega(\mu(\Omega)/2)}\int_\Omega |\nabla f|\, d\mu, \text{ for all } f \in S_{L^1}(\Omega). \tag{3.2.2}$$

Our next result is an extension of the Poincaré type inequality (3.2.2) by interpolation (*i.e.*, using $K-$functionals).

Theorem 6. — *Let X be a r.i space on Ω. Then for all $f \in X$,*

$$\frac{1}{\mu(\Omega)}\int_\Omega |f - f_\Omega|\, d\mu \leq 2\frac{K\left(\frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}, f; X, S_X\right)}{\phi_X(\mu(\Omega))}. \tag{3.2.3}$$

Proof. — For an arbitrary decomposition $f = (f - h) + h$, with $h \in S_X$, we have

$$\begin{aligned} \frac{1}{\mu(\Omega)}\int_\Omega |f - m(h)|\, d\mu &= \frac{1}{\mu(\Omega)}\int_\Omega |f - h + (h - m(h))|\, d\mu \\ &\leq \frac{1}{\mu(\Omega)}\int_\Omega |f - h|\, d\mu + \frac{1}{\mu(\Omega)}\int_\Omega |h - m(h)|\, d\mu \\ &\leq \frac{1}{\mu(\Omega)}\|f - h\|_{L^1} + \frac{1}{2I_\Omega(\mu(\Omega)/2)}\||\nabla h|\|_{L^1} \text{ (by (3.2.2))} \\ &\leq \frac{1}{\mu(\Omega)}\|f - h\|_X\, \phi_{X'}(\mu(\Omega)) + \\ &\quad \frac{1}{2I_\Omega(\mu(\Omega)/2)}\||\nabla h|\|_X\, \phi_{X'}(\mu(\Omega)) \text{ (by Hölder's inequality)} \\ &= \frac{1}{\mu(\Omega)}\phi_{X'}(\mu(\Omega))\left(\|f - h\|_X + \frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}\||\nabla h|\|_X\right) \\ &= \frac{1}{\phi_X(\mu(\Omega))}\left(\|f - h\|_X + \frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}\||\nabla h|\|_X\right). \end{aligned}$$

Combining the last inequality with (3.2.1), we obtain that for all decompositions $f = (f - h) + h$, with $h \in S_X$,

$$\frac{1}{\mu(\Omega)} \int_{\Omega} |f - f_{\Omega}| \, d\mu \leq \frac{2}{\phi_X(\mu(\Omega))} \left(\|f - h\|_X + \frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)} \, |||\nabla h|||_X \right).$$

Consequently,

$$\begin{aligned} \frac{1}{\mu(\Omega)} \int_{\Omega} |f - f_{\Omega}| \, d\mu &\leq \frac{2}{\phi_X(\mu(\Omega))} \inf_{h \in S_X} \left(\|f - h\|_X + \frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)} \, |||\nabla h|||_X \right) \\ &= \frac{2}{\phi_X(\mu(\Omega))} K\left(\frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)}, f; X, S_X \right), \end{aligned}$$

as we wished to show. □

The main result of this section is the following

***Theorem* 7.** — *Let X be a r.i. space on Ω. Then (3.1.1) and (3.1.2) hold for all $f \in X + S_X$.*

Proof. — It will be useful to note for future use that if $\|\cdot\|$ denotes either $\|\cdot\|_X$ or $\|\cdot\|_{S_X}$, we have

$$|||f||| \leq \|f\| . \tag{3.2.4}$$

Let $\varepsilon > 0$, and consider any decomposition $f = f - h + h$ with $h \in S_X$, such that

$$\|f - h\|_X + t \, |||\nabla h|||_X \leq K(t, f; X, S_X) + \varepsilon. \tag{3.2.5}$$

Since by (3.2.4), $h \in S_X$ implies that $|h| \in S_X$, this decomposition of f produces the following decomposition of $|f|$:

$$|f| = |f| - |h| + |h| .$$

Therefore, by (3.2.4) and (3.2.5) we have

$$\begin{aligned} |||f| - |h|||_X + t \, |||\nabla |h|||_X &\leq \|f - h\|_X + t \, |||\nabla h|||_X \\ &\leq K(t, f; X, S_X) + \varepsilon. \end{aligned}$$

Consequently,

$$\inf_{0 \leq h \in S_X} \{ |||f| - h\|_X + t \, |||\nabla h|||_X \} \leq K(t, f; X, S_X) . \tag{3.2.6}$$

In what follows we shall use the following notation:

$$K(t, f) := K(t, f; X, S_X) .$$

We shall start by proving (3.1.1). The proof will be divided in two parts.

Part 1: $t \in (0, \mu(\Omega)/4]$. — Given $0 \le h \in S_X$ consider the decomposition

$$|f| = (|f| - h) + h.$$

By (2.1.2) we have

$$|f|_\mu^{**}(t) \le ||f| - h|_\mu^{**}(t) + |h|_\mu^{**}(t),$$

and by (2.1.1) we get

$$|h|_\mu^{*}(2t) - ||f| - h|_\mu^{*}(t) \le |f|_\mu^{*}(t).$$

Combining the previous estimates we can write

(3.2.7)

$$\begin{aligned}
|f|_\mu^{**}(t) - |f|_\mu^{*}(t) &\le ||f| - h|_\mu^{**}(t) + |h|_\mu^{**}(t) - (|h|_\mu^{*}(2t) - ||f| - h|_\mu^{*}(t)) \\
&= ||f| - h|_\mu^{**}(t) + ||f| - h|_\mu^{*}(t) + |h|_\mu^{**}(t) - |h|_\mu^{*}(2t) \\
&\le 2\,||f| - h|_\mu^{**}(t) + \left(|h|_\mu^{**}(t) - |h|_\mu^{*}(2t)\right) \\
&= 2\,||f| - h|_\mu^{**}(t) + \left(|h|_\mu^{**}(t) - |h|_\mu^{**}(2t)\right) + \left(|h|_\mu^{**}(2t) - |h|_\mu^{*}(2t)\right) \\
&= (I) + (II) + (III).
\end{aligned}$$

We first show that $(II) \le (III)$. Recall that $\frac{d}{dt}(-|g|_\mu^{**}(t)) = \frac{|g|_\mu^{**}(t) - |g|_\mu^{*}(t)}{t}$, then using the fundamental theorem of Calculus, and then the fact that $t\left(|g|_\mu^{**}(t) - |g|_\mu^{*}(t)\right) = \int_{|g|_\mu^{*}(t)}^{\infty} \mu_{|g|}(s)\,ds$ is increasing, to estimate (II) as follows:

$$\begin{aligned}
(II) &= |h|_\mu^{**}(t) - |h|_\mu^{**}(2t) \\
&= \int_t^{2t} \left(|h|_\mu^{**}(s) - |h|_\mu^{*}(s)\right) \frac{ds}{s} \\
&\le 2t\left(|h|_\mu^{**}(2t) - |h|_\mu^{*}(2t)\right) \int_t^{2t} \frac{ds}{s^2} \\
&= \left(|h|_\mu^{**}(2t) - |h|_\mu^{*}(2t)\right) \\
&= (III).
\end{aligned}$$

Inserting this information in (3.2.7), applying (2.1.5), and using the fact that $I_\Omega(s)$ is increasing on $(0, \mu(\Omega)/2)$, yields,

$$\begin{aligned}
|f|_\mu^{**}(t) - |f|_\mu^{*}(t) &\le 2\left(||f| - h|_\mu^{**}(t) + \frac{2t}{I_\Omega(2t)} |\nabla h|^{**}(2t)\right) \\
&\le 4\left(||f| - h|_\mu^{**}(t) + \frac{t}{I_\Omega(t)} |\nabla h|^{**}(t)\right) \qquad (3.2.8) \\
&= 4\,(A(t) + B(t)).
\end{aligned}$$

We now estimate the two terms on the right hand side of (3.2.8). For the term $A(t)$: Note that for any $g \in X$,

$$|g|_{\mu}^{**}(t) = \frac{1}{t}\int_0^t |g|_{\mu}^{*}(s)\,ds = \frac{1}{t}\int_0^1 |g|_{\mu}^{*}(s)\chi_{(0,t)}(s)\,ds.$$

Therefore, by Hölder's inequality (cf. (2.2.2) and (2.2.4)) we have

$$\begin{aligned} ||f| - h|_{\mu}^{**}(t) &= \frac{1}{t}\int_0^1 ||f| - h|_{\mu}^{*}(s)\chi_{(0,t)}(s)\,ds \\ &\leq \|(|f| - h)\|_X \frac{\phi_{X'}(t)}{t} \\ &= \|(|f| - h)\|_X \frac{1}{\phi_X(t)}. \end{aligned} \tag{3.2.9}$$

Similarly, for $B(t)$ we get

$$B(t) = \frac{t}{I(t)}|\nabla h|_{\mu}^{**}(t) \leq \frac{t}{I_{\Omega}(t)}\frac{\||\nabla h|\|_X}{\phi_X(t)}. \tag{3.2.10}$$

Inserting (3.2.9) and (3.2.10) back in (3.2.8) we find that,

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq \frac{4}{\phi_X(t)}\left(\||f| - h\|_X + \frac{t}{I_{\Omega}(t)}\||\nabla h|\|_X\right).$$

Therefore, by (3.2.6),

$$\begin{aligned} |f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) &\leq \frac{4}{\phi_X(t)}\inf_{0\leq h\in S_X}\left(\||f| - h\|_X + \frac{t}{I_{\Omega}(t)}\||\nabla h|\|_X\right) \\ &\leq 4\frac{K\left(\frac{t}{I_{\Omega}(t)}, f\right)}{\phi_X(t)}. \end{aligned}$$

Part II: $t \in (\mu(\Omega)/4, \mu(\Omega)/2]$. — Using that the function $t(|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t))$ is increasing, we get,

$$\begin{aligned} t(|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t)) &\leq \mu(\Omega)/2(|f|_{\mu}^{**}(\mu(\Omega)/2) - |f|_{\mu}^{*}(\mu(\Omega)/2)) \\ &= \int_0^{\frac{\mu(\Omega)}{2}}\left(|f|_{\mu}^{*}(s) - |f|_{\mu}^{*}(\mu(\Omega)/2)\right)ds \\ &= I. \end{aligned}$$

Now we use the following elementary inequality to estimate the difference $|f|_{\mu}^{*}(s) - |f|_{\mu}^{*}(\mu(\Omega)/2)$ (cf. [**40**, p. 94], [**55**, (2.5)]): For any $\sigma \in \mathbb{R}$, $0 < r \leq \tau < \mu(\Omega)$, we have

$$|f|_{\mu}^{*}(r) - |f|_{\mu}^{*}(\tau) \leq |f - \sigma|_{\mu}^{*}(r) + |f - \sigma|_{\mu}^{*}(\mu(\Omega) - \tau). \tag{3.2.11}$$

Consequently, if $0 < s < \mu(\Omega)/2$ and we let $r = s, \tau = \mu(\Omega)/2$, then (3.2.11) yields

$$|f|_\mu^* (s) - |f|_\mu^* (\mu(\Omega)/2) \leq |f - \sigma|_\mu^* (s) + |f - \sigma|_\mu^* (\mu(\Omega)/2).$$

The term I above can be now be estimated as follows. For any $\sigma \in \mathbb{R}$, we have

$$\begin{aligned} I &\leq \int_0^{\frac{\mu(\Omega)}{2}} |f - \sigma|_\mu^* (s) + |f - \sigma|_\mu^* (\mu(\Omega)/2)\, ds \\ &\leq 2 \int_0^{\frac{\mu(\Omega)}{2}} |f - \sigma|_\mu^* (s)\, ds \\ &\leq 2 \int_0^{\mu(\Omega)} |f - \sigma|_\mu^* (s)\, ds. \end{aligned}$$

Selecting $\sigma = f_\Omega$, yields

$$\begin{aligned} I &\leq 2 \int_0^{\mu(\Omega)} |f - f_\Omega|_\mu^* (s)\, ds \\ &= 2 \int_\Omega |f - f_\Omega|\, d\mu. \end{aligned}$$

At this point we can apply (3.2.3) to obtain

$$I \leq 2\mu(\Omega) \frac{K\left(\frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}, f\right)}{\phi_X(\mu(\Omega))}.$$

A well known elementary fact about $K-$functionals is that they are concave functions (cf. [**13**]), in particular $\frac{K(t,f)}{t}$ is a decreasing function. Thus, since $I_\Omega(t)$ is increasing on $(0, \mu(\Omega)/2)$, we see that

$$\begin{aligned} K\left(\frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}, f\right) &\leq \frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)} \frac{I_\Omega(t)}{t} K\left(\frac{t}{I_\Omega(t)}, f\right) \\ &\leq \frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)} \frac{I_\Omega(\mu(\Omega)/2)}{t} K\left(\frac{t}{I_\Omega(t)}, f\right) \\ &\leq 2K\left(\frac{t}{I_\Omega(t)}, f\right) \text{ (since } t > \mu(\Omega)/4). \end{aligned}$$

Therefore,

$$\begin{aligned} |f|_\mu^{**} (t) - |f|_\mu^* (t) &\leq 4 \frac{\mu(\Omega)}{t} \frac{K\left(\frac{t}{I_\Omega(t)}, f\right)}{\phi_X(\mu(\Omega))} \\ &\leq 16 \frac{K\left(\frac{t}{I_\Omega(t)}, f\right)}{\phi_X(\mu(\Omega))} \text{ (since } t > \mu(\Omega)/4) \\ &\leq 16 \frac{K\left(\frac{t}{I_\Omega(t)}, f\right)}{\phi_X(t)} \left(\text{since } \frac{1}{\phi_X} \text{ decreases}\right). \end{aligned}$$

Let us now to prove (3.1.2). Once again we divide the proof in two cases. For $t \in (0, \mu(\Omega)/2)$ we claim that (3.1.2) is a consequence of (3.1.1). Indeed, if we apply (3.1.1) to the function $f - f_\Omega$ and then use the fact that the K–functional is subadditive and zero on constant functions, we obtain

$$\begin{aligned} |f - f_\Omega|_\mu^{**}(t) - |f - f_\Omega|_\mu^{*}(t) &\leq 16K\left(\frac{t}{I_\Omega(t)}, f - f_\Omega\right) \\ &\leq 16K\left(\frac{t}{I_\Omega(t)}, f\right) + K\left(\frac{t}{I_\Omega(t)}, f_\Omega\right) \\ &= 16K\left(\frac{t}{I_\Omega(t)}, f\right). \end{aligned}$$

Suppose now that $t \in (\mu(\Omega)/2, \mu(\Omega))$. We have

$$\begin{aligned} |f - f_\Omega|_\mu^{**}(t) - |f - f_\Omega|_\mu^{*}(t) &\leq |f - f_\Omega|_\mu^{**}(t) \\ &\leq \frac{1}{t}\int_0^t |f - f_\Omega|_\mu^{*}(s)\, ds \\ &\leq \frac{2}{\mu(\Omega)}\int_0^{\mu(\Omega)} |f - f_\Omega|_\mu^{*}(s)\, ds \\ &= \frac{2}{\mu(\Omega)}\int_\Omega |f - f_\Omega|\, d\mu \\ &\leq 4\frac{K\left(\frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}, f\right)}{\phi_X(\mu(\Omega))} \quad \text{(by (3.2.3))}. \end{aligned}$$

Recalling that $\frac{t}{I_\Omega(t)}$ is increasing and $\frac{1}{\phi_X(t)}$ is decreasing, we can continue with

$$\begin{aligned} 4\frac{K\left(\frac{\mu(\Omega)/2}{I_\Omega(\mu(\Omega)/2)}, f\right)}{\phi_X(\mu(\Omega))} &\leq 4\frac{K\left(\frac{t}{I_\Omega(t)}, f\right)}{\phi_X(\mu(\Omega))} \\ &\leq 4\frac{K\left(\frac{t}{I_\Omega(t)}, f\right)}{\phi_X(t)}, \end{aligned}$$

an the desired result follows. □

A useful variant of the previous result can be stated as follows (cf. [**73**] for a somewhat weaker result).

Theorem 8. — *Let $X = X(\Omega)$ be a r.i. space with $\underline{\alpha}_X > 0$. Then, there exists a constant $c = c(X) > 0$ such that for all $f \in X$,*

(3.2.12)
$$\left\| \left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2)\right)\chi_{(0,t/2)}(\cdot)\right\|_{\bar{X}} \leq cK\left(\frac{t}{I_\Omega(t)}, f; X, S_X\right), \quad 0 < t \leq \mu(\Omega).$$

Proof. — Fix $t \in (0, \mu(\Omega)]$. We will first suppose that f is bounded. We claim that in the computation of $K(t, f; X, S_X)$ we can restrict ourselves to consider decompositions with bounded h. In fact, we have

(3.2.13)
$$\inf_{0 \leq h \leq \|f\|_\infty, h \in S_X} \{\||f| - h\|_X + t \||\nabla h|\|_X\} \leq 2 \inf_{0 \leq h \in S_X} \{\||f| - h\|_X + t \||\nabla h|\|_X\}.$$

To see this consider any competing decomposition with $0 \leq h \in S_X$. Let

$$g = \min(h, \|f\|_\infty) \in S_X.$$

Then,

$$|\nabla g| \leq |\nabla h|,$$

and, moreover, we have

$$\begin{aligned} \||f| - g\|_X + t \||\nabla g|\|_X \leq & \left\|(|f| - h) \chi_{\{h \leq \|f\|_\infty\}}\right\|_X + \\ & \left\|(|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}}\right\|_X + t \||\nabla h|\|_X \\ \leq & \||f| - h\|_X + \left\|(|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}}\right\|_X + t \||\nabla h|\|_X. \end{aligned}$$

Now, since

$$\left|(|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}}\right| = (\|f\|_\infty - |f|) \chi_{\{h > \|f\|_\infty\}} \leq (h - |f|) \chi_{\{h > \|f\|_\infty\}},$$

we see that

$$\left\|(|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}}\right\|_X = \left\|(h - |f|) \chi_{\{h > \|f\|_\infty\}}\right\|_X \leq \||f| - h\|_X.$$

Consequently

$$\||f| - g\|_X + t \||\nabla g|\|_X \leq 2 \||f| - h\|_X + t \||\nabla h|\|_X,$$

and (3.2.13) follows.

Let $0 \leq h$ be a bounded $Lip(\Omega)$ function, and fix $g \in \bar{X}'$ with $\|g\|_{\bar{X}'} = 1$. Recall that $\bar{X}'$ is a r.i. space on $([0, \mu(\Omega)], m)$, where m denotes Lebesgue measure. We let $|g|^* := |g|_m^*$. Consider the decomposition

$$|f| = (|f| - h) + h.$$

Writing $h = |f| + (-(|f| - h))$, we see that $|h|_\mu^*(t) \leq |f|_\mu^*(t/2) + ||f| - h|_\mu^*(t/2)$. We use this inequality to provide a lower bound for $|f|_\mu^*(t/2)$, namely

$$|h|_\mu^*(t) - ||f| - h|_\mu^*(t/2) \leq |f|_\mu^*(t/2).$$

Then we have

$$\begin{aligned}&\int_0^{t/2}(|f|_\mu^*(s)-|f|_\mu^*(t/2))\chi_{(0,t/2)}(s)\,|g|^*(s)\,ds\\ &=\int_0^{t/2}|f|_\mu^*(s)\,|g|^*(s)\,ds-|f|_\mu^*(t/2)\int_0^{t/2}|g|^*(s)\,ds \qquad (3.2.14)\\ &\leq\int_0^{t/2}|f|_\mu^*(s)\,|g|^*(s)\,ds-\Big(|h|_\mu^*(t)-||f|-h|_\mu^*(t/2)\Big)\int_0^{t/2}|g|^*(s)\,ds.\end{aligned}$$

To estimate the first integral on the right hand side without sacrificing precision on the range of the variable *t* requires an argument. We shall use the majorization principle (2.2.1) as follows. Since the operation $f\to f^{**}$ is sub-additive, for any $r>0$ we have

$$\begin{aligned}\int_0^r|f|_\mu^*(s)\chi_{(0,\frac{t}{2})}(s)\,ds&=\int_0^{\min\{r,\frac{t}{2}\}}|f|_\mu^*(s)\,ds\\ &\leq\int_0^{\min\{r,\frac{t}{2}\}}||f|-h|_\mu^*(s)\,ds+\int_0^{\min\{r,\frac{t}{2}\}}|h|_\mu^*(s)\,ds\\ &=\int_0^{\min\{r,\frac{t}{2}\}}\Big(||f|-h|_\mu^*(s)+|h|_\mu^*(s)\Big)\,ds\\ &\leq\int_0^r\Big(||f|-h|_\mu^*(s)+|h|_\mu^*(s)\Big)\,\chi_{(0,\frac{t}{2})}(s)\,ds.\end{aligned}$$

Now, since for each fixed t the functions $H_1=|f|_\mu^*(\cdot)\chi_{(0,\frac{t}{2})}(\cdot)$ and $H_2=(||f|-h|_\mu^*(\cdot)+|h|_\mu^*(\cdot))\chi_{(0,\frac{t}{2})}(\cdot)$ are decreasing, we can apply the Calderón-Hardy Lemma to the (Lorentz) function seminorm defined by (cf. [**62**, Theorem 1])

$$\|H\|_{\Lambda g}=\int_0^{\mu(\Omega)}|H|^*(s)\,|g|^*(s)\,ds.$$

We obtain

$$\|H_1\|_{\Lambda g}\leq\|H_2\|_{\Lambda g}.$$

It follows that

$$\begin{aligned}\int_0^{t/2}|f|_\mu^*(s)\,|g|^*(s)\,ds&=\int_0^{\mu(\Omega)}|f|_\mu^*(s)\chi_{(0,\frac{t}{2})}(s)\,|g|^*(s)\,ds\\ &\leq\int_0^{\mu(\Omega)}\Big(||f|-h|_\mu^*(s)+|h|_\mu^*(s)\Big)\,\chi_{(0,\frac{t}{2})}(s)\,|g|^*(s)\,ds\\ &=\int_0^{t/2}\Big(||f|-h|_\mu^*(s)+|h|_\mu^*(s)\Big)\,|g|^*(s)\,ds.\end{aligned}$$

Inserting this estimate back in (3.2.14) we have,

$$\begin{aligned}
&\int_0^{t/2} (|f|_\mu^*(s) - |f|_\mu^*(t/2))\chi_{(0,t/2)}(s)|g|^*(s)\,ds \\
&\leq \int_0^{t/2} \left(||f| - h|_\mu^*(s) + |h|_\mu^*(s)\right)|g|^*(s)\,ds - \\
&\quad \left(|h|_\mu^*(t) - ||f| - h|_\mu^*(t/2)\right)\int_0^{t/2}|g|^*(s)\,ds \\
&= \int_0^{t/2} ||f| - h|_\mu^*(s)|g|^*(s)\,ds + \int_0^{t/2}|h|_\mu^*(s)|g|^*(s)\,ds + \\
&\quad \left(||f| - h|_\mu^*(t/2) - |h|_\mu^*(t)\right)\int_0^{t/2}|g|^*(s)\,ds \\
&= \int_0^{t/2}\left(||f| - h|_\mu^*(s) + ||f| - h|_\mu^*(t/2)\right)|g|^*(s)\,ds + \\
&\quad \int_0^{t/2}\left(|h|_\mu^*(s) - |h|_\mu^*(t)\right)|g|^*(s)\,ds \\
&\leq 2\int_0^{\mu(\Omega)} ||f| - h|_\mu^*(s)\chi_{(0,t/2)}(s)\,|g|^*(s)\,ds + \\
&\quad \int_0^{\mu(\Omega)}\left(|h|_\mu^*(s) - |h|_\mu^*(t)\right)\chi_{(0,t/2)}(s)\,|g|^*(s)\,ds \\
&\leq 2\,\||f| - h\|_{\bar{X}} + \left\|(|h|_\mu^*(\cdot) - |h|_\mu^*(t))\chi_{(0,t/2)}\right\|_{\bar{X}} \quad \text{(by (2.2.8))}.
\end{aligned}$$

Now, let $\{h_n\}_{n\in N}$ be the sequence provided by Lemma 2, Chapter 2. Then,

$$\begin{aligned}
\left(|h_n|_\mu^*(s) - |h_n|_\mu^*(t)\right)\chi_{(0,t/2)}(s) &= \int_s^t \left(-|h_n|_\mu^*\right)'(r)dr\chi_{(0,t/2)}(s) \\
&\leq \int_s^t \left(-|h_n|_\mu^*\right)'(r)I_\Omega(r)\frac{dr}{I_\Omega(r)} \\
&\leq \frac{t}{I_\Omega(t)}\int_s^t \left(-|h_n|_\mu^*\right)'(r)I_\Omega(r)\frac{dr}{r} \\
&\leq \frac{t}{I_\Omega(t)}\int_s^{\mu(\Omega)} \left(-|h_n|_\mu^*\right)'(r)I_\Omega(r)\frac{dr}{r} \\
&= \frac{t}{I_\Omega(t)}Q(\left(-|h_n|_\mu^*\right)' I_\Omega)(s).
\end{aligned}$$

Applying $\|\cdot\|_{\bar{X}}$ (in the variable s) and using the fact that $\underline{\alpha}_X > 0$, we see that

$$\begin{aligned}
\left\|\left(|h_n|_\mu^*(\cdot) - |h_n|_\mu^*(t)\right)\chi_{(0,t/2)}(\cdot)\right\|_{\bar{X}} &\leq c\frac{t}{I_\Omega(t)}\left\|\left(-|h_n|_\mu^*\right)'(\cdot)I_\Omega(\cdot)\right\|_{\bar{X}} \\
&\leq c\frac{t}{I_\Omega(t)}\||\nabla h_n|\|_X \quad \text{(by (2.1.8))} \\
&\leq c\frac{t}{I_\Omega(t)}\left(1 + \frac{1}{n}\right)\||\nabla h_n|\|_X \quad \text{(by (2.1.6))}.
\end{aligned}$$

On the other hand, by (2.1.7), $h_n \underset{n\to\infty}{\to} h$ in L^1, and by Lemma 1 we get $|h_n|_\mu^*(s) \to |h|_\mu^*(s)$, thus applying Fatou's Lemma in the X norm (cf. Section 2.2), we find that

(3.2.15)

$$\left\| \left(|h|_\mu^*(\cdot) - |h|_\mu^*(t) \right) \chi_{(0,t)}(\cdot) \right\|_{\bar{X}} = \liminf_{n\to\infty} \left\| \left(|h_n|_\mu^*(\cdot) - |h_n|_\mu^*(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} \leq c \frac{t}{I_\Omega(t)} \left\| |\nabla h| \right\|_X .$$

Combining (3.2.14) and (3.2.15) we get

(3.2.16)

$$\int_0^{\mu(\Omega)} (|f|_\mu^*(s) - |f|_\mu^*(t/2))\chi_{(0,t/2)}(s) |g|^*(s)\, ds \leq c(\left\| |f| - h \right\|_X + \frac{t}{I_\Omega(t)} \left\| |\nabla h| \right\|_X).$$

Since $\left(|f|_\mu^*(s) - |f|_\mu^*(t/2) \right)\chi_{(0,t/2)}(s)$ is a decreasing function of s, we can write

$$\left(|f|_\mu^*(s) - |f|_\mu^*(t/2) \right) \chi_{(0,t/2)}(s) = \left(\left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2) \right) \chi_{(0,t/2)}(\cdot) \right)^*(s).$$

Combining successively duality, the last formula and (3.2.16), we get

$$\begin{aligned} &\left\| \left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2) \right) \chi_{(0,t/2)}(s) \right\|_{\bar{X}} \\ &= \sup_{\|g\|_{\bar{X}'}\leq 1} \int_0^{\mu(\Omega)} \left[\left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2) \right) \chi_{(0,t/2)}(\cdot) \right]^*(s) |g|^*(s)\, ds \\ &= \sup_{\|g\|_{\bar{X}'}\leq 1} \int_0^{\mu(\Omega)} \left(|f|_\mu^*(s) - |f|_\mu^*(t/2) \right) \chi_{(0,t/2)}(s) |g|^*(s)\, ds \\ &\leq 2c \left(\left\| |f| - h \right\|_X + \frac{t}{I_\Omega(t)} \left\| |\nabla h| \right\|_X \right), \end{aligned}$$

where c is an absolute constant that depends only on X. Consequently, if f is bounded there exists an absolute constant $c > 0$ such that

$$\begin{aligned} \left\| \left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2) \right) \chi_{(0,t/2)}(\cdot) \right\|_{\bar{X}} &\leq c \inf_{0\leq h\in S_X} \left\{ \left\| |f| - h \right\|_X + \frac{t}{I_\Omega(t)} \left\| |\nabla h| \right\|_X \right\} \\ &\leq cK\left(\frac{t}{I_\Omega(t)}, f; X, S_X \right) \quad \text{(by (3.2.6))}. \end{aligned}$$

Suppose now that f is not bounded. Let $f_n = \min(|f|, n) \nearrow |f|$. By the first part of the proof we have,

$$|f_n|_\mu^{**}(t) - |f_n|_\mu^*(t) \leq cK\left(\frac{t}{I_\Omega(t)}, f_n; X, S_X \right).$$

Fix n. For any $0 \leq h \in S_X$, let $\tilde{h}_n = \min(h, n)$. Then,

$$K\left(\frac{t}{I(t)}, f_n; X, S_X\right) \leq \left\| |f_n| - \tilde{h}_n \right\|_X + \frac{t}{I_\Omega(t)} \left\| \left| \nabla \tilde{h}_n \right| \right\|_X$$
$$\leq \| |f| - h \|_X + \frac{t}{I_\Omega(t)} \| |\nabla h| \|_X .$$

Taking infimum it follows that, for all $n \in \mathbb{N}$,

$$K\left(\frac{t}{I_\Omega(t)}, f_n; X, S_X\right) \leq K\left(\frac{t}{I_\Omega(t)}, f; X, S_X\right).$$

Since $f_n = \min(|f|, n) \nearrow |f|$, then by the Fatou property of the norm we have

$$\begin{aligned}\left\|\left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2)\right)\chi_{(0,t/2)}(\cdot)\right\|_{\bar{X}} &= \lim_n \left\|\left(|f_n|_\mu^*(\cdot) - |f_n|_\mu^*(t/2)\right)\chi_{(0,t/2)}(\cdot)\right\|_X \\ &\leq c \lim_n K\left(\frac{t}{I_\Omega(t)}, f_n; X, S_X\right) \\ &\leq cK\left(\frac{t}{I_\Omega(t)}, f; X, S_X\right),\end{aligned}$$

as we wished to show. □

Remark 2. — *For perspective we now show that, under the extra assumption that the r.i. space satisfies* $\underline{\alpha}_X > 0$, *(3.2.12) can be used to give a direct proof of (3.1.1).*

Proof. — We have,

$$\begin{aligned}|f|_\mu^{**}(t/2) - |f|_\mu^*(t/2) &= \frac{2}{t}\int_0^{\mu(\Omega)} \left(|f|_\mu^*(s) - |f|_\mu^*(t/2)\right)\chi_{(0,t/2)}(s)\, ds \\ &\leq \left\|\left(|f|_\mu^*(\cdot) - |f|_\mu^*(t/2)\right)\chi_{(0,t/2)}(\cdot)\right\|_{\bar{X}} \frac{2\phi_{X'}(t/2)}{t} \\ &\quad \text{(by Hölder's inequality)} \\ &\leq 2cK\left(\frac{t}{I_\Omega(t)}, f; X, S_X\right)\frac{\phi_{X'}(t)}{t} \quad \text{(by (3.2.12))} \\ &= 2c\frac{K\left(\frac{t}{I_\Omega(t)}, f; X, S_X\right)}{\phi_X(t)}.\end{aligned}$$

□

Example 1. — *For familiar spaces (3.2.12) takes a more concrete form. For example, if* $X = L^p$, $1 \leq p < \infty$, *then* $\underline{\alpha}_{L^p} > 0$, *and (3.2.12) becomes*

$$\int_0^{t/2} \left(|f|_\mu^*(s) - |f|_\mu^*(t/2)\right)^p ds \leq c_p \left(K\left(\frac{t}{I_\Omega(t)}, f; L^p, S_{L^p}\right)\right)^p. \tag{3.2.17}$$

In particular, when $p = 1$, *the left hand side of (3.2.17) becomes*

$$\frac{t}{2}(|f|_\mu^{**}(t/2) - |f|_\mu^*(t/2)) = \int_0^{t/2} \left(f_\mu^*(s) - f_\mu^*(t/2)\right) ds.$$

As a consequence, when $X = L^1$ and $0 < t < \mu(\Omega)/4$, (3.2.12) and (3.1.1) represent the same inequality, modulo constants.

The next easy variant of Theorem 7 gives more flexibility for some applications.

Theorem 9. — *Let X and Y be a r.i. spaces on Ω. Then, for each $f \in X + S_Y$ we have*

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \leq c\frac{K\left(\frac{t}{I_\Omega(t)}\frac{\phi_X(t)}{\phi_Y(t)}, f; X, S_Y\right)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega). \tag{3.2.18}$$

Proof. — Since the proof is almost the same as the proof of Theorem 7 we shall only briefly indicate the necessary changes. Let $f = f_0 + f_1$ be a decomposition of f, then using estimate (3.2.10), with Y instead of X, we get

$$|\nabla f_1|_\mu^{**}(t) \leq \frac{\||\nabla f_1|\|_Y}{\phi_Y(t)}.$$

Therefore,

$$\frac{t}{I_\Omega(t)}|\nabla f_1|_\mu^{**}(t) \leq \frac{\phi_X(t)}{\phi_X(t)}\frac{t}{I_\Omega(t)}\frac{\||\nabla f_1|\|_Y}{\phi_Y(t)}. \tag{3.2.19}$$

Inserting (3.2.9) and (3.2.19) back in (3.2.8) we find that, for $0 < t < \mu(\Omega)$,

$$\begin{aligned} |f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) &\leq \frac{\|f_0\|_X}{\phi_X(t)} + \frac{\phi_X(t)}{\phi_X(t)}\frac{t}{I_\Omega(t)}\frac{\||\nabla f_1|\|_Y}{\phi_Y(t)} \\ &\leq \frac{1}{\phi_X(t)}\left(\|f_0\|_X + \phi_X(t)\frac{t}{I_\Omega(t)}\frac{\||\nabla f_1|\|_Y}{\phi_Y(t)}\right). \end{aligned}$$

The desired result follows taking infimum over all decompositions of f. □

Remark 3. — *Obviously if there exists a constant $c > 0$ such that for all t*

$$\phi_X(t) \leq c\phi_Y(t), \tag{3.2.20}$$

then for each $f \in X + S_Y$,

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \leq 2\frac{K\left(\frac{ct}{I_\Omega(t)}, f; X, S_Y\right)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega).$$

Note that $Y \subset X$ implies that (3.2.20) holds.

3.3. A variant of Theorem 7

This section is devoted to the proof of an improvement of Theorem 7 that will play an important role in Chapter 4. In this variant we shall replace the variable inside the $K-$functional, namely $\frac{t}{I_\Omega(t)}$, by a smaller function that depends on the space X. The relevant functions are defined next.

Definition 2. — *Let X be a r.i. space on Ω. For $t \in (0, \mu(\Omega))$ we let*

$$\psi_{X,\Omega}(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, \tag{3.3.1}$$

$$\Psi_{X,\Omega}(t) = \phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}. \tag{3.3.2}$$

If either X and/or Ω are clear from the context we shall drop the corresponding sub-index. In the next Lemma we collect a few elementary remarks connected with these functions.

Lemma 3

(i) The function $\psi_X(t)$ is always finite, and in fact

$$\psi_X(t) \leq \frac{t}{I_\Omega(t)}, \quad t \in (0, \mu(\Omega)).$$

(ii) In the isoperimetric case there is no improvement: When $X = L^1$,

$$\psi_{L^1}(t) = \frac{t}{I_\Omega(t)}.$$

(iii) We always have

$$\psi_X \leq \Psi_X.$$

(iv) The function Ψ_X is increasing

(v) A necessary and sufficient condition for $\Psi_X(t)$ to take only finite values is

$$\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} < \infty. \tag{3.3.3}$$

Proof

(i) Since $\frac{s}{I_\Omega(s)}$ is increasing,

$$\begin{aligned}
\frac{\phi_X(t)}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} &\leq \frac{\phi_X(t)}{t} \frac{t}{I_\Omega(t)} \left\| \chi_{(0,t)}(s) \right\|_{\bar{X}'} \\
&= \frac{\phi_X(t)}{t} \frac{t}{I_\Omega(t)} \phi_{X'}(t) \\
&= \frac{t}{I_\Omega(t)}.
\end{aligned}$$

(ii) For $X = L^1$,

$$\begin{aligned}
\psi_{L^1}(t) &= \frac{\phi_{L^1}(t)}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{L^\infty} \\
&= \frac{t}{t} \sup_{s<t} \left\{ \frac{s}{I_\Omega(s)} \right\} \\
&= \frac{t}{I_\Omega(t)}.
\end{aligned}$$

(iii) Since $s \uparrow$

$$\begin{aligned}\psi_X(t) &= \frac{\phi_X(t)}{t}\left\|\frac{s}{I_\Omega(s)}\chi_{(0,t)}(s)\right\|_{\bar{X}'}\\ &\leq \phi_X(t)\left\|\frac{1}{I_\Omega(s)}\chi_{(0,t)}(s)\right\|_{\bar{X}'}\\ &= \Psi_X(t).\end{aligned}$$

(iv) By inspection $\left\|\frac{1}{I_\Omega(s)}\chi_{(0,t)}(s)\right\|_{\bar{X}'}$ increases in the variable t.

(v) By (iv) for $t \in (0, \mu(\Omega))$,

$$\Psi_X(t) \leq \Psi_X(\mu(\Omega)) = \phi_X(\mu(\Omega))\left\|\frac{1}{I_\Omega(s)}\right\|_{\bar{X}'}.$$

On the other hand, by the triangle inequality

$$\Psi_X(\mu(\Omega)/2) \geq \phi_X(\mu(\Omega)/2)\left\|\frac{1}{I_\Omega(s)}\right\|_{\bar{X}'} - \phi_X(\mu(\Omega)/2)\left\|\frac{\chi_{(\mu(\Omega)/2,\mu(\Omega))}(s)}{I_\Omega(s)}\right\|_{\bar{X}'}$$

It follows that

$$\phi_X(\mu(\Omega)/2)\left\|\frac{1}{I_\Omega(s)}\right\|_{\bar{X}'} \leq 2\Psi_X(\mu(\Omega)/2). \qquad \square$$

Remark 4. — *Unless mention to the contrary, we shall always assume that (3.3.3) holds when dealing with the functions introduced in Definition 2.*

Theorem 10. — *Let X be a r.i. space on Ω. Then, for all $f \in X + S_X$,*

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \leq 8\frac{K(\psi_X(2t), f; X, S_X)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega)/2. \tag{3.3.4}$$

Remark 5. — *Lemma 3 (i) implies that (3.3.4) is stronger than (3.1.1).*

Proof of Theorem 10. — Let $f \in X + S_X$, be bounded. Proceeding as in the proof of Theorem 7 we can show that for any $h \in S_X$, with $0 \leq h \leq \|f\|_{L^\infty}$,

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \leq 4\,\||f| - h|_\mu^{**}(t) + 2(|h|_\mu^{**}(2t) - |h|_\mu^{*}(2t)), \quad 0 < t < \mu(\Omega)/2.$$

The first term on the right hand side was estimated in (3.2.9)

$$4\,\||f| - h|_\mu^{**}(t) \leq 4\,\|(|f| - h)\|_X \frac{1}{\phi_X(t)}.$$

To estimate $|h|_{\mu}^{**}(2t) - |h|_{\mu}^{*}(2t)$, consider $\{h_n\}_{n\in N}$, the sequence of *Lip* functions associated to h that is provided by Lemma 2. Then

$$\begin{aligned}
|h_n|_{\mu}^{**}(2t) - |h_n|_{\mu}^{*}(2t) &= \frac{1}{2t}\int_0^{2t} s\left(-|h_n|_{\mu}^{*}\right)'(s)\,ds \text{ (integration by parts)}\\
&= \frac{1}{2t}\int_0^{2t} s\left(-|h_n|_{\mu}^{*}\right)'(s) I_{\Omega}(s)\frac{ds}{I(s)}\\
&\leq \frac{1}{2t}\left\|\frac{s}{I_{\Omega}(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\left\|\left(-|h_n|_{\mu}^{*}\right)'(s)I_{\Omega}(s)\right\|_{\bar{X}}\\
&\quad \text{(by Hölder's inequality)}\\
&\leq \frac{1}{2t}\left\|\frac{s}{I_{\Omega}(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\||\nabla h_n|\|_X \quad \text{(by (2.1.8))}\\
&\leq \frac{1}{2t}\left\|\frac{s}{I_{\Omega}(s)}\chi_{(0,2t)}(z)\right\|_{\bar{X}'}(1+\frac{1}{n})\||\nabla h|\|_X\\
&\quad \text{(by (2.1.6))}.
\end{aligned}$$

On the other hand, from (2.1.7) and Lemma 1, we have $|h_n|_{\mu}^{*}(s) \to |h|_{\mu}^{*}(s)$, and $|h_n|_{\mu}^{**}(s) \to |h|_{\mu}^{**}(s)$. Consequently,

$$|h|_{\mu}^{**}(2t) - |h|_{\mu}^{*}(2t) \leq \frac{1}{2t}\left\|\frac{s}{I_{\Omega}(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\||\nabla h|\|_X .$$

Summarizing,

$$\begin{aligned}
|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) &\leq 4\frac{\||f|-h\|_X}{\phi_X(t)} + \frac{1}{2t}\left\|\frac{s}{I_{\Omega}(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\||\nabla h|\|_X\\
&= \frac{4}{\phi_X(t)}\left(\||f|-h\|_X + \frac{\phi_X(t)}{2t}\left\|\frac{s}{I_{\Omega}(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\||\nabla h|\|\right)_X\\
&\leq \frac{4}{\phi_X(t)}\left(\||f|-h\|_X + \psi_X(2t)\||\nabla h|\|\right)_X .
\end{aligned}$$

Thus,

$$\begin{aligned}
|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) &\leq \frac{4}{\phi_X(t)}\inf_{0\leq h\leq \|f\|_{\infty}, h\in S_X}\{\||f|-h\|_X + \psi(2t)\||\nabla h|\|\}\\
&\leq \frac{8}{\phi_X(t)}\frac{K(\psi_X(2t), f; X, S_X)}{\phi_X(t)} \quad \text{(by (3.2.13))}.
\end{aligned}$$

When f is not bounded we consider the sequence $f_n = \min(|f|, n) \nearrow |f|$, and we proceed as in the proof of Theorem 8. □

3.4. Isoperimetry

In this section we show the connection of (3.1.1) and (3.3.4) with isoperimetry. Observe that (3.1.1) holds for all r.i. spaces. In particular, it holds for $X = L^1$.

Theorem 11. — *Let G be a continuous function on $(0,\mu(\Omega))$ which is zero at zero and symmetric around $\mu(\Omega)/2$. Then the following are equivalent*

i) Isoperimetric inequality*: There exists an absolute constant $c>0$, such that*

$$G(t)\leq cI_{\Omega}(t),\ 0<t\leq\mu(\Omega). \tag{3.4.1}$$

ii) There exists an absolute constant $c>0$ such that for each $f\in L^1$,

$$|f|_{\mu}^{**}(t)-|f|_{\mu}^{*}(t)\leq c\frac{K\left(\frac{t}{G(t)},f;L^1,S_{L^1}\right)}{t},\ 0<t\leq\mu(\Omega)/2. \tag{3.4.2}$$

Proof

(i) $\Rightarrow$(ii). Since

$$|f|_{\mu}^{**}(t)-|f|_{\mu}^{*}(t)\leq c\frac{K\left(\frac{t}{I_{\Omega}(t)},f;L^1,S_{L^1}\right)}{t},\ 0<t\leq\mu(\Omega)/2.$$

it follows that (3.4.1) implies (3.4.2).

(ii) $\Rightarrow$ (i). Suppose that A is a Borel set with $0<\mu(A)<\mu(\Omega)/2$. We may assume, without loss, that $P(A;\Omega)<\infty$. By [**18**, Lemma 3.7] we can select a sequence $\{f_n\}_{n\in N}$ of *Lip* functions such that $f_n\underset{L^1}{\rightarrow}\chi_A$, and

$$P(A;\Omega)\geq\lim\sup_{n\rightarrow\infty}\left\||\nabla f_n|\right\|_{L^1}.$$

Going through a subsequence, if necessary, we can actually assume that for all n we have

$$P(A;\Omega)\geq\left\||\nabla f_n|\right\|_{L^1}.$$

From (3.4.2) we know that there exists a constant $c>0$ such that for all $0<t\leq\mu(\Omega)/2$,

$$|f_n|_{\mu}^{**}(t)-|f_n|_{\mu}^{*}(t)\leq c\frac{K\left(\frac{t}{G(t)},f_n;L^1,S_{L^1}\right)}{t}.$$

We take limits when $n\rightarrow\infty$ on both sides of this inequality. To compute on the left hand side we observe that, since, $f_n\underset{L^1}{\rightarrow}\chi_A$, Lemma 1 implies that:

$$|f_n|_{\mu}^{**}(t)\rightarrow|\chi_A|_{\mu}^{**}(t),\text{ uniformly for }t\in[0,1],\text{ and}$$
$$|f_n|_{\mu}^{*}(t)\rightarrow|\chi_A|_{\mu}^{*}(t)\text{ for }t\in(0,1).$$

Fix $1/2>r>\mu(A)$. We have

$$\begin{aligned}\lim_{n\rightarrow\infty}\left(|f_n|_{\mu}^{**}(r)-|f_n|_{\mu}^{*}(r)\right)&=(\chi_A)_{\mu}^{**}(r)-(\chi_A)_{\mu}^{*}(r)\\&=(\chi_A)_{\mu}^{**}(r)\ (\text{since }\ (\chi_A)_{\mu}^{*}(r)=\chi_{(0,\mu(A))}(r)=0)\\&=\frac{\mu(A)}{r}.\end{aligned}$$

Now, to estimate the right hand side we observe that, for each n, $f_n \in L^1 \cap S_{L^1}$. Consequently, by the definition of K-functional, we have

$$\begin{aligned}\frac{K(\frac{t}{G(t)}, f_n; L^1, S_{L^1})}{t} &\leq \min\left\{\frac{\|f_n\|_{L^1}}{t}, \frac{1}{G(t)}\,|||\nabla f_n|||_{L^1}\right\}\\ &\leq \min\left\{\frac{\|f_n\|_{L^1}}{t}, \frac{1}{G(t)}P(A;\Omega)\right\}.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{K(\frac{t}{G(t)}, f_n; L^1, S_{L^1})}{t} \leq \min\left\{\frac{\mu(A)}{t}, \frac{1}{G(t)}P(A;\Omega)\right\}.$$

Combining these estimates we find that for all $1/2 > r > \mu(A)$,

$$\frac{\mu(A)}{r} \leq c\frac{1}{G(r)}P(A;\Omega).$$

Let $r \to \mu(A)$ then, by the continuity of G, we find

$$1 \leq c\frac{1}{G(\mu(A))}P(A;\Omega),$$

or

$$G(\mu(A)) \leq cP(A;\Omega).$$

Thus,

$$\begin{aligned}G(\mu(A)) &\leq c\inf\{P(B;\Omega) : \mu(B) = \mu(A)\}\\ &= cI_\Omega(\mu(A)).\end{aligned}$$

Suppose now that $t \in (\mu(\Omega)/2, \mu(\Omega))$. Then $1 - t \in (0, \mu(\Omega))$ and by symmetry,

$$G(t) = G(1-t) \leq cI(1-t) = cI(t),$$

and we are done. □

CHAPTER 4

EMBEDDING INTO CONTINUOUS FUNCTIONS

4.1. Introduction and Summary

In this chapter we obtain a general version of the Morrey-Sobolev theorem on metric measure spaces (Ω, d, μ) satisfying the usual assumptions.

4.1.1. Inequalities for signed rearrangements. — Let (Ω, d, μ) be a metric measure space satisfying the usual assumptions. We collect a few more facts about signed rearrangements. First let us note that for $c \in \mathbb{R}$,

$$(f + c)_{\mu}^{*}(t) = f_{\mu}^{*}(t) + c. \tag{4.1.1}$$

Moreover, if $X(\Omega)$ is a r.i. space, we have

$$\| \, |f|_{\mu}^{*} \, \|_{\bar{X}(0,\mu(\Omega))} = \| |f| \|_{X(\Omega)} = \|f\|_{X(\Omega)} = \|f_{\mu}^{*}\|_{\bar{X}(0,\mu(\Omega))},$$

where $\bar{X}(0, \mu(\Omega))$ is the representation space of $X(\Omega)$.

The results of the previous chapter can be easily formulated in terms of signed rearrangements. In particular, we shall now discuss in detail the following extension (variant) of Theorem 10.

Theorem 12. — *Let X be a r.i. space on Ω. Then, for all $f \in X + S_X$, we have,*

$$f_{\mu}^{**}(t) - f_{\mu}^{*}(t) \leq 8 \frac{K(\psi(2t), f; X, S_X)}{\phi_X(t)}, \quad 0 < t \leq \mu(\Omega)/2, \tag{4.1.2}$$

where $\psi(t) := \psi_{X,\Omega}(t) = \frac{\phi_X(t)}{s} \left\| \frac{s}{I_{\Omega}(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}$ is the function introduced in (3.3.1)).

Proof. — Let us first further assume that f is bounded below, and let $c = \inf_{\Omega} f$. We can then apply Theorem 10 to the positive function $f - c$, and we obtain

$$(f - c)_{\mu}^{**}(t) - (f - c)_{\mu}^{*}(t) \leq 8 \frac{K(\psi(2t), f - c; X, S_X)}{\phi_X(t)}. \tag{4.1.3}$$

We can simplify the left hand side of (4.1.3) using (4.1.1)

$$(f-c)_\mu^{**}(t)-(f-c)_\mu^{*}(t)=f_\mu^{**}(t)-f_\mu^{*}(t).$$

On the other hand, the sub-additivity of the $K-$functional, and the fact that it is zero on constant functions, allows us to estimate the right hand side of (4.1.3) as follows

$$\begin{aligned} K(\psi(t),f-c;X,S_X) &\leq K(\psi(t),f;X,S_X)+K(\psi(t),c;X,S_X) \\ &= K(\psi(t),f;X,S_X). \end{aligned}$$

Combining these observations we see that,

$$f_\mu^{**}(t)-f_\mu^{*}(t)\leq 8\frac{K(\psi(2t),f;X,S_X)}{\phi_X(t)}.$$

If f is not bounded from below, we use an approximation argument. Let

$$f_n=\max(f,-n),\quad n=1,2,\ldots$$

Then by the previous discussion we have

$$(f_n)_\mu^{**}(t)-(f_n)_\mu^{*}(t)\leq 8\frac{K(\psi(2t),f_n;X,S_X)}{\phi_X(t)}.$$

Now $f_n(x)\to f(x)$ $\mu-$a.e., with $|f_n|\leq|f|$, therefore, L^1 convergence follows by dominated convergence, and we can then apply Lemma 1 to conclude that

$$f_\mu^{**}(t)-f_\mu^{*}(t)\leq 8\frac{K(\psi(2t),f_n;X,S_X)}{\phi_X(t)}.$$

We estimate the right hand side as follows. Given $\varepsilon>0$, select $h^\varepsilon\in S_X$ such that

$$\|f-h^\varepsilon\|_X+\psi(t)\,\||\nabla h^\varepsilon|\|_X\leq K(\psi(t),f;X,S_X)+\varepsilon. \tag{4.1.4}$$

For each $n\in\mathbb{N}$ let us define $h_n^\varepsilon=\max(h^\varepsilon,-n)$. Then

$$h_n^\varepsilon\in S_X\quad\text{with}\quad|\nabla h_n^\varepsilon|\leq|\nabla h^\varepsilon|. \tag{4.1.5}$$

By a straightforward analysis of all possible cases we see that

$$\|f_n-h_n^\varepsilon\|_X\leq\|f-h^\varepsilon\|_X. \tag{4.1.6}$$

Therefore, combining (4.1.4), (4.1.5) and (4.1.6), we obtain

$$K(\psi(t),f_n;X,S_X)\leq K(\psi(t),f;X,S_X),$$

thus

$$f_\mu^{**}(t)-f_\mu^{*}(t)\leq 8\frac{K(\psi(2t),f;X,S_X)}{\phi_X(t)},\quad 0<t\leq\mu(\Omega)/2.$$

and (4.1.2) follows. □

4.2. Continuity *via* rearrangement inequalities

In this section we consider the following problems: Characterize, in terms of $K-$functional conditions, the functions in $f \in X(\Omega) + S_X(\Omega)$ that are bounded, or essentially continuous. One can rephrase these questions as suitable embedding theorems for Besov type spaces.

We consider boundedness first.

Lemma 4. — *Let (Ω, d, μ) be a metric measure space, and let $X(\Omega)$ be a r.i. space. Then if $f \in X + S_X$ is such that*

$$\int_0^{\mu(\Omega)} \frac{K(\Psi(t), f; X, S_X)}{\phi_X(t/2)} \frac{dt}{t} < \infty,$$

then f is essentially bounded, where $\Psi(t) := \Psi_{X,\Omega}(t) = \phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}$ is the function introduced in (3.3.2).

Proof. — To simplify the notation we shall let $K(t, f) := K(t, f; X, S_X)$. By Lemma 3 (i) and Theorem 10, we have

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \leq 8 \frac{K(\Psi(t), f)}{\phi_X(t/2)}, \quad 0 < t \leq \mu(\Omega). \tag{4.2.1}$$

Fix $0 < r < \frac{\mu(\Omega)}{2}$; then integrating both sides of (4.2.1) from r to $\mu(\Omega)$, we find

$$\int_r^{\mu(\Omega)} \left(|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \right) \frac{dt}{t} \leq 8 \int_r^{\mu(\Omega)} \frac{K(\Psi(t), f)}{\phi_X(t/2)} \frac{dt}{t}.$$

We can compute the left hand side using the fundamental theorem of calculus

$$\int_r^{\frac{\mu(\Omega)}{2}} \left(|f|_\mu^{**}(u) - |f|_\mu^{*}(u) \right) \frac{du}{u} = f^{**}(r) - f^{**}\left(\frac{\mu(\Omega)}{2} \right),$$

thus,

$$f^{**}(r) - f^{**}\left(\frac{\mu(\Omega)}{2} \right) \leq 8 \int_r^{\mu(\Omega)} \frac{K(\Psi(t), f)}{\phi_X(t/2)} \frac{dt}{t}.$$

Therefore,

$$f^{**}(r) \leq \frac{2}{\mu(\Omega)} \int_0^{\frac{\mu(\Omega)}{2}} |f|_\mu^{*}(s)\, ds + 8 \int_r^{\mu(\Omega)} \frac{K(\Psi(t), f; X, S_X)}{\phi_X(t/2)} \frac{dt}{t}.$$

Thus, letting $r \to 0$,

$$\|f\|_{L^\infty} \leq \frac{2}{\mu(\Omega)} \int_0^{\frac{\mu(\Omega)}{2}} |f|_\mu^{*}(s)\, ds + 8 \int_0^{\mu(\Omega)} \frac{K(\Psi(t), f; X, S_X)}{\phi_X(t/2)} \frac{dt}{t},$$

and the result follows. □

To study essential continuity it will be useful to introduce some notation. Let G be an open subset of Ω. Recall (cf. Section 2.2) that $X_r(G) = X(G, d_{|G}, \mu_{|G})$. When the open set G is understood from the context, we shall simply write X_r and

S_{X_r}. We shall denote by $\bar{X}_r = \bar{X}_r(0, \mu(G))$ the representation space of X_r, and we let X_r' denote the corresponding associated space of X_r (For more information see Section 2.2, (2.2.13).)

If $f \in X(\Omega) + S_X(\Omega)$, then we obviously have that $f\chi_G \in X_r(G) + S_{X_r}(G)$. However, we can not apply our fundamental inequalities (3.1.1), (3.3.2) since we are now working in the metric space $(G, d_{|G}, \mu_{|G})$ and therefore the isoperimetric profile has changed.

Given $G \subset \Omega$ an open subset, and let $A \subset G$. The **perimeter** of A **relative** to G is defined by

$$P(A; G) = \lim\inf_{h \to 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in G : d(x, A) < h\}$. Obviously

$$P(A; G) \leq P(A; \Omega).$$

The **relative isoperimetric profile** of $G \subset \Omega$ is defined by (see for example [**4**] and the references quoted therein)

$$I_G(s) = I_{(G,d,\mu)}(s) = \inf\{P(A; G) : \ A \subset G, \ \mu(A) = s\}, \quad 0 < s < \mu(G).$$

We say that an **isoperimetric inequality relative** to G holds true if there exists a positive constant C_G such that

$$I_G(s) \geq C_G \min(I_\Omega(s), I_\Omega(\mu(G) - s)) = J_G(t), \quad 0 < s < \mu(G),$$

where I_Ω is the isoperimetric profile of (Ω, d, μ). Notice that, if $\mu(G) \leq \mu(\Omega)/2$, then $J_G : [0, \mu(G)] \to [0, \infty)$ is increasing on $(0, \mu(G)/2)$, symmetric around the point $\mu(G)/2$, and such that

$$I_G \geq J_G,$$

i.e., J_G is an isoperimetric estimator for the metric space $(G, d_{|G}, \mu_{|G})$.

Definition 3. — *We will say that a metric measure space (Ω, d, μ) has the* ***relative uniform isoperimetric property*** *if there is a constant C such that for any ball B in Ω, its* ***relative isoperimetric profile*** *I_B satisfies:*

$$I_B(s) \geq C \min(I_\Omega(s), I_\Omega(\mu(B) - s)), \quad 0 < s < \mu(B).$$

The following proposition will be useful in what follows

Proposition 2. — *Let J be an isoperimetric estimator of (Ω, d, μ). Let $G \subset \Omega$ be an open set with $\mu(G) \leq \mu(\Omega)/2$, and let $Z = Z([0, \mu(G)])$ be a r.i. space on $[0, \mu(G)]$. Then*

$$R(t) := \left\| \frac{1}{\min(J(s), J(\mu(G) - s))} \chi_{(0,t)}(s) \right\|_Z \leq 2 \left\| \frac{1}{J(s)} \chi_{(0,t)}(s) \right\|_Z, \quad 0 < t \leq \mu(G).$$

Proof. — Since $\min(I_\Omega(s), I_\Omega(\mu(G)-s))$ is an isoperimetric estimator for $(G, d_{|G}, \mu_{|G})$, we have

$$\begin{aligned} R(t) &\leq \left\| \frac{1}{\min(I_\Omega(s), I_\Omega(\mu(G)-s))} \chi_{(0,\mu(G)/2)}(s) \right\|_Z + \\ &\quad \left\| \frac{1}{\min(I_\Omega(s), I_\Omega(\mu(G)-s))} \chi_{(\mu(G)/2,t)}(s) \right\|_Z \\ &= \left\| \frac{1}{I_\Omega(s)} \chi_{(0,\mu(G)/2)}(s) \right\|_Z + \left\| \frac{1}{I_\Omega(\mu(G)-s)} \chi_{(\mu(G)/2,t)}(s) \right\|_Z \\ &\leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_Z + \left\| \frac{1}{I_\Omega(\mu(G)-s)} \chi_{(\mu(G)/2,t)}(s) \right\|_Z . \end{aligned}$$

To estimate the second term on the right hand side, notice that, by the properties of $I_\Omega(s)$, the functions

$$\frac{1}{I_\Omega(\mu(G)-s)} \chi_{(\mu(G)/2,t)}(s) \text{ and } \frac{1}{I_\Omega(s)} \chi_{(\mu(G)-t,\mu(G)/2)}(s)$$

are equimeasurable (with respect to the Lebesgue measure), consequently

$$\begin{aligned} \left\| \frac{1}{I_\Omega(\mu(G)-s)} \chi_{(\mu(G)/2,t)}(s) \right\|_Z &= \left\| \frac{1}{I_\Omega(s)} \chi_{(\mu(G)-t,\mu(G)/2)}(s) \right\|_Z \\ &= \left\| \left(\frac{1}{I_\Omega(s)} \chi_{(\mu(G)-t,\mu(G)/2)} \right)^* (s) \right\|_Z . \end{aligned}$$

Since $1/I_\Omega(s)$ is decreasing on $(0, \mu(G)/2)$,

$$\left\| \left(\frac{1}{I_\Omega(s)} \chi_{(\mu(G)-t,\mu(G)/2)} \right)^* (s) \right\|_Z \leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t-\mu(G)/2)}(s) \right\|_Z \leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_Z .$$

Consequently,

$$R(t) \leq 2 \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_Z . \qquad \square$$

Theorem 13. — *Let (Ω, d, μ) be a metric measure space with the relative uniform isoperimetric property and let X be a r.i. space on Ω. Then all the functions in $f \in X + S_X$ that satisfy the condition*

$$\int_0^{\mu(\Omega)} \frac{K(\Psi_{X,\Omega}(t), f; X, S_X)}{\phi_X(t)} \frac{dt}{t} < \infty, \tag{4.2.2}$$

are essentially continuous.

Proof. — We will show that there exists a universal constant $c > 0$ such that for any function that satisfies (4.2.2) we have: For all balls B with $\mu(B) < \mu(\Omega)/2$,

$$|f(x) - f(y)| \leq c \int_0^{2\mu(B)} \frac{K(\Psi_{X,\Omega}(t), f; X, S_X)}{\phi_X(t)} \frac{dt}{t},$$

for μ–almost every $x, y \in B$.

We shall use the following notation: $X = X(\Omega)$, $S_X = S_X(\Omega)$, $X_r = X_r(B)$, $S_{X_r} = S_{X_r}(B)$, $\psi_B(t) = \psi_{B,X_r}(t)$, and $\psi_B = \psi_{X_r,B}$, $\Psi_B(t) = \Psi_{X_r,B}$, $\Psi_X = \Psi_{X,\Omega}$.

Given $f \in X + S_X$, then $f\chi_B \in X_r + S_{X_r}$. By Theorem 12

$$(4.2.3)\quad \begin{aligned}(f\chi_B)^{**}_{\mu}(t) - (f\chi_B)^{*}_{\mu}(t) &= (f\chi_B)^{**}_{\mu_{|B}}(t) - (f\chi_B)^{*}_{\mu_{|B}}(t)\\ &\le 8\frac{K\left(\psi_B(2t), f\chi_B; X_r, S_{X_r}\right)}{\phi_{X_r}(t)},\quad 0 \le t \le \mu(B)/2.\end{aligned}$$

By (2.2.12),

$$\frac{K\left(\psi_B(2t), f\chi_B; X_r, S_{X_r}\right)}{\phi_{X_r}(t)} \le \frac{K\left(\psi_B(2t), f; X, S_X\right)}{\phi_X(t)},\quad 0 \le t \le \mu(B)/2.$$

On the other hand, for $0 \le t \le \mu(B)/2$,

$$(4.2.4)\quad \begin{aligned}\psi_B(2t) &= \frac{\phi_{X_r}(2t)}{2t}\left\|\frac{s}{I_B(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'_r}\\ &\le \phi_{X_r}(2t)\left\|\frac{1}{I_B(s)}\chi_{(02,t)}(s)\right\|_{\bar{X}'_r}\\ &= \phi_X(2t)\left\|\frac{1}{I_B(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\\ &\le c\phi_X(2t)\left\|\frac{1}{\min(I_\Omega(B), I_\Omega(\mu(B)-s))}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\\ &\le c\phi_X(2t)\left\|\frac{1}{I_\Omega(s)}\chi_{(0,2t)}(s)\right\|_{\bar{X}'}\quad \text{(by Proposition 2)}\\ &= c\Psi_X(2t).\end{aligned}$$

Consequently for $0 < t \le \mu(B)/2$, we have

$$(4.2.5)\quad \begin{aligned}(f\chi_B)^{**}_{\mu}(t) - (f\chi_B)^{*}_{\mu}(t) &\le 8\frac{K\left(c\Psi_X(2t), f; X, S_X\right)}{\phi_X(t)}\\ &\le 8C\frac{K\left(\Psi_X(2t), f; X, S_X\right)}{\phi_X(t)}\\ &= A(2t).\end{aligned}$$

Change variables: Let $t = zr$, with $r = \frac{\mu(B)}{2\mu(\Omega)}$, $0 < z \le \mu(\Omega)$. Then,

$$(f\chi_B)^{**}_{\mu}(zr) - (f\chi_B)^{*}_{\mu}(zr) \le A(2zr),\qquad 0 < z \le \mu(\Omega).$$

Integrating the previous inequality we obtain

$$\begin{aligned}\int_0^{\mu(\Omega)}\left[(f\chi_B)^{**}_{\mu}(zr) - (f\chi_B)^{*}_{\mu}(zr)\right]\frac{dz}{z} &\le \int_0^{\mu(\Omega)} A(2zr)\frac{dz}{z}\\ &= \int_0^{\mu(B)} A(u)\frac{du}{u}.\end{aligned}$$

Using the formula

$$\frac{d}{dz}\left\{-\left(f\chi_B\right)_\mu^{**}(zr)\right\} = \frac{\left(f\chi_B\right)_\mu^{**}(zr) - \left(f\chi_B\right)_\mu^{*}(zr)}{z},$$

we get

$$\begin{aligned} ess\sup(f\chi_B) - \left(f\chi_B\right)_\mu^{**}(\mu(B)/2) &= \left(f\chi_B\right)_\mu^{**}(0) - \left(f\chi_B\right)_\mu^{**}(\mu(B)/2) \\ &= \int_0^{\mu(\Omega)} \left[\left(f\chi_B\right)_\mu^{**}(zr) - \left(f\chi_B\right)_\mu^{*}(zr)\right]\frac{dz}{z} \\ &\le \int_0^{\mu(B)} A(z)\frac{dz}{z}. \end{aligned}$$

Similarly, considering $-f\chi_B$ instead of $f\chi_B$, we obtain

$$ess\sup(-f\chi_B) - \left(-f\chi_B\right)_\mu^{**}(\mu(B)/2) \le \int_0^{\mu(B)} A(z)\frac{dz}{z}.$$

Adding both inequalities

$$ess\sup(f\chi_B) - ess\inf(f\chi_B) \le \tag{4.2.6}$$
$$2\int_0^{\mu(B)} A(z)\frac{dz}{z} + \left[\left(f\chi_B\right)_\mu^{**}(\mu(B)/2) + \left(-f\chi_B\right)_\mu^{**}(\mu(B)/2)\right]$$

To estimate the last term on the right hand side we let $t = \mu(B)/2$ in (4.2.3),

$$\left[\left(f\chi_B\right)_\mu^{**}(\mu(B)/2) + \left(-f\chi_B\right)_\mu^{**}(\mu(B)/2)\right] \le 8\frac{K\left(\psi_B(\mu(B)), f\chi_B; X_r, S_{X_r}\right)}{\phi_{X_r}(\mu(B)/2)},$$

and we do the same thing for the corresponding estimate for $-f$,

$$\left[\left(-f\chi_B\right)_\mu^{**}(\mu(B)/2) + \left(-f\chi_B\right)_\mu^{**}(\mu(B)/2)\right] \le 8\frac{K\left(\psi_B(\mu(B)), f\chi_B; X_r, S_{X_r}\right)}{\phi_{X_r}(\mu(B)/2)}.$$

Adding these inequalities, and recalling that, since $(-f\chi_B)_\mu^*(t) = (-f\chi_B)_{\mu_{|B}}^*(t) = -(f\chi_B)_{\mu_{|B}}^*(\mu(B)-t) = -(f\chi_B)_\mu^*(\mu(B)-t)$, we have

$$(-f\chi_B)_\mu^*(\mu(B)/2) = -(f\chi_B)_\mu^*(\mu(B)/2),$$

we see that

$$\left(f\chi_B\right)_\mu^{**}(\mu(B)/2) + \left(-f\chi_B\right)_\mu^{**}(\mu(B)/2) \le 16\frac{K\left(\psi_B(\mu(B)), f\chi_B; X_r, S_{X_r}\right)}{\phi_{X_r}(\mu(B)/2)}.$$

Inserting this information back in (4.2.6) we obtain

$$\begin{aligned}
&ess\ \sup(f\chi_B) - ess\ \inf(f\chi_B)\\
&\leq 2\int_0^{\mu(B)} A(z)\frac{dz}{z} + 16\frac{K\left(\psi_B(\mu(B)), f\chi_B; X_r, S_{X_r}\right)}{\phi_{X_r}(\mu(B)/2)}\\
&\leq C\left(\int_0^{\mu(B)} \frac{K\left(\Psi_X(t), f; X, S_X\right)}{\phi_X(t/2)}\frac{dz}{z} + 16\frac{K\left(\Psi_X(\mu(B)), f; X, S_X\right)}{\phi_X(\mu(B)/2)}\right)\\
&\leq C\left(\int_0^{\mu(B)} \frac{K\left(\Psi_X(t), f; X, S_X\right)}{\phi_X(t)}\frac{dz}{z} + \frac{K\left(\Psi_X(\mu(B)), f; X, S_X\right)}{\phi_X(\mu(B))}\right).
\end{aligned}$$

Elementary considerations show that the second term on the right hand side can be controlled by the first term. Indeed, we use that $K(t, f)$ increases and $\Psi_X(t)$ increases (cf. Lemma 3 (iv) above) to derive that $K\left(\Psi_X(t), f; X, S_X\right)$ increases, which we combine with the fact that $\phi_{X'}(t)$ increases, and $\phi_{X'}(t)\phi_X(t) = t$, then we obtain

$$\begin{aligned}
\int_0^{2\mu(B)} \frac{K\left(\Psi_X(t), f; X, S_X\right)}{\phi_X(t)}\frac{dt}{t} &= \int_0^{2\mu(B)} \frac{\phi_{X'}(t)K\left(\Psi_X(t), f; X, S_X\right)}{t}\frac{dt}{t}\\
&\geq \int_{\mu(B)}^{2\mu(B)} \frac{\phi_{X'}(t)K\left(\Psi_X(t), f; X, S_X\right)}{t}\frac{dt}{t}\\
&\geq \phi_{X'}(\mu(B))K\left(\Psi_X(\mu(B)), f; X, S_X\right)\int_{\mu(B)}^{2\mu(B)} \frac{1}{t}\frac{dt}{t}\\
&= K\left(\Psi_X(\mu(B)), f; X, S_X\right)\frac{\phi_{X'}(\mu(B))}{2\mu(B)}\\
&= \frac{1}{2}\frac{K\left(\Psi_X(\mu(B)), f; X, S_X\right)}{\phi_X(\mu(B))}
\end{aligned}$$

Thus,

$$ess\ \sup(f\chi_B) - ess\ \inf(f\chi_B) \leq c\int_0^{2\mu(B)} \frac{K\left(\Psi_X(t), f; X, S_X\right)}{\phi_X(t)}\frac{dt}{t}.$$

It follows that for $\mu-$almost every $x, y \in B$,

$$|f(x) - f(y)| \leq c\int_0^{2\mu(B)} \frac{K\left(\Psi_X(t), f; X, S_X\right)}{\phi_X(t)}\frac{dt}{t},$$

and the essential continuity of f follows. □

CHAPTER 5

EXAMPLES AND APPLICATIONS

5.1. Summary

We verify the relative uniform isoperimetric property for a number of concrete examples. As a consequence we shall show in detail how our methods provide a unified treatment of embeddings of Sobolev and Besov spaces into spaces of continuous functions in different contexts.

5.2. Euclidean domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (*i.e.*, a bounded, open and connected set). For a measurable function $u : \Omega \to \mathbb{R}$, let

$$u^+ = \max(u, 0) \text{ and } u^- = \min(u, 0).$$

Let $X = X(\Omega)$ be a r.i. space on Ω. The Sobolev space $W^1_X(\Omega) := W^1_X$ is the space of real-valued weakly differentiable functions on Ω that, together with their first order derivatives, belong to X.

In this setting the basic rearrangement inequality holds for all $f \in W^1_{L^1}$

$$|f|^{**}(t) - |f|^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla f|^*(s)\, ds, \; 0 < t < |\Omega|, \tag{5.2.1}$$

(rearrangements are taken with respect to the Lebesgue measure). We indicate briefly the proof using the method of [**70**].

It is well known (see for example [**5**], [**97**, Theorem 2.1.4]) that if $u \in W^1_{L^1}$ $(= W^1_1)$ then u^+, $u^- \in W^1_1$ and

$$\nabla u^+ = \nabla u \chi_{\{u>0\}} \text{ and } \nabla u^- = \nabla u \chi_{\{u<0\}}.$$

For given a measurable function g and $0 < t_1 < t_2$, the truncation $g_{t_1}^{t_2}$ of g is defined by

$$g_{t_1}^{t_2} = \min\{\max\{0, g - t_1\}, t_2 - t_2\}\}.$$

It follows that if $g \in W_1^1$, then $g_{t_1}^{t_2} \in W_1^1$ and, in fact,

$$\nabla g_{t_1}^{t_2} = \nabla g \chi_{\{t_1 < g < t_2\}}.$$

In other words, W_1^1 is invariant by truncation. On the other hand, given $g \in W_1^1$, the Federer-Fleming-Rishel co-area formula (cf. [**37**]) states that

$$\int_\Omega |\nabla g(x)|\, dx = \int_{-\infty}^{\infty} P_\Omega(g > s)\, ds.$$

Applying this result to $|g|_{t_1}^{t_2}$, we get

$$\begin{aligned}
(5.2.2) \quad \int_{\{t_1 < |g| < t_2\}} |\nabla |g|(x)|\, dx &= \int_0^\infty P_\Omega(|g|_{t_1}^{t_2} > s)\, ds \\
&\geq \int_0^\infty I_\Omega(\mu_{|g|_{t_1}^{t_2}}(s))\, ds \text{ (isoperimetric inequality)} \\
&= \int_0^{t_2 - t_1} I_\Omega(\mu_{|g|_{t_1}^{t_2}}(s))\, ds.
\end{aligned}$$

Observe that, for $0 < s < t_2 - t_1$,

$$|\{|f| \geq t_2\}| \leq \mu_{|f|_{t_1}^{t_2}}(s) \leq |\{|f| > t_1\}|.$$

Consequently, by the properties of I_Ω, we have

$$\int_0^{t_2 - t_1} I(\mu_{|g|_{t_1}^{t_2}}(s))\, ds \geq (t_2 - t_1) \min\left(I_\Omega(|\{|g| \geq t_1\}|), I_\Omega(|\{|g| \geq t_2\}|)\right).$$

For $s > 0$ and $h > 0$, pick $t_1 = |g|^*(s+h)$, $t_2 = |g|^*(s)$, then

$$(5.2.3) \qquad s \leq \left|\left\{|g| \geq |g|^*(s)\right\}\right| \leq \mu_{|g|_{t_1}^{t_2}}(s) \leq \left|\left\{|g(x)| > |g|^*(s+h)\right\}\right| \leq s + h.$$

Combining (5.2.2) and (5.2.3) we have,

$$\left(|g|^*(s) - |g|^*(s+h)\right) \min(I_\Omega(s+h), I_\Omega(s)) \leq \int_{\{|g|^*(s+h) < |g| < |g|^*(s)\}} |\nabla |g|(x)|\, dx.$$

At this stage we can continue as in [**70**], and we obtain that if $f \in W_1^1$, then (5.2.1) holds. Moreover, $|f|^*$ is locally absolutely continuous, and

$$(5.2.4) \qquad \int_0^t \left|(-|f|^*)'(\cdot) I_\Omega(\cdot)\right|^*(s) \leq \int_0^t |\nabla f|^*(s)\, ds.$$

From here, using the same approximation method provided in the proof of Theorem 12 Chapter 4, we find that, if $f \in W_1^1$, then for $0 < t < |\Omega|$,

$$f^{**}(t) - f^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla f|^* (s)\, ds. \tag{5.2.5}$$

Indeed, first assume that f is bounded from below, and let $c = \inf_\Omega f$, then, since $f - c \geq 0$, we can apply (5.2.1) to $f - c$, and we obtain

$$|f - c|^{**} (t) - |f - c|^* (t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla (f - c)|^* (s)\, ds.$$

Since $|f - c|^* (t) = (f - c)^* (t)$ and $f^{**}(t) - f^*(t) = (f - c)^{**} (t) - (f - c)^* (t)$, we get

$$f^{**}(t) - f^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla f|^* (s)\, ds.$$

If f is not bounded from below, let $f_n = \max(f, -n)$, $n = 1, 2, \ldots$ The previous discussion gives

$$\begin{aligned} (f_n)^{**}(t) - (f_n)^*(t) &\leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla f_n|^* (s)\, ds \\ &\leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla f|^* (s)\, ds. \end{aligned}$$

We now take limits. To compute the left hand side we observe that $f_n(x) \to f(x)$ $\mu-$a.e., and $|f_n| \leq |f|$, then by dominated convergence $f_n \to_{L^1} f$. Consequently, by Lemma 1, we have the pointwise convergence $(f_n)^{**}(t) - (f_n)^*(t) \underset{n\to\infty}{\to} f^{**}(t) - f^*(t)$, $(0 < t < \mu(\Omega))$, concluding the proof.

Let $X = X(\Omega)$ be a r.i. space on Ω. The homogeneous Sobolev space $\dot{W}_X^1$ is defined by means of the seminorm

$$\|u\|_{\dot{W}_X^1} := \| |\nabla u| \|_X .$$

We consider the corresponding $K-$functional

$$K(t, f; X, \dot{W}_X^1) = \inf\{\|f - g\|_X + t \|g\|_{\dot{W}_X^1}\}.$$

The previous discussion shows that all the results of Chapters 3 and 4 remain valid for functions in $\dot{W}_X^1$ or $X + \dot{W}_X^1$.

5.2.1. Sobolev spaces defined on Lipschitz domains of $\mathbb{R}^n$.

— We now discuss assumptions on the domain that translate into good estimates for the corresponding isoperimetric profiles.

In this section we consider Sobolev spaces defined on Lipschitz domains of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then the isoperimetric profile, satisfies (see for example [**75**])

$$I_\Omega(t) = c(n)\left(\min(t, |\Omega| - t)\right)^{\frac{n-1}{n}}. \tag{5.2.6}$$

For any open ball that $B_\alpha \subset \Omega$ with $|B_\alpha| = \alpha$, we know that (see for example [**75**] or [**97**])

$$I_{B_\alpha}(t) \geq q(n)\left(\min(t, \alpha - t)\right)^{\frac{n-1}{n}}, \qquad 0 < t < \alpha,$$

where $q(n)$ is a constant that only depends on n. Moreover, since

$$c(n)\left(\min(t, \alpha - t)\right)^{\frac{n-1}{n}} = \min(I_\Omega(t), I_\Omega(\alpha - t)) \qquad 0 < t < \alpha,$$

we see that there is a constant $C = C(n)$ such that

$$I_{B_\alpha}(t) \geq C \min(I_\Omega(t), I_\Omega(\alpha - t)) \qquad 0 < t < \alpha.$$

In particular the metric space $(\Omega, |\cdot|, dm)$ has the relative uniform isoperimetric property.

Theorem 14. — *Let $X = X(\Omega)$ be a r.i. space on Ω, then*

$$\dot{W}_X^1(\Omega) \subset L^\infty \Leftrightarrow \left\| t^{1/n-1}\chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} < \infty \Longleftrightarrow \dot{W}_X^1(\Omega) \subset C_b(\Omega).$$

(Here $C_b(\Omega)$ denotes the space of real valued continuous bounded functions defined on Ω.)

Proof. — Let us first observe that the condition of the Theorem can be reformulated in terms of the isoperimetric profile of Ω as follows,

$$\left\| t^{1/n-1}\chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} \simeq \left\| \frac{1}{I_\Omega(t)} \right\|_{\bar{X}'} < \infty. \tag{5.2.7}$$

Indeed, since I_Ω is symmetric around the point $|\Omega|/2$, it follows that

$$\begin{aligned} \left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} &\leq \left\| \frac{1}{I_\Omega(s)}\chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} + \left\| \frac{1}{I_\Omega(s)}\chi_{(|\Omega|/2,|\Omega|)} \right\|_{\bar{X}'} \\ &= 2\left\| \frac{1}{I_\Omega(s)}\chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} \\ &\leq 2\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'}. \end{aligned}$$

Now (5.2.7) follows since, in view of (5.2.6), we have

$$\left\| \frac{1}{I_\Omega(s)}\chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} \simeq \left\| t^{1/n-1}\chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'}.$$

Suppose that $\left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \simeq \left\|t^{1/n-1}\chi_{(0,|\Omega|/2)}\right\|_{\bar{X}'} < \infty$. Let $f \in \dot{W}^1_X(\Omega)$. Since we have shown in the previous section that $|f|^*$ is locally absolutely continuous (cf. [**60**] and [**70**]) we can write

(5.2.8)

$$\begin{aligned}\|f\|_{L^\infty} - |f|^*(|\Omega|) = |f|^*(0) - |f|^*(|\Omega|) &= \int_0^{|\Omega|} (-|f^*|)'(s)\, ds \\ &= \int_0^{|\Omega|} (-|f^*|)'(s) I_\Omega(s) \frac{ds}{I_\Omega(s)} \\ &\leq \|(-|f^*|)'(s)I_\Omega(s)\|_{\bar{X}} \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \quad \text{(by Hölder's inequality)} \\ &\leq \||\nabla f|\|_X \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \quad \text{(by (5.2.4))}.\end{aligned}$$

We have thus obtained

$$\|f\|_{L^\infty} \leq \|f\|_{L^1} + \||\nabla f|\|_X \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'},$$

which applied to $f - \int_\Omega f$ yields

$$\begin{aligned}\left\|f - \int_\Omega f\right\|_{L^\infty} &\leq \left\|f - \int_\Omega f\right\|_{L^1} + \||\nabla f|\|_X \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \\ &\leq c(|\Omega|)\, \||\nabla f|\|_{L^1} + \||\nabla f|\|_X \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \quad \text{(by Poincaré's inequality)} \\ &\leq c(|\Omega|)\phi_{X'}(|\Omega|)\, \||\nabla f|\|_X + \||\nabla f|\|_X \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \\ &\quad \text{(by Hölder's inequality)} \\ &= C(|\Omega|) \left\|\frac{1}{I_\Omega(t)}\right\|_{\bar{X}'} \||\nabla f|\|_X,\end{aligned}$$

where $C(|\Omega|)$ is a constant that depends on X and the measure of Ω.

Conversely, suppose that $\dot{W}^1_X(\Omega) \subset L^\infty$, then

$$\left\|f - \int_\Omega f\right\|_{L^\infty} \leq c\, \||\nabla f|\|_X.$$

Since Ω has bounded Lipschitz boundary, this is equivalent (cf. [**70**] and [**68**, Theorem 2]) to the existence of an absolute constant $C > 0$ such that for all $g \in \bar{X}$, $g \geq 0$

$$\left\|\int_t^{|\Omega|/2} \frac{g(s)}{I_\Omega(s)} ds\right\|_{L^\infty} \leq C\, \|g\|_{\bar{X}}.$$

Thus,

$$\sup_{\|g\|_{\bar{X}}\leq 1}\int_0^{|\Omega|/2}\frac{|g|(s)}{I_\Omega(s)}\,ds=\sup_{\|g\|_{\bar{X}}\leq 1}\int_0^{|\Omega|}|g|(s)\frac{\chi_{(0,|\Omega|/2)}(s)}{I_\Omega(s)}\,ds<C,$$

which, by duality, implies that

$$\left\|\frac{1}{I_\Omega(s)}\chi_{(0,|\Omega|/2)}\right\|_{\bar{X}'}<\infty.$$

To conclude the proof we show that (5.2.7) and the relative uniform isoperimetric property imply that

$$\dot{W}^1_X(\Omega)\subset C_b(\Omega).$$

Let $f\in\dot{W}^1_X(\Omega)$. Consider any open ball B_α contained in Ω with $|B_\alpha|=\alpha$. An easy computation shows that

$$\left\|\frac{1}{\min(t,\alpha-t)^{\frac{n-1}{n}}}\right\|_{\bar{X}'}\simeq\left\|t^{1/n-1}\chi_{(0,\alpha/2)}\right\|_{\bar{X}'}.$$

Applying the inequality (5.2.5) to $f\chi_{B_\alpha}$ and integrating, we get

$$\begin{aligned}
\mathit{ess}\sup\,(f\chi_{B_\alpha})-\frac{1}{\alpha}\int f\chi_{B_\alpha}(x)\,dx&=\int_0^\alpha\left((f\chi_{B_\alpha})^{**}(t)-(f\chi_{B_\alpha})^{*}(t)\right)\frac{dt}{t}\\
&\leq\int_0^\alpha\left(\frac{t}{I_{B_\alpha}(t)}\frac{1}{t}\int_0^t|\nabla f\chi_{B_\alpha}|^{*}(s)\,ds\right)\frac{dt}{t}\\
&\leq\left\|\frac{1}{t}\int_0^t|\nabla f\chi_{B_\alpha}|^{*}(s)\,ds\right\|_{\bar{X}}\left\|\frac{1}{I_{B_\alpha}(t)}\right\|_{\bar{X}'}\\
&\leq c(n,X)\,\||\nabla f\chi_{B_\alpha}|\|_X\left\|t^{1/n-1}\chi_{(0,\alpha/2)}\right\|_{\bar{X}'}.
\end{aligned}$$

Similarly, considering $-f\chi_{B_\alpha}$, we get

$$-\mathit{ess}\inf f\,(f\chi_{B_\alpha})+\frac{1}{\alpha}\int f\chi_{B_\alpha}(x)\,dx\leq c(n,X)\,\||\nabla f\chi_{B_\alpha}|\|_X\left\|t^{1/n-1}\chi_{(0,\alpha/2)}\right\|_{\bar{X}'}.$$

Adding these inequalities we see that

$$\mathit{ess}\inf\,(f\chi_{B_\alpha})-\mathit{ess}\inf\,(f\chi_{B_\alpha})\leq c\,\||\nabla f\chi_{B_\alpha}|\|_X\left\|t^{1/n-1}\chi_{(0,\alpha/2)}\right\|_{\bar{X}'}.$$

Thus, for almost every $x,y\in B_\alpha$,

$$|f(x)-f(y)|\leq c(n)\,\||\nabla f\chi_{B_\alpha}|\|_X\left\|t^{1/n-1}\chi_{(0,\alpha/2)}\right\|_{\bar{X}'}.$$

The essential continuity of f follows. □

Remark 6. — *Let us consider the case when $X=L^p$, with $p>n$. An elementary computation shows that*

$$\left\|t^{1/n-1}\chi_{(0,\alpha/2)}\right\|_{p'}\leq c_{(n,p)}\alpha^{\frac{1}{n}(1-\frac{n}{p})}.$$

Let B_α be a ball with $|B_\alpha| = \alpha$, then $\alpha^{1/n}$ is c_n times the radius of the ball, thus, for almost every $y, z \in B_\alpha$ such that $|y - z| = c_n \alpha^{1/n}$, we get

$$|f(y) - f(z)| \leq c(n,p)\, |y - z|^{(1-\frac{n}{p})}\, \||\nabla f|\|_p .$$

The method of proof fails if $p = n$. However, if we consider the smaller Lorentz[1] *space $X = L^{n,1} \subset L^n$, then $X' = L^{\frac{n}{n-1},\infty}$, and*

$$\left\| t^{1/n-1} \chi_{(0,\alpha/2)} \right\|_{L^{\frac{n}{n-1},\infty}} = \sup_{0<s<\alpha/2} s^{\frac{1}{n}-1} s^{1-\frac{1}{n}} = 1.$$

Thus, for almost every $y, z \in B_\alpha$ such that $|y - z| = c_n \alpha^{1/n}$, we have that

$$|f(y) - f(z)| \leq c(n)\, \||\nabla f \chi_{B_\alpha}|\|_{L^{n,1}} = c(n) \int_0^{|y-z|^n} s^{1/n} |\nabla f|^*(s) \frac{ds}{s}.$$

The essential continuity of f follows. Thus we recover the classical result independently due to Stein [**90**] *and C. P. Calderón* [**23**].

Remark 7. — *See* [**25**] *for a related result, using a different method and involving Orlicz norms.*

5.2.2. Spaces defined in terms of the modulus of continuity on Lipschitz domains of $\mathbb{R}^n$.

— For Euclidean domains Ω with Lipschitz boundary it is known that (cf. [**54**, Theorem 1], [**13**, Chapter 5, exercise 13, p. 430]),

$$K(t, g; X(\Omega), \dot{W}_X^1(\Omega)) \simeq \omega_X(g,t),\ 0 < t < |\Omega|,$$

where

$$\omega_X(f,t) = \sup_{0<|h|\leq t} \left\| (f(\cdot + h) - f(\cdot)) \chi_{\Omega(h)} \right\|_{L^p(\Omega)},$$

with $\Omega(h) = \{x \in \Omega : x + \rho h \in \Omega,\ 0 \leq \rho \leq 1\}$ and $h \in \mathbb{R}^n$.

Moreover, as we have seen, $(\Omega, |\cdot|, dm)$ has the relative uniform isoperimetric property. Consequently, by Theorem 13, we have

Theorem 15. — *Let X be a r.i. space on Ω. If $f \in X + \dot{W}_X^1$ satisfies*

$$\int_0^{\mu(\Omega)} \frac{\omega_X\left(f, \phi_X(t) \left\| \frac{1}{(\min(t,|\Omega|-t))^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'}\right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$

then, f is essentially bounded and essentially continuous.

1. See (7.1.3), (7.1.4).

In particular, when $X = L^p$, we obtain

Theorem 16. — *If $n/p < 1$, then there exists a constant $c > 0$, such that*

$$|f(y) - f(z)| \leq C \int_0^{|y-z|} \frac{\omega_p(f,t)}{t^{n/p}} \frac{dt}{t}.$$

Proof. — Let B_α an open ball contained in Ω of measure $\alpha \leq |\Omega|/2$. Since $p > n$ an elementary computation shows that

$$\left\| \frac{1}{I_{B_\alpha}(s)} \chi_{(0,t)}(s) \right\|_{L^{p'}(B_\alpha)} \leq c_{(n,p)} t^{\frac{1}{n} - \frac{1}{p}}.$$

Thus, Theorem 13 ensures for almost every $y, z \in B_\alpha$,

$$\begin{aligned} |f(y) - f(z)| &\leq c \int_0^{2\alpha} \frac{K\left(t^{1/n}, f; L^p(\Omega), \dot{W}_p^1(\Omega)\right)}{t^{1/p}} \frac{dt}{t} \\ &\simeq c \int_0^{2\alpha} \frac{\omega_p(f, t^{1/n})}{t^{1/p}} \frac{dt}{t} \\ &= c \int_0^{(2\alpha)^{1/n}} \frac{\omega_p(f,t)}{t^{n/p}} \frac{dt}{t}. \end{aligned}$$

Since $|B_\alpha| = \alpha$, $\alpha^{1/n}$ is a constant times the radius of the ball, therefore, for almost every $y, z \in B_\alpha$ such that $|y - z| = c\alpha^{1/n}$,

$$|f(y) - f(z)| \preceq \int_0^{|y-z|} \frac{\omega_p(f,t)}{t^{n/p}} \frac{dt}{t}.$$

The essential continuity of f follows. □

Remark 8. — *In Chapter 10 we shall discuss the connection with A. Garsia's work.*

Theorem 17. — *Let X be a r.i. space such that $\overline{\alpha}_{\Lambda(X')} < \frac{1}{n}$. If f satisfies*

$$\int_0^{|\Omega|} \frac{\omega_X(f,t)}{\phi_X(t^{1/n})} \frac{dt}{t} < \infty,$$

then, f is essentially continuous.

Proof. — It is enough to prove

$$R(t) = \left\| \frac{1}{(\min(s, (|\Omega| - s)))^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \leq c_{(n,X)} t^{\frac{1}{n}} \phi_{\bar{X}'}(t), \quad 0 < t < |\Omega|.$$

Recall that if $\underline{\alpha}_{\Lambda(X')} > 0$, the fundamental function of X' satisfies (see [**88**, Theorem 2.4])

$$d\phi_{X'}(s) \simeq \frac{\phi_{X'}(s)}{s},$$

and, moreover, for every $0 < \gamma < \underline{\alpha}_{\Lambda(X')}$ the function $\phi_{X'}(s)/s^\gamma$ is almost increasing (*i.e.*, $\exists c > 0$ s.t. $\phi_{X'}(s)/s^\gamma \leq c\phi_{X'}(t)/t^\gamma$ whenever $t \geq s$). Pick $0 < \beta$ such that (notice that $\overline{\alpha}_{\Lambda(X)} < \frac{1}{n}$ implies that $\underline{\alpha}_{\Lambda(X')} > 1 - \frac{1}{n}$)

$$1 - \frac{1}{n} + \beta < \underline{\alpha}_{\Lambda(X')}.$$

Since I_Ω is symmetric around the point $|\Omega|/2$, we get

$$R(t) \simeq \left\| \frac{1}{s^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'},$$

and

$$\begin{aligned} R(t) &\leq \int_0^t s^{\frac{1}{n}-1} d\phi_{X'}(s) \\ &\simeq \int_0^t s^{\frac{1}{n}-1} \frac{\phi_{X'}(s)}{s} ds \\ &= \int_0^t \frac{\phi_{X'}(s)}{s^{1-\frac{1}{n}+\beta}} \frac{ds}{s^{1-\beta}} \\ &\preceq \frac{\phi_{X'}(t)}{t^{1-\frac{1}{n}+\beta}} \int_0^t \frac{ds}{s^{1-\beta}} \\ &\simeq \frac{\phi_{X'}(t)}{t^{1-\frac{1}{n}}} \\ &= t^{\frac{1}{n}} \phi_{\bar{X}}(t). \end{aligned}$$

□

5.3. Domains of Maz'ya's class $\mathcal{J}_\alpha$ $(1 - 1/n \leq \alpha < 1)$

***Definition 4* (See [74], [75]).** — *A domain $\Omega \subset \mathbb{R}^n$ (with finite measure) belongs to the class $\mathcal{J}_\alpha$ $(1 - 1/n \leq \alpha < 1)$ if there exists a constant $M \in (0, |\Omega|)$ such that*

$$U_\alpha(M) = \sup \frac{|\mathcal{S}|^\alpha}{P_\Omega(\mathcal{S})} < \infty,$$

where the sup *is taken over all $\mathcal{S}$ open bounded subsets of Ω such that $\Omega \cap \partial\mathcal{S}$ is a manifold of class C^∞ and $|\mathcal{S}| \leq M$, (in which case we will say that $\mathcal{S}$ is an admissible subset) and where for a measurable set $E \subset \Omega$, $P_\Omega(E)$ is the De Giorgi perimeter of E in Ω defined by*

$$P_\Omega(E) = \sup \left\{ \int_E \mathrm{div}\varphi \, dx : \varphi \in [C_0^1(\Omega)]^n, \ \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

By an approximation process it follows that if Ω is a bounded domain in $\mathcal{J}_\alpha$, then there exists a constant $c_\Omega > 0$ such that, for all measurable set $E \subset \Omega$ with $|E| \leq |\Omega|/2$, we have

$$P_\Omega(E) \geq c_\Omega |E|^\alpha.$$

Since E and $\Omega\backslash E$ have the same boundary measure, we obtain the following isoperimetric inequality

$$I_\Omega(t) \geq c_\Omega \left(\min(t, |\Omega| - t)\right)^\alpha := J_\Omega(t), \quad 0 < t < |\Omega| .$$

Example 2. — *If Ω is a bounded domain, star shaped with respect to a ball, then Ω belongs to the class $\mathcal{J}_{1-1/n}$; if Ω is a bounded domain with the cone property then Ω belongs to the class $\mathcal{J}_{1-1/n}$; if Ω is a bounded Lipschitz domain, then Ω belongs to the class $\mathcal{J}_{1-1/n}$; if Ω is a $s-$John domain, then $\Omega \in \mathcal{J}_{(n-1)s/n}$; if Ω is a domain with one $\beta-$cusp ($\beta \geq 1$), then it belongs to the Maz'ya class $\mathcal{J}_{\frac{\beta(n-1)}{\beta(n-1)+1}}$.*

Theorem 18. — *Let Ω be a domain in the Maz'ya class $\mathcal{J}_\alpha$, and let X a r.i. space on Ω. Suppose that*

$$\left\| \frac{1}{J_\Omega(t)} \right\|_{X'} < \infty. \tag{5.3.1}$$

Then,

1. $$\dot{W}_X^1(\Omega) \subset C_b(\Omega).$$
2. *If $f \in X + \dot{W}_X^1$ satisfies*
$$\int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{J_\Omega(t)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, \dot{W}_X^1\right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$
then f is essentially bounded and essentially continuous.

Proof

Part 1. — The inclusion $\dot{W}_X^1(\Omega) \subset L^\infty$ follows in the same way as the corresponding part of Theorem 14 (cf. inequality (5.2.8)). To prove the essential continuity we proceed as follows. Let B be any open ball contained in Ω with $|B| \leq \min(1, |\Omega|/2)$. Notice that if $f \in \dot{W}_X^1(\Omega)$, then $f\chi_B \in \dot{W}_X^1(B)$. Now, since B is a *Lip* domain, by Theorem 14 we just need to verify that

$$\left\| t^{1/n-1} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} < \infty.$$

Since $1 - 1/n \leq \alpha < 1$, and $0 < t < |B|/2 < 1$, we have

$$\sup_{0<t<|B|/2} t^{\alpha+1/n-1} = \left(\frac{|B|}{2}\right)^{\alpha+1/n-1}.$$

Thus,

$$\begin{aligned} \left\| t^{1/n-1} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} &= \left\| \frac{t^{\alpha+1/n-1}}{t^\alpha} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} \\ &\leq \left(\frac{|B|}{2}\right)^{\alpha+1/n-1} \left\| \frac{1}{t^\alpha} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} < \infty \quad \text{(by (5.3.1))}. \end{aligned}$$

Part 2. — Let B be an open ball contained in Ω with $|B| \leq \min(1, |\Omega|/2)$. Then

$$K\left(t, f\chi_B; X(B), \dot{W}_X^1(B)\right) \leq K\left(t, f; X, \dot{W}_X^1\right).$$

Using the same argument given in the first part of the proof, we obtain

$$\begin{aligned} \left\| \frac{\chi_{(0,t)}(s)}{(\min(t, |B|-t))^{\frac{n-1}{n}}} \right\|_{\bar{X}'} &\leq 2 \left\| \frac{\chi_{(0,t)}(s)}{t^{\frac{n-1}{n}}} \right\|_{\bar{X}'} \\ &\preceq \left\| \frac{1}{J_\Omega(t)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, \qquad 0 < t < |B|. \end{aligned}$$

Therefore,

$$\int_0^{|B|} \frac{K\left(\phi_X(t) \left\| \frac{\chi_{(0,t)}(s)}{(\min(t,|B|-t))^{\frac{n-1}{n}}} \right\|_{\bar{X}'}, f\chi_B; X(B), \dot{W}_X^1(B)\right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$

and Theorem 15 applies. □

5.4. Ahlfors Regular Metric Measure Spaces

Let (Ω, d, μ) be a complete connected metric Borel measure space and let $k > 1$. We shall say that (Ω, d, μ) is Ahlfors $k-$regular if there exist absolute constants c_Ω, C_Ω such that

$$c_\Omega r^k \leq \mu(B(x,r)) \leq C_\Omega r^k, \quad \forall x \in \Omega, \quad r \in (0, diam(\Omega)). \tag{5.4.1}$$

We will consider Ahlfors $k-$regular spaces (Ω, d, μ) that support a weak $(1,1)-$Poincaré inequality. In other words we shall assume the existence of constants $C > 0$ and $\lambda \geq 1$ such that for all $u \in Lip(\Omega)$,

$$\int_{B(x,r)} \left| u(y) - u_{B_{x,r}} \right| d\mu(y) \leq Cr \int_{B(x,\lambda r)} |\nabla u(y)| \, d\mu(y), \tag{5.4.2}$$

where $u_{B_{x,r}}$ denotes the mean value of u in B, *i.e.*, $u_{B_{x,r}} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) \, d\mu(y)$.

Examples of spaces supporting a (weak) $(1,1)-$Poincaré inequality include Riemannian manifolds with nonnegative Ricci curvature, Carnot-Carathéodory groups, and more generally (in the case of doubling spaces) Carnot-Carathéodory spaces associated to smooth (or locally Lipschitz) vector fields satisfying Hörmander's condition (see for example [**4**], [**48**] and the references quoted therein).

By a well known result of Hajlasz and Koskela (cf. [**48**]), (5.4.1) and (5.4.2) imply

$$\left(\int_{B(x,r)} \left| u(y) - u_{B_{x,r}} \right|^{k/(k-1)} d\mu(y) \right)^{(k-1)/k} \leq D \int_{B(x,2\lambda r)} |\nabla u(y)| \, d\mu(y),$$

with $C = (2C)^{(k-1)/k}$.

According to [**79**] (see also the references quoted therein), given a Borel set $E \subset \Omega$, and $A \subset \Omega$ open, the relative perimeter of E in A, denoted by $P(E, A)$, is defined by

$$P(E,A) = \inf \left\{ \liminf_{h\to\infty} \int_A |\nabla u_h| \, d\mu : u_h \in Lip_{loc}(A), \; u_h \to \chi_E \text{ in } L^1_{loc}(A) \right\}.$$

Lemma 5. — *The following relative isoperimetric inequality holds:*

$$\min\left(\mu(E\cap B(x,r)), \mu(E^c \cap B(x,r))\right) \leq D\left(P(E, B(x,r))\right)^{k/(k-1)}. \tag{5.4.3}$$

Proof. — The proof of this result is contained in the proof of [**4**, Theorem 4.3]. □

Theorem 19. — *Let $k > 1$ be the exponent satisfying (5.4.1), and let $B := B(x,r)$ be a ball. The following statements are equivalent.*

1. *For every set of finite perimeter E in Ω,*
$$c(k,C)\left(\min(\mu(E\cap B), \mu(E^c\cap B))\right)^{\frac{k}{k-1}} \leq P(E,B),$$
where the constant $c(k,C)$ does not depend on B.
2. *$\forall u \in Lip(\Omega)$, the function $|u\chi_B|^*_\mu$ is locally absolutely continuous and, for $0 < t < \mu(B)$,*
$$c(k,C)\int_0^t \left| \left((-|u\chi_B|)^*_\mu \right)'(\cdot) \left(\min(\cdot, \mu(B) - \cdot)\right)^{\frac{k-1}{k}} \right|^*(s)\, ds \leq \int_0^t |\nabla u\chi_B|^*_\mu (s)\, ds.$$
3. *Oscillation inequality: $\forall u \in Lip(\Omega)$ and, for $0 < t < \mu(B)$,*
$$\left(|u\chi_B|^{**}_\mu(t) - |u\chi_B|^*_\mu(t)\right) \leq \frac{t}{c(k,C)\left(\min(t, \mu(B)-t)\right)^{\frac{k-1}{k}}} |\nabla u\chi_B|^{**}_\mu(t).$$

Proof. — Consider the metric space $(B, d_{|B}, \mu_{|B})$, then the Theorem is a particular case of Theorem 1 of [**70**]. □

The local version of Theorem 12 is

Theorem 20. — *Let X be a r.i. space on Ω and let $B \subset \Omega$ be an open ball. Then, for each $f \in X + S_X$,*

$$(f\chi_B)^{**}_\mu(t/2) - (f\chi_B)^*_\mu(t/2) \leq 4\frac{K(\psi(t), f\chi_B; X, S_X)}{\phi_X(t)}, \quad 0 < t < \mu(B),$$

where

$$\psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{c(k,C)\left(\min(s,\mu(B)-s)\right)^{\frac{k-1}{k}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'}.$$

Proof. — Let $f \in X + S_X$, then $f\chi_B \in X_r(B) + S_{X_r}(B)$, where B is the metric space $(B, d_{|B}, \mu_{|B})$. By Lemma 5 we know that

$$c(k,C)\left(\min(\mu(E\cap B), \mu(E^c\cap B))\right)^{\frac{k}{k-1}} \leq P(E,B).$$

Thus, for any Borel set $E \subseteq B$,

$$c(k,C)\left(\min(\mu(E),\mu(B)-\mu(E))\right)^{\frac{k}{k-1}} \leq P_B(E).$$

Consequently, $J_B(t) = c(k,C)\left(\min(t,\mu(B)-t)\right)^{\frac{k}{k-1}}$ $(0 < t < \mu(B))$ is an isoperimetric estimator of $\left(B, d_{|B}, \mu_{|B}\right)$, and now we finish the proof in the same way as in Theorem 12. □

Theorem 21. — *Let $f \in X + S_X$ and let B be an open ball, if*

$$\int_0^{\mu(B)} \frac{K\left(\phi_X(t)\left\|\left(\min(s,\mu(B)-s)\right)^{1-1/k}\chi_{(0,t)}(s)\right\|_{\bar{X}'}, f\chi_B; X, S_X\right)}{\phi_X(t)}\frac{dt}{t} < \infty$$

then, $f\chi_B$ is essentially bounded and essentially continuous.

Proof. — By the proof of the previous Theorem we know that

$$J_B(t) = c(k,C)\left(\min\left(t,\mu(B)-t\right)\right)^{\frac{k}{k-1}}$$

is an isoperimetric estimator of $\left(B, d_{|B}, \mu_{|B}\right)$. For any open ball $B(x,r) \subset B$, it follows from Lemma 5 that, for $0 < s < \mu(Q_{B(x,r)})$,

$$\begin{aligned} c(k,C)\left(\min\left(t,\mu(B(x,r))-t\right)\right)^{\frac{k}{k-1}} &\preceq c(k,C)\min(J_B(t), J_B(\mu(Q_{B(x,r)})-t)) \\ &\leq P_{B(x,r)}(s). \end{aligned}$$

Therefore $\left(B, d_{|B}, \mu_{|B}\right)$ has the relative isoperimetric property and Theorem 13 applies. □

Remark 9. — *In the particular case $X = L^p$, we can thus use the same argument given in Theorem 16 to obtain that for $k/p < 1$, there exists an absolute constant such that*

$$|f(y)-f(z)| \preceq \int_0^{|y-z|} \frac{K\left(t, f\chi_B; X, S_X\right)}{t^{k/p}}\frac{dt}{t}, \quad y,z \in B.$$

CHAPTER 6

FRACTIONAL SOBOLEV INEQUALITIES IN GAUSSIAN MEASURES

6.1. Introduction and Summary

As another application of our theory, in this chapter we consider in detail fractional logarithmic Sobolev inequalities. We will deal not only with Gaussian measures but also with measures that interpolate between Gaussian and exponential.

In the context of classical Gaussian measures a typical result in this chapter is given by the following fractional logarithmic Sobolev inequality. Let $d\gamma_n$ be the Gaussian measure on $\mathbb{R}^n$, let $1 \leq q < \infty$, $\theta \in (0,1)$; then, there exists an absolute constant $c > 0$, independent of the dimension, such that (cf. Theorem 23 below)

$$\left\{ \int_0^{1/2} |f|_{\gamma_n}^* (t)^q \left(\log \frac{1}{t} \right)^{\frac{q\theta}{2}} dt \right\}^{1/q} \leq c \, \|f\|_{B_{L^q}^{\theta,q}(\gamma_n)}, \tag{6.1.1}$$

where $B_{L^q}^{\theta,q}(\gamma_n)$ is the Gaussian Besov space, see (6.3.1) below. Note that if $q = 2$, (6.1.1) interpolates between the embedding that follows from the classical logarithmic Sobolev inequality (which corresponds to the case $\theta = 1$) and the trivial embedding $L^2 \subset L^2$ (the case $\theta = 0$). For related inequalities using semigroups see [**6**] and also [**39**].

More generally, we will also prove fractional Sobolev inequalities for tensor products of measures that, on the real line are defined as follows. Let $\alpha \geq 0$, $r \in [1,2]$ and $\gamma = \exp(2\alpha/(2-r))$, ($\alpha = 0$ if $r = 2$) and let

$$d\mu_{r,\alpha}(x) = Z_{r,\alpha}^{-1} \exp\left(- |x|^r \left(\log(\gamma + |x|)^\alpha\right)\right) dx,$$
$$\mu_{r,\alpha,n} = \mu_{p,\alpha}^{\otimes n},$$

where $Z_{r,\alpha}^{-1}$ is chosen to ensure that $\mu_{r,\alpha}(\mathbb{R}) = 1$. The corresponding results are apparently new and give fractional Sobolev inequalities, that just like the logarithmic Sobolev inequalities of [**70**], exhibit logarithmic gains of integrability that are directly related to the corresponding isoperimetric profiles. For example, if $\alpha = 0$, then the corresponding fractional Sobolev inequalities take the following form. Let $1 \leq q < \infty$,

$\theta \in (0,1)$, then there exists an absolute constant $c > 0$, independent of the dimension, such that (cf. Theorem 23 below)

$$\left(\int_0^{1/4} |f|^*_{\mu_{r,0,n}}(t)^q \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right)^{1/q} \leq c \|f\|_{B^{\theta,q}_{L^q}(\mu_{r,0,n})}.$$

Likewise, for $q = \infty$ (cf. (6.3.4) below)

$$\sup_{t \in (0,\frac{1}{4})} \left(|f|^{**}_{\mu_{r,0,n}}(t) - |f|^*_{\mu_{r,0,n}}(t) \right) \left(\log \frac{1}{t} \right)^{(1-\frac{1}{r})\theta} \leq c \|f\|_{\dot{B}^{\theta,\infty}_{L^\infty}(\mu_{r,0,n})}.$$

We also explore the scaling of fractional inequalities for Gaussian Besov spaces based on exponential Orlicz spaces. We show that in this context the gain of integrability can be measured directly in the power of the exponential.

We start by considering the corresponding embeddings of Gaussian-Sobolev spaces into L^∞.

6.2. Boundedness of functions in Gaussian-Sobolev spaces

Let $\alpha \geq 0$, $r \in [1,2]$ and $\gamma = \exp(2\alpha/(2-r))$ ($\alpha = 0$ if $r = 2$), and let $\mu_{r,\alpha}$ be the probability measure on $\mathbb{R}$ defined by

$$d\mu_{r,\alpha}(x) = Z_{r,\alpha}^{-1} \exp\left(-|x|^r \left(\log(\gamma + |x|)^\alpha\right) dx = \varphi_{r,\alpha}(x)\, dx,\ x \in \mathbb{R},$$

where $Z_{r,\alpha}^{-1}$ is chosen to ensure that $\mu_{r,\alpha}(\mathbb{R}) = 1$. Then we let

$$\varphi^n_{\alpha,r}(x) = \varphi_{r,\alpha}(x_1) \cdots \varphi_{r,\alpha}(x_n),\ x \in \mathbb{R}^n,$$

and $\mu_{r,\alpha,n} = \mu_{r,\alpha}^{\otimes n}$. In other words

$$d\mu_{r,\alpha,n}(x) = \varphi^n_{r,\alpha}(x)\, dx.$$

In particular, $\mu_{2,0,n} = \gamma_n$ (Gaussian measure).

It is known that the isoperimetric problem for $\mu_{r,\alpha}$ is solved by half-lines (cf. [**19**] and [**16**]) and the isoperimetric profile is given by

$$I_{\mu_{r,\alpha}}(t) = \varphi\left(H^{-1}(\min(t, 1-t))\right) = \varphi\left(H^{-1}(t)\right), \quad t \in [0,1],$$

where $H : \mathbb{R} \to (0,1)$ is the increasing function given by

$$H(r) = \int_{-\infty}^r \varphi(x)\, dx.$$

Moreover (cf. [**7**] and [**8**]), there exist constants c_1, c_2 such that, for all $t \in [0,1]$,

$$c_1 L_{\mu_{r,\alpha}}(t) \leq I_{\mu_{r,\alpha}}(t) \leq c_2 L_{\mu_{r,\alpha}}(t), \tag{6.2.1}$$

where

$$L_{\mu_{r,\alpha}}(t) = \min(t, 1-t) \left(\log \frac{1}{\min(t,1-t)} \right)^{1-\frac{1}{r}} \left(\log\log \left(e + \frac{1}{\min(t,1-t)} \right) \right)^{\frac{\alpha}{r}}.$$

Moreover, we have

$$I_{\mu_{r,\alpha,n}}(t) \simeq t\left(\log\frac{1}{t}\right)^{1-\frac{1}{r}}\left(\log\log\left(e+\frac{1}{t}\right)\right)^{\alpha/r}, \text{ for } t\in\left(0,\frac{1}{2}\right). \tag{6.2.2}$$

For the rest of the section we shall let μ denote the measure $\mu_{r,\alpha,n}$ on $\mathbb{R}^n$. For a given r.i. space $X := X(\mathbb{R}^n,\mu)$, let $W_X^1(\mathbb{R}^n,\mu)$ be the classical Sobolev space endowed with the norm $\|u\|_{W_X^1(\mathbb{R}^n,\mu)} = \|u\|_X + \||\nabla u|\|_X$. The homogeneous Sobolev space $\dot{W}_X^1(\mathbb{R}^n,\mu)$ is defined by means of the quasi norm $\|u\|_{\dot{W}_X^1} := \||\nabla u|\|_X$.

The discussion of Section 5.2 applies and therefore we see that $W_{L_1}^1(\mathbb{R}^n,\mu)$ is invariant under truncation. Moreover, if $u\in W_{L_1}^1(\mathbb{R}^n,\mu)$ then the following co-area formula holds:

$$\int_{\mathbb{R}^n}|\nabla u(x)|\,d\mu(x) = \int_{\mathbb{R}^n}|\nabla u(x)|\,\varphi_{\alpha,p}^n(x)\,dx = \int_{-\infty}^{\infty}P_\mu(u>s)\,ds.$$

From here we see that inequalities (5.2.1), (5.2.4) and (5.2.5) hold for all $W_{L_1}^1(\mathbb{R}^n,\mu)$ functions (of course, the rearrangements are now with respect to the measure μ). Finally, if we consider

$$K(t,f;X,\dot{W}_X^1) = \inf\left\{\|f-g\|_X + t\,\|g\|_{\dot{W}_X^1(\mathbb{R}^n,\mu)}\right\},$$

all the results that we have obtained in Chapter 3, remain true.

Theorem 22. — *If $\mu=\mu_{r,\alpha,n}$ then*

$$\dot{W}_X^1(\mathbb{R}^n,\mu)\not\subseteq L^\infty.$$

Proof. — By [**70**, Theorem 6] the embedding $\dot{W}_X^1(\mathbb{R}^n,\mu)\subset L^\infty$ is equivalent to the existence of a positive constant $c>0$, such that for all $f\in\bar{X}$, supported on $(0,\frac{1}{2})$ we have

$$\sup_{t\geq 0}\int_t^{1/2}\frac{|f(s)|}{L_{\mu_{r,\alpha,n}}(s)}\,ds \leq c\,\|f\|_{\bar{X}}.$$

In particular this implies that

$$\int_0^{1/2}\frac{ds}{L_{\mu_{r,\alpha,n}}(s)} \leq c.$$

But this is not possible since $1/L_{\mu_{r,\alpha,n}}(s)\notin L^1$. □

It follows that the results of Chapter 5 cannot be applied directly to obtain the continuity of functions in the space $\dot{W}_X^1$.

Remark 10. — *Let us remark that since continuity is a local property, a weak version of the Morrey-Sobolev theorem (that depends on the dimension) is available.*

Let $\mu = \mu_{r,\alpha,n}$, and let $X = X(\mathbb{R}^n, \mu)$ be a r.i space on $(\mathbb{R}^n, \mu)$ such that

$$\left\| \frac{1}{\min(1, 1-t)^{1-1/n}} \right\|_{\bar{X}'} < \infty.$$

Then every function in $\dot{W}^1_X(\mathbb{R}^n, \mu)$ is essentially continuous.

Proof. — Let $f \in \dot{W}^1_X(\mathbb{R}^n, \mu)$ and let $B \subset \mathbb{R}^n$ be an arbitrary ball with Lebesgue measure equal to 1. To prove that f is continuous on B let us note that $f\chi_B \in \dot{W}^1_X(B, \mu)$, *i.e.*,

$$\| |\nabla f \chi_B| \|_X < \infty.$$

Let m be the Lebesgue measure on $\mathbb{R}^n$, it is plain that for all $t > 0$,

$$c_B m\{x \in B : |\nabla f| > t\} \leq \mu\{x \in B : |\nabla f| > t\} \leq C_B m\{x \in B : |\nabla f| > t\},$$

where $c_B = \inf_{x \in B} \varphi^n_{\alpha,p}(x)$ and $C_B = \max_{x \in B} \varphi^n_{\alpha,p}(x)$. Therefore,

$$c_Q \| |\nabla f \chi_B| \|_{X(B,m)} \leq \| |\nabla f \chi_B| \|_{X(\mathbb{R}^n,\mu)} \leq C_Q \| |\nabla f \chi_B| \|_{X(B,m)}.$$

Consequently, $f\chi_B \in \dot{W}^1_X(B, m)$, and by Theorem 14, $f\chi_B \in C(B)$. □

6.3. Embeddings of Gaussian Besov spaces

In what follows unless it is necessary to be more specific we shall let $\mu := \mu_{r,\alpha,n}$. We consider the Besov spaces $\dot{B}^{\theta,q}_X(\mu)$, $B^{\theta,q}_X(\mu)$ can be defined using real interpolation (cf. [**14**], [**94**]). In other words for $1 \leq q \leq \infty$, $\theta \in (0,1)$, and let

$$\dot{B}^{\theta,q}_X(\mu) = \{f : \|f\|_{\dot{B}^{\theta,q}_X(\mu)} < \infty\},$$

$$B^{\theta,q}_X(\mu) = \{f : \|f\|_{B^{\theta,q}_X(\mu)} = \|f\|_{\dot{B}^{\theta,q}_X(\mu)} + \|f\|_X < \infty\}, \tag{6.3.1}$$

where

$$\|f\|_{\dot{B}^{\theta,q}_X(\mu)} = \begin{cases} \left(\int_0^1 \left(K\left(s, f; X(\mu), \dot{W}^1_X(\mu)\right) s^{-\theta} \right)^q \frac{ds}{s} \right)^{1/q} & \text{if } q < \infty \\ \sup_s \left(K\left(s, f; X(\mu), \dot{W}^1_X(\mu)\right) s^{-\theta} \right) & \text{if } q = \infty. \end{cases}$$

The embeddings we prove in this section will follow from

(6.3.2)

$$|f|^{**}_\mu(t) - |f|^*_\mu(t) \leq c \frac{K\left(\frac{t}{I_\mu(t)}, f; X(\mu), \dot{W}^1_X(\mu)\right)}{\phi_X(t)}, \quad 0 < t \leq 1/2, \quad (f \in X + \dot{W}^1_X).$$

To simplify the presentation we shall state and prove our results only for the Gaussian measures $\mu_{r,0,n}$, $r \in (1,2]$, which include the most important examples: Gaussian measures and the so called interpolation measures between exponential and Gaussian.

Theorem 23. — *Let $1 \leq q < \infty$, $\theta \in (0,1)$, $r \in (1,2]$. Then there exists an absolute constant $c = c(q,\theta,r) > 0$ such that,*

$$\left\{ \int_0^{1/2} |f|^*_{\mu_{r,0,n}}(t)^q \left(\log \frac{1}{t}\right)^{q\theta(1-1/r)} dt \right\}^{1/q} \leq c \|f\|_{B^{\theta,q}_{L^q}(\mu_{r,0,n})}. \tag{6.3.3}$$

Let $q = \infty$, then there exists an absolute constant $c = c(\theta,r) > 0$ such that

$$\sup_{t\in(0,\frac{1}{2})} \left(|f|^{**}_{\mu_{r,0,n}}(t) - |f|^*_{\mu_{r,0,n}}(t)\right) \left(\log \frac{1}{t}\right)^{(1-\frac{1}{r})\theta} \leq c \|f\|_{\dot{B}^{\theta,\infty}_{L^\infty}(\mu_{r,0,n})}. \tag{6.3.4}$$

Proof. — We shall let $\mu =: \mu_{r,0,n}$, $K(s,f) := K\big(s,f;L^q(\mu),\dot{W}^1_{L^q}(\mu)\big)$. Suppose that $1 \leq q < \infty$. We start by rewriting the term we want to estimate

$$\begin{aligned}
&\left\{ \int_0^{1/2} |f|^*_{\mu}(t)^q \left(\log \frac{1}{t}\right)^{q\theta(1-1/r)} dt \right\}^{1/q} \\
&\preceq \left\{ \int_0^{1/2} |f|^*_{\mu}(t)^q \left(\int_t^{1/2} \left(\log \frac{1}{s}\right)^{q\theta(1-1/r)-1} \frac{ds}{s} + (\log 2)^{q\theta(1-1/r)} \right) dt \right\}^{1/q} \\
&\preceq \left\{ \int_0^{1/2} \left(\log \frac{1}{s}\right)^{q\theta(1-1/r)-1} \frac{1}{s}\int_0^s |f|^*_{\mu}(t)^q dt ds \right\}^{1/q} + \left\{ \int_0^{1/2} |f|^*_{\mu}(t)^q dt \right\}^{1/q} \\
&= (I) + (II)
\end{aligned}$$

The term (II) is under control since

$$(II) \leq \|f\|_{L^q} \leq \|f\|_{B^{\theta,q}_{L^q}(\mu)}.$$

To estimate (I) we first note that the elementary inequality [1]: $|x|^q \leq 2^{q-1}(|x-y|^q + |y|^q)$, yields

$$\frac{1}{s}\int_0^s |f|^*_{\mu}(t)^q dt \preceq \frac{1}{s}\int_0^s \left(f^*_{\mu}(t) - f^*_{\mu}(s)\right)^q dt + f^*_{\mu}(s)^q.$$

Consequently,

$$\begin{aligned}
(I) &\preceq \left\{ \int_0^{1/2} \left(\log \frac{1}{s}\right)^{q\theta(1-1/r)-1} \left(\frac{1}{s}\int_0^s \left(|f|^*_{\mu}(t) - |f|^*_{\mu}(s)\right)^q dt \right) ds \right\}^{1/q} + \\
&\quad \left\{ \int_0^{1/2} \left(\log \frac{1}{s}\right)^{q\theta(1-1/r)-1} |f|^*_{\mu}(s)^q ds \right\}^{1/q} \\
&= (I_1) + (I_2), \text{ say.}
\end{aligned}$$

1. Which follows readily by Jensen's inequality.

To control (I_1) we first use Example 1 in Chapter 3 and (6.2.2) to estimate the inner integral as follows

$$\frac{1}{s}\int_0^s \left(|f|_\mu^*(t) - |f|_\mu^*(s)\right)^q dt \preceq \frac{1}{s}\left(K\left(\left(\log\frac{1}{s}\right)^{1/r-1}, f\right)\right)^q, \quad 0 < s \leq 1/2.$$

Thus,

$$(I_1) \preceq \left\{\int_0^{1/2}\left(\left(\log\frac{1}{s}\right)^{\theta(1-1/r)}\left(K\left(\left(\log\frac{1}{s}\right)^{1/r-1}, f\right)\right)\right)^q \frac{ds}{s\log\frac{1}{s}}\right\}^{1/q}.$$

The change of variables $u = (\log\frac{1}{s})^{1/r-1}$ then yields

$$(I_1) \preceq \|f\|_{B^{\theta,q}_{L^q}(\mu)}.$$

It remains to estimate (I_2). We write

$$(I_2) \leq \left\{\int_0^{1/2}\left(\log\frac{1}{s}\right)^{q\theta(1-1/r)-1} |f|_\mu^{**}(s)^q ds\right\}^{1/q},$$

then, using the fundamental theorem of calculus, we have

$$\begin{aligned}(I_2) &\leq \left\{\int_0^{1/2}\left(\log\frac{1}{s}\right)^{q\theta(1-1/r)-1}\left(\int_s^{1/2}\left(|f|_\mu^{**}(z) - |f|_\mu^*(z)\right)\frac{dz}{z} + |f|_\mu^{**}(1/2)\right)^q ds\right\}^{1/q} \\ &\leq \left\{\int_0^{1/2}\left(\left(\log\frac{1}{s}\right)^{\theta(1-1/r)-1/q}\int_s^{1/2}\left(|f|_\mu^{**}(z) - |f|_\mu^*(z)\right)\frac{dz}{z}\right)^q ds\right\}^{1/q} + \\ &\quad |f|_\mu^{**}(1/2)\left\{\int_0^{1/2}\left(\log\frac{1}{t}\right)^{q\theta(1-1/r)} dt\right\}^{1/q} \\ &= (A) + (B), \text{ say.}\end{aligned}$$

To use the Hardy logarithmic inequality of [**12**, (6.7)] we first write

$$(A) = \left\{\int_0^{1/2}\left(\left(\log\frac{1}{s}\right)^{\theta(1-1/r)-1/q} s^{1/q}\int_s^{1/2}\left(|f|_\mu^{**}(z) - |f|_\mu^*(z)\right)\frac{dz}{z}\right)^q \frac{ds}{s}\right\}^{1/q}$$

and then find that

$$(A) \preceq \left\{\int_0^{1/2}\left(\left(|f|_\mu^{**}(s) - |f|_\mu^*(s)\right) s^{1/q}\left(\log\frac{1}{s}\right)^{\theta(1-1/r)-1/q}\right)^q \frac{ds}{s}\right\}^{1/q}.$$

Now we use the fact that in the region of integration $s^{1/q} \leq 1$, combined with (6.3.2) and (6.2.2), to conclude that

$$\left\{\int_0^{1/2}\left(\left(|f|_\mu^{**}(s)-|f|_\mu^{*}(s)\right)s^{1/q}\left(\log\frac{1}{s}\right)^{\theta(1-1/r)-1/q}\right)^q\frac{ds}{s}\right\}^{1/q}$$

$$\preceq\left\{\int_0^{1/2}\left(K\left(\left(\log\frac{1}{s}\right)^{\frac{1}{r}-1},f\right)\right)^q\left(\log\frac{1}{s}\right)^{q\theta(1-1/r)}\frac{ds}{s\left(\log\frac{1}{s}\right)}\right\}^{1/q}$$

$$\simeq\left\{\int_0^{(\log 2)^{\frac{1}{r}-1}}(K(u,f))^q\,u^{-\theta q}\frac{du}{u}\right\}^{1/q}\left(\text{change of variables } u=\left(\log\frac{1}{s}\right)^{\frac{1}{r}-1}\right)$$

$$\leq\|f\|_{B_{L^q}^{\theta,q}(\mu)}.$$

Finally it remains to estimate (B):

$$(B)=\frac{1}{2}\left(2\,|f|_\mu^{**}(1/2)\right)\left\{\int_0^{1/2}t^{1/2}\left(\log\frac{1}{t}\right)^{q\theta(1-1/r)-1}\frac{dt}{t^{1/2}}\right\}^{1/q}$$

$$\leq 4\,\|f\|_{L^1}\left(\sup_{t\in(0,1/2]}t^{1/2}\left(\log\frac{1}{t}\right)^{q\theta(1-1/r)-1}\right)\left\{\int_0^{1/2}\frac{dt}{t^{1/2}}\right\}$$

$$\preceq\|f\|_{L^1}\leq\|f\|_{L^q}$$

$$\leq\|f\|_{B_{L^q}^{\theta,q}(\mu)}.$$

We consider now the case $q=\infty$. We apply (6.3.2), observing that for $X=L^\infty$, we have $\phi_{L^\infty}(t)=1$, and obtain that for $t\in\left(0,\frac{1}{2}\right]$,

$$|f|_\mu^{**}(t)-|f|_\mu^{*}(t)\preceq K\left(\left(\log\frac{1}{t}\right)^{\frac{1}{r}-1},f\right)$$

$$=K\left(\left(\log\frac{1}{t}\right)^{\frac{1}{r}-1},f\right)\left(\log\frac{1}{t}\right)^{-(\frac{1}{r}-1)\theta}\left(\log\frac{1}{t}\right)^{(\frac{1}{r}-1)\theta}$$

$$\leq\left(\log\frac{1}{t}\right)^{(\frac{1}{r}-1)\theta}\left(\sup_u\left(K(u,f)u^{-\theta}\right)\right)$$

$$=\left(\log\frac{1}{t}\right)^{(\frac{1}{r}-1)\theta}\|f\|_{\dot{B}_{L^\infty}^{\theta,\infty}(\mu)},$$

as we wished to show. □

Remark 11. — *Gaussian measure corresponds to $r = 2$, in this case, for $q = 2$, (6.3.3) yields the logarithmic Sobolev inequality*

$$\left\{ \int_0^{1/2} |f|_{\gamma_n}^* (t)^2 \left(\log \frac{1}{t} \right)^\theta dt \right\}^{1/2} \leq c \|f\|_{B^{\theta,2}_{L^2}(\gamma_n)}, \ \theta \in (0, 1).$$

*Formally the case $\theta = 1$ corresponds to an L^2 Logarithmic Sobolev inequality, while the case $\theta = 0$, corresponds to the trivial $L^2 \subset L^2$ embedding. One could formally approach such inequalities by complex interpolation (cf. [**6**] as well as the calculations provided in [**76**])*

$$[L^2, \dot{W}^1_{L^2}]_\theta \subset [L^2, L^2 LogL]_\theta = L^2(LogL)^\theta.$$

*The case $r = 2$, $q = 1$, corresponds to a fractional version of Ledoux's inequality (cf. [**59**]). Besides providing a unifying approach our method can be applied to deal with more general domains and measures.*

Remark 12. — *When $q = \infty$ the inequality (6.3.4) reflects a refined estimate of the exponential integrability of f. In particular, note that the case $\theta = 1$, formally gives the following inequality (cf. [**17**] and the references therein)*

$$\|f\|_{L[\infty,\infty]} \simeq \sup_{t \in (0, \frac{1}{2}]} \left(|f|_{\gamma_n}^{**} (t) - |f|_{\gamma_n}^* (t) \right) \leq c \|f\|_{\dot{W}^1_{e^{L^2}}(\gamma_n)}$$

(cf. (7.1.4) below for the definition of the $L^{[p,q]}$ spaces). The previous inequality can be proved readily using

$$|f|_{\gamma_n}^{**} (t) - |f|_{\gamma_n}^* (t) \leq c \frac{1}{(\log \frac{1}{t})^{1/2}} |\nabla f|_{\gamma_n}^{**} (t), \ t \in (0, 1/2].$$

Remark 13. — *Using the transference principle of [**70**] the Gaussian results can be applied to derive results related to the dimensionless Sobolev inequalities on Euclidean cubes studied by Krbec-Schmeisser (cf. [**56**], [**57**]) and Triebel [**95**].*

6.4. Exponential Classes

There is a natural connection between Gaussian measure and the exponential class e^{L^2}. Likewise, this is also true with more general exponential measures and other exponential spaces. Although there are many nice inequalities associated with this topic that follow from our theory, we will not develop the matter in great detail here. Instead, we shall only give a flavor of possible results by considering Besov embeddings connected with the Sobolev space $\dot{W}^1_{e^{L^2}} := \dot{W}^1_{e^{L^2}}(\mathbb{R}^n, \gamma_n)$.

In this setting (6.3.2) takes the form

$$|f|_{\gamma_n}^{**} (t) - |f|_{\gamma_n}^* (t) \leq c \frac{K\left(\left(\log \frac{1}{t}\right)^{-\frac{1}{2}}, f; e^{L^2}, \dot{W}^1_{e^{L^2}} \right)}{\phi_{e^{L^2}}(t)}, \ t \in \left(0, \frac{1}{2}\right].$$

Now, since $\phi_{e^{L^2}}(t) = \left(\log\frac{1}{t}\right)^{-\frac{1}{2}}$, $t \in (0, \frac{1}{2})$ we formally have

$$\left(|f|^{**}_{\gamma_n}(t) - |f|^{*}_{\gamma_n}(t)\right) \leq cK\left(\left(\log\frac{1}{t}\right)^{-\frac{1}{2}}, f; e^{L^2}, \dot{W}^1_{e^{L^2}}\right)\left(\log\frac{1}{t}\right)^{\frac{1}{2}}$$
$$\leq c\,\|f\|_{\dot{B}^1_{e^{L^2},\infty}(\gamma_n)},$$

or

$$\|f\|_{L^{[\infty,\infty]}(\gamma_n)} \leq c\,\|f\|_{\dot{B}^1_{e^{L^2},\infty}(\gamma_n)}. \tag{6.4.1}$$

More generally,

$$\left(|f|^{**}_{\gamma_n}(t) - |f|^{*}_{\gamma_n}(t)\right)\left(\log\frac{1}{t}\right)^{-\frac{1}{2}+\frac{\theta}{2}} \leq cK\left(\left(\log\frac{1}{t}\right)^{-\frac{1}{2}}, f; e^{L^2}, \dot{W}^1_{e^{L^2}}\right)\left(\log\frac{1}{t}\right)^{\frac{\theta}{2}}$$
$$\leq c\,\|f\|_{\dot{B}^{\theta}_{e^{L^2},\infty}},$$

which shows directly the improvement on the exponential integrability in the $\dot{B}^{\theta}_{e^{L^2},\infty}$ scale.

CHAPTER 7

ON LIMITING SOBOLEV EMBEDDINGS AND *BMO*

7.1. Introduction and Summary

The discussion in this chapter is connected with the role of *BMO* in some limiting Sobolev inequalities. We start by reviewing some definitions, and then proceed to describe Sobolev inequalities which follow readily from our symmetrization inequalities, and will be relevant for our discussion.

Let (Ω, d, μ) be a metric measure space satisfying the usual assumptions, including the relative uniform isoperimetric property. The space $BMO(\Omega) = BMO$, introduced by John-Nirenberg, is the space of integrable functions $f : \Omega \to \mathbb{R}$, such that

$$\|f\|_{BMO} = \sup_{B} \left\{ \inf_{c} \left(\frac{1}{\mu(B)} \int_{B} |f - c| \, d\mu \right) : B \text{ ball in } \Omega \right\} < \infty.$$

In fact, it is enough to consider averages $f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$, or a median $m(f)$ of f (cf. [Definition 1, Chapter 3]),

$$\|f\|_{BMO} \simeq \sup_{B} \left\{ \frac{1}{\mu(B)} \int_{B} |f - f_B| \, d\mu : B \text{ ball in } \Omega \right\} < \infty.$$

To obtain a norm we may set

$$\|f\|_{BMO_*} = \|f\|_{BMO} + \|f\|_{L^1}.$$

Remark 14. — *One can also control* $\|f\|_*$ *through the use of maximal operators (cf.* [**38**], [**26**], [**1**]*). Let*

$$f^{\#}(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_{B} |f - f_B| \, d\mu,$$

where the sup *is taken over all open balls containing* x. *Then we have*

$$\|f\|_{BMO} \simeq \left\| f^{\#} \right\|_{\infty}.$$

Let $\theta \in (0,1), 1 \leq p \leq \infty, 1 \leq q \leq \infty$. Consider the Besov spaces $\dot{b}_p^{\theta,q}(\Omega)$ (resp. $b_p^{\theta,q}(\Omega)$), defined by

$$\|f\|_{\dot{b}_p^{\theta,q}(\Omega)} = \left(\int_0^{\mu(\Omega)} \left(t^{-\theta} K(t,f;L^p(\Omega),S_{L^p}(\Omega))\right)^q \frac{dt}{t}\right)^{1/q} \tag{7.1.1}$$

$$\|f\|_{b_p^{\theta,q}(\Omega)} = \|f\|_{\dot{b}^{\theta}_{p,q}(\Omega)} + \|f\|_{L^p}.$$

For ready comparison with classical embedding theorems, from now on in this section, unless explicitly stated to the contrary, we shall consider metric measure spaces (Ω, d, μ) such that the corresponding isoperimetric profiles satisfy

$$t^{1-1/n} \preceq I_\Omega(t), \ t \in (0, \mu(\Omega)/2). \tag{7.1.2}$$

We now recall the definition of the $L^{p,q}$ spaces. Moreover, in order to incorporate in a meaningful way the limiting cases that correspond to the index $p = \infty$, we also recall the definition of the modified $L^{[p,q]}$ spaces (1). Let $1 \leq p < \infty, 1 \leq q \leq \infty$ (cf. [**11**], [**9**]), and let (2)

$$L^{p,q}(\Omega) = \left\{ f : \|f\|_{L^{p,q}} = \left(\int_0^{\mu(\Omega)} \left(|f|_\mu^*(s)s^{1/p}\right)^q \frac{ds}{s}\right)^{1/q} < \infty \right\}. \tag{7.1.3}$$

For $1 \leq p \leq \infty, 1 \leq q \leq \infty$, we let
(7.1.4)

$$L^{[p,q]}(\Omega) = \left\{ f : \|f\|_{L^{[p,q]}} = \left(\int_0^{\mu(\Omega)} \left((|f|_\mu^{**}(s) - |f|_\mu^*(s))s^{1/p}\right)^q \frac{ds}{s}\right)^{1/q} < \infty \right\}.$$

It is known that (cf. [**71**] and the references therein)

$$L^{p,q}(\Omega) = L^{[p,q]}(\Omega), \ \text{for } 1 \leq p < \infty, 1 \leq q \leq \infty.$$

Then, under our current assumptions on the isoperimetric profile of Ω, Theorem 7 states that

$$|f|_\mu^{**}(t) - |f|_\mu^*(t) \leq c \frac{K(t^{1/n}, f; L^p(\Omega), S_{L^p}(\Omega))}{t^{1/p}}, \ t \in (0, \mu(\Omega)/2). \tag{7.1.5}$$

The following basic version of the Sobolev embedding follows readily

Proposition 3

$$b_p^{\theta,q}(\Omega) \subset L^{\bar{p},q}(\Omega), \ \textit{where } \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\theta}{n}, \ \theta \in (0,1), \ 1 \leq q \leq \infty, \ \theta p \leq n. \tag{7.1.6}$$

1. The $L^{p,q}$ and $L^{[p,q]}$ spaces are equivalent for $p < \infty$.
2. With the usual modifications when $q = \infty$.

Proof. — Indeed, from the relationship between the indices and (7.1.5), we can write

$$\left(|f|_{\mu}^{**}(t)-|f|_{\mu}^{*}(t)\right)t^{1/\bar{p}}\preceq t^{-\frac{\theta}{n}}K(t^{1/n},f;L^{p}(\Omega),S_{L^{p}}(\Omega)),\ t\in(0,\mu(\Omega)/2).$$

If $q=\infty$, (7.1.6) follows taking supremum on both sides of the inequality above. Likewise, if $q<\infty$, then the desired result follows raising both sides to the power q and integrating from 0 to $\mu(\Omega)/2$. In reference to the role of the $L^{[\infty,q]}$ spaces here let us remark that, in the limiting case $\theta p=n$, we have $\bar{p}=\infty$. □

We consider the limiting case, $\theta=\frac{n}{p}$, $p>n$, in more detail. In this case (7.1.6) reads (cf. [**65**])

$$b_p^{n/p,q}(\Omega)\subset L^{[\infty,q]}(\Omega),\ p>n,\ 1\leq q\leq\infty.$$

Note that when $q=1$, $L^{[\infty,1]}(\Omega)=L^{\infty}(\Omega)$, and we recover the well known result (for Euclidean domains),

$$b_p^{n/p,1}(\Omega)\subset L^{\infty}(\Omega). \tag{7.1.7}$$

On the other hand, when $q=\infty$, from (7.1.6) we only get

$$\dot{b}_p^{n/p,\infty}(\Omega)\subset L^{[\infty,\infty]}(\Omega). \tag{7.1.8}$$

In the Euclidean world better results are known. Recall that given a domain $\Omega\subset\mathbb{R}^n$ the Besov spaces $\dot{B}_p^{\theta,q}(\Omega)$ (resp. $B_p^{\theta,q}(\Omega)$), are defined by

$$\|f\|_{\dot{B}_p^{\theta,q}(\Omega)}=\left(\int_0^{|\Omega|}\left(t^{-\theta}K(t,f;L^p(\Omega),\dot{W}^1_{L^p}(\Omega))\right)^q\frac{dt}{t}\right)^{1/q} \tag{7.1.9}$$

$$\|f\|_{B_p^{\theta,q}(\Omega)}=\|f\|_{\dot{B}_p^{\theta,q}(\Omega}+\|f\|_{L^p}.$$

Indeed, for smooth domains, we have a better result than (7.1.7), namely

$$B_p^{n/p,1}(\Omega)\subset C(\Omega), \tag{7.1.10}$$

and, moreover, it is well known that (cf. [**20**])

$$\dot{B}_p^{n/p,\infty}(\Omega)\subset BMO(\Omega). \tag{7.1.11}$$

We note that since we have [3] $BMO([0,1]^n)\subset L^{[\infty,\infty]}$: *i.e.*,

$$\sup_t\left(|f|^{**}(t)-|f|^{*}(t)\right)\leq C\|f\|_{BMO},$$

then (7.1.11) is stronger than (7.1.8).

In Chapter 4 we have shown that for Sobolev and Besov spaces that are based on metric probability spaces with the relative uniform isoperimetric property,

3. This is an easy consequence of (7.1.15) below.

the rearrangement inequality (7.1.7) self-improves to (7.1.10). Let X be a r.i. space on Ω, we will show that the $K-$Poincaré inequality (cf. Theorem 6, Chapter 3)

$$\frac{1}{\mu(\Omega)}\int_{\Omega}|f-f_{\Omega}|\,d\mu\leq c\frac{K\left(\frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)},f;X,S_X\right)}{\phi_X(\mu(\Omega))},f\in X+S_X, \tag{7.1.12}$$

combined with the relative uniform isoperimetric property self improves to (7.1.11). In fact, the self improved result reads

$$\|f\|_{BMO(\Omega)}\leq C\sup_{0<t<\mu(\Omega)}\frac{K\left(\frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)},f;X,S_X\right)}{\phi_X(\mu(\Omega))}, \tag{7.1.13}$$

and is valid for the general class of isoperimetric profiles considered in this paper. Indeed, the result exhibits a new connection between the geometry of the ambient space and the embedding of Besov and *BMO* spaces. For example, for an Ahlfors $k-$regular space (Ω,d,μ) (cf. Section 5.4) given a ball B, consider the metric space $(B,d_{|B},\mu_{|B})$ then

$$\|f\|_{BMO(B)}\leq c\,\|f\|_{\dot{b}_p^{k/p,\infty}(B)},\ p>k. \tag{7.1.14}$$

We shall also discuss a connection between our development in this paper and a characterization of *BMO* provided by John [**53**] and Stromberg [**91**].

Finally, re-interpreting *BMO* as a limiting *Lip* space we were lead to an analog of (3.1.1) which we now describe. We argue that in $\mathbb{R}^n$ the natural replacement of (1.1.2) involving the space *BMO* is given by the Bennett-DeVore-Sharpley inequality (cf. [**11**], [**13**], [**1**], [**2**])

$$|f|^{**}(t)-|f|^{*}(t)\leq c(f^{\#})^{*}(t),\ 0<t<\frac{|B|}{6},\text{ where } B \text{ is a ball on } \mathbb{R}^n. \tag{7.1.15}$$

Variants of this inequality are known to hold in more general contexts. For our purposes here the following inequality will suffice

$$|f|_{\mu}^{**}(t)-|f|_{\mu}^{*}(t)\leq C\,\|f\|_{BMO},\ 0<t<\mu(\Omega). \tag{7.1.16}$$

We shall therefore assume for this particular discussion that our metric measure space (Ω,d,μ) also satisfies the following condition: There exists a constant $C>0$ such that (7.1.16) holds for all $f\in BMO$. For example, in [**86**, see (3.8)] it is shown that (7.1.16) holds for doubling measures on Euclidean domains. More general results can be found in [**1**].

Assuming the validity of (7.1.16), and using the method of proof of Theorem 7, we will show below (cf. Theorem 27) that if $X(\Omega)$ is a r.i. space, then we have [4]

$$|f|^{**}(t)-|f|^{*}(t)\leq c\frac{K(\phi_X(t),f;X(\Omega),BMO(\Omega))}{\phi_X(t)},\ 0<t<\mu(\Omega)/2. \tag{7.1.17}$$

4. On $\mathbb{R}^n$ (7.1.17) is known and can be obtained by combining (7.1.15) with [**13**, theorem 8.8].

This result should be compared with Theorem 7 above. For perspective, we now show a different road to a special case of (7.1.17). Recall that for Euclidean domains it is shown in [**13**, (8.11)] that

$$\frac{K(t,f;L^1,BMO)}{t} \simeq (f^{\#})^*(t).$$

Combining this inequality with (7.1.15), we obtain a different approach to (7.1.17) in the special case $X = L^1$, at least when t is close to zero.

7.1.1. Self Improving inequalities and ***BMO*****.** — We show that (7.1.12) combined with the relative uniform isoperimetric property yields the following embedding

Theorem 24. — *Let (Ω, d, μ) be a metric space satisfying the standard assumptions and with the relative uniform isoperimetric property. Let X be a r.i. space on Ω, then, there exists an absolute constant $C > 0$ such that,*

$$\|f\|_{BMO(\Omega)} \leq C \sup_{0<t<\mu(\Omega)} \frac{K\left(\frac{t/2}{I_\Omega(t/2)}, f; X, S_X\right)}{\phi_X(t)}.$$

Proof. — Given an integrable function f and a ball B in Ω, consider $f\chi_B$. By Theorem 6, applied to the metric space $(B, d_{|B}, \mu_{|B})$, we have

$$\frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \leq c \frac{K\left(\frac{\mu(B)/2}{I_B(\mu(B)/2)}, f\chi_B; X_r(B), S_{X_r}(B)\right)}{\phi_{X_r}(\mu(B))}.$$

Since (Ω, d, μ) has the relative uniform isoperimetric property, we have

$$\begin{aligned}\frac{K\left(\frac{\mu(B)/2}{I_B(\mu(B)/2)}, f\chi_B; X_r(B), S_{X_r}(B)\right)}{\phi_{X_r}(\mu(B))} &\leq C \frac{K\left(\frac{\mu(B)/2}{I_\Omega(\mu(B)/2)}, f\chi_B; X, S_X\right)}{\phi_X(\mu(B))} \\ &\leq C \sup_{0<t<\mu(\Omega)} \frac{K\left(\frac{t/2}{I_\Omega(t/2)}, f; X, S_X\right)}{\phi_X(t)}.\end{aligned}$$

Consequently,

$$\sup_B \frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \leq C \sup_{0<t<\mu(\Omega)} \frac{K\left(\frac{t/2}{I_\Omega(t/2)}, f; X, S_X\right)}{\phi_X(t)}. \qquad \square$$

We now give a concrete application of the previous result.

Corollary 1. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain that belongs to the Maz'ya's class $\mathcal{J}_{1-1/n}$ (cf. Section 5.3). Suppose that $p > n$, then*

$$\dot{B}_p^{n/p,\infty}(\Omega) \subset BMO(\Omega).$$

Proof. — Since Ω belongs to the Maz'ya's class $\mathcal{J}_{1-1/n}$ the following isoperimetric estimate holds.

$$t^{1-1/n} \preceq I_{\Omega}(t), \; t \in (0, |\Omega|/2).$$

On the other hand, for any ball B in Ω, we have

$$I_B(t) \geq c(n) \min(s, (|B| - s))^{\frac{n-1}{n}}, \qquad 0 < s < |B|.$$

Since $\Omega \subset \mathbb{R}^n$, using the same argument we provided in Section 5.2, it follows readily that the inequality (3.2.3) remains valid for all functions in $f \in X + \dot{W}_X^1$, *i.e.*,

$$\text{(7.1.18)} \qquad \frac{1}{\mu(\Omega)} \int_{\Omega} |f - f_{\Omega}| \, d\mu \leq c \frac{K\left(\frac{\mu(\Omega)/2}{I_{\Omega}(\mu(\Omega)/2)}, f; X, \dot{W}_X^1\right)}{\phi_X(\mu(\Omega))}.$$

Thus, by the argument given in the previous Theorem, we see that

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu &\leq C(n) \sup_{0<t<(|\Omega|/2)^{1/n}} \frac{K\left((t/2)^{1/n}, f; L^p, \dot{W}_{L^p}^1\right)}{t^{1/p}} \\ &\leq C(n) \sup_{0<t<|\Omega|} t^{-n/p} K\left(t, f; L^p, \dot{W}_{L^p}^1\right) \\ &= C(n) \|f\|_{\dot{B}_p^{n/p,\infty}(\Omega)}. \end{aligned}$$

□

7.2. On the John-Stromberg characterization of *BMO*

Our discussion in this chapter is closely connected with a characterization of $BMO([0,1]^n)$ using rearrangements due to John [**53**] and Stromberg [**91**]. Let $\lambda \in (0, \frac{1}{2}]$, then

$$\|f\|_{BMO_*} \simeq \sup_{Q \subset [0,1]^n} \inf_{c \in \mathbb{R}} ((f-c)\chi_Q)^* (\lambda |Q|).$$

See also Jawerth-Torchinsky [**52**], Lerner [**61**], [**30**], and the references therein.

Theorem 25. — *Let (Ω, d, μ) be a measure metric space satisfying our standard assumptions, let $f : \Omega \to \mathbb{R}$ be an integrable function. For a measurable set $Q \subset \Omega$, $(f\chi_Q)_\mu^* (\mu(Q)/2)$ is a median of f on Q.*

Proof. — It is easy to convince oneself that Definition 1 of median in is equivalent to

$$\mu\{f > m(f)\} \leq \mu(Q); \text{ and } \mu\{f < m(f)\} \leq \mu(Q).$$

Now

$$\mu\{f\chi_Q < (f\chi_Q)_\mu^*(\mu(Q)/2)\} = \mu\{-f\chi_Q > -(f\chi_Q)_\mu^*(\mu(Q)/2)\}$$

But since

$$(-f\chi_Q)_\mu^* (t) = -(f\chi_Q)_\mu^*(\mu(Q) - t)$$

it follows that

$$(-f\chi_Q)_\mu^* (\mu(Q)/2) = -(f\chi_Q)_\mu^*(\mu(Q)/2).$$

Consequently

$$\begin{aligned}\mu\{f\chi_Q < (f\chi_Q)^*_\mu(\mu(Q)/2)\} &= \mu\{-f\chi_Q > -(f\chi_Q)^*_\mu(\mu(Q)/2)\}\\ &= \mu\{-f\chi_Q > (-f\chi_Q)^*_\mu(\mu(Q)/2)\}\\ &\leq \mu(Q)/2 \text{ (by definition).}\end{aligned}$$

Therefore $(f\chi_Q)^*_\mu(\mu(Q)/2)$ is a median as we wished to show. □

As a consequence we have the following John-Stromberg inequality: for any ball B,

$$\text{(7.2.1)}\qquad \left((f\chi_B)^{**}_\mu(\mu(B)/2) - (f\chi_B)^*_\mu(\mu(B)/2)\right) \leq \frac{1}{2}\|f\|_{BMO}.$$

Theorem 26. — *Let (Ω, d, μ) be a metric space satisfying the standard assumptions. Then there exists a constant $C > 0$ such that for all f, it holds*

$$\|f\|_{BMO(\Omega)} \leq C \sup_{B \text{ ball in } \Omega} \left\{(f\chi_B)^{**}_\mu(\mu(B)) - (f\chi_B)^*_\mu(\mu(B))\right\}.$$

Proof. — For $t \in (0, \mu(\Omega))$ let us write

$$\begin{aligned}t(f^{**}_\mu(t) - f^*_\mu(t)) &= \int_0^t (f^*_\mu(x) - f^*_\mu(t))\, d\mu\\ &= \int_0^{\mu(\Omega)} \max\left(0, f^*_\mu(x) - f^*_\mu(t)\right) d\mu\\ &= \int_{\{s: f(s) > f^*_\mu(t)\}} \max\left(0, f(x) - f^*_\mu(t)\right) d\mu.\end{aligned}$$

Fix a ball B and apply the preceding equality to $f\chi_B$ and $t = \mu(B)$:

$$\begin{aligned}&\mu(B)\left((f\chi_B)^{**}_\mu(\mu(B)) - (f\chi_B)^*_\mu(\mu(B))\right)\\ &= \int_{\{s\in B: f(s) > (f\chi_B)^*_\mu(\mu(B))\}} \max(0, f\chi_B(s) - (f\chi_B)^*_\mu(\mu(B)))\, d\mu.\end{aligned}$$

To estimate the right hand side from below we observe that

$$f_B := \frac{1}{\mu(B)}\int_B f(x)\, d\mu = \frac{1}{\mu(B)}\int_0^{\mu(B)} (f\chi_B)^*_\mu(s)\, ds \geq (f\chi_B)^*_\mu(\mu(B)),$$

therefore

$$f(s) - (f\chi_Q)^*_\mu(\mu(B)) \geq f(s) - f_B.$$

Consequently,

$$\begin{aligned}&\int_{\{s\in B: f(s) > (f\chi_B)^*_\mu(\mu(B))\}} \max(0, f\chi_B(s) - (f\chi_B)^*_\mu(\mu(B)))\, d\mu.\\ &\geq \int_{\{s\in B:\ f(s) > f_B\}} \max(0, f\chi_B(s) - (f\chi_B)^*_\mu(\mu(B)))\, d\mu\\ &\geq \int_{\{x\in B:\ f(s) > f_B\}} [f(s) - f_B]\, d\mu.\end{aligned}$$

We will verify in a moment that

$$\frac{1}{\mu(B)}\int_{\{x\in B:\ f(s)>f_B\}}[f(s)-f_B]\,d\mu=\frac{1}{2}\frac{1}{\mu(B)}\int_B|f(s)-f_B|\,d\mu \tag{7.2.2}$$

Combining (7.2.2) with the previous estimates we see that

$$\left((f\chi_B)^{**}_\mu(\mu(B))-(f\chi_B)^{*}_\mu(\mu(B))\right)\geq\frac{1}{2}\frac{1}{\mu(B)}\int_B|f(s)-f_B|\,d\mu.$$

Hence

$$\begin{aligned}\|f\|_{BMO}&=\sup_B\frac{1}{\mu(B)}\int_B|f(s)-f_B|\,d\mu\\&\leq 2\sup_B\left((f\chi_B)^{**}_\mu(\mu(B))-(f\chi_B)^{*}_\mu(\mu(B))\right),\end{aligned}$$

as we wished to show.

It remains to see (7.2.2). Since

$$\int_{\{x\in B:\ f(s)>f_B\}}[f(s)-f_B]\,d\mu+\int_{\{x\in B:\ f(s)<f_B\}}[f(s)-f_B]\,d\mu=0,$$

we have that

$$\int_{\{x\in B:\ f(s)>f_B\}}[f(s)-f_B]\,d\mu=\int_{\{x\in B:\ f(s)<f_B\}}[f_B-f(s)]\,d\mu.$$

Consequently,

$$\begin{aligned}\int_B|f(s)-f_B|\,d\mu&=\int_{\{x\in B:\ f(s)>f_B\}}[f(s)-f_B]\,d\mu+\int_{\{x\in B:\ f(s)<f_B\}}[f_B-f(s)]\,d\mu\\&=2\int_{\{x\in B:\ f(s)>f_B\}}[f(s)-f_B]\,d\mu.\end{aligned}$$

□

7.3. Oscillation, *BMO* and $K-$functionals

As is well known in the Euclidean world or even for fairly general metric spaces (cf. [**26**]) one can realize *BMO* as a limiting *Lip* space. The easiest way to see this is through the equivalence

$$\|f\|_{Lip_\alpha}\simeq\sup_Q\frac{1}{|Q|^{1-\alpha/n}}\int_Q|f-f_Q|\,dx<\infty.$$

From this point of view *BMO* corresponds to a *Lip* space of order $\alpha=0$.

This observation leads naturally to consider the analogs of the results of Chapter 4 in the context of *BMO*.

Theorem 27. — *Suppose that* (Ω,d,μ) *is a metric space with finite measure and such that there exists an absolute constant* $C>0$ *such that for all* $f\in L^1(\Omega)$ *we have*

$$|f|^{**}_\mu(t)-|f|^{*}_\mu(t)\leq C\,\|f\|_{BMO},\ \ 0<t<\mu(\Omega). \tag{7.3.1}$$

Then, for every r.i. space $X(\Omega)$ there exists a constant $c > 0$ such that

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \le c\frac{K(\phi_X(t), f; X(\Omega), BMO(\Omega))}{\phi_X(t)}, \ 0 < t < \mu(\Omega), \tag{7.3.2}$$

where

$$K(t, f; X(\Omega), BMO(\Omega)) = \inf_{h\in BMO}\{\|f-h\|_X + t\,\|h\|_{BMO}\}.$$

Proof. — The proof follows exactly the same lines as the proof of Theorem 7, so we shall be brief. We start by noting three important properties that functional $\|f\|_{BMO}$ shares with $\||\nabla f|\|$:

(i) for any constant c, $\|f+c\|_{BMO} = \|f\|_{BMO}$,
(ii) $\||f|\|_{BMO} \le \|f\|_{BMO}$, and more generally
(iii) for any *Lip* 1 function Ψ, $\|\Psi(f)\|_{BMO} \le \|f\|_{BMO}$.

Let $t > 0$, then using the corresponding arguments in Theorem 7 shows that we have

$$\inf_{0\le h\in BMO}\{\||f|-h\|_X + \phi_X(t)\,\|h\|_{BMO}\} \le K(\phi_X(t), f; X(\Omega), BMO(\Omega)), \tag{7.3.3}$$

To prove (7.3.2) we proceed as in the proof of Theorem 7 until we arrive to

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \le ||f|-h|_\mu^{**}(t) + ||f|-h|_\mu^{*}(t) + |h|_\mu^{**}(t) - |h|_\mu^{*}(2t)$$

which we now estimate as

$$|f|_\mu^{**}(t/2) - |f|_\mu^{*}(t/2) \le 2\,||f|-h|_\mu^{**}(t) + \left(|h|_\mu^{**}(t) - |h|_\mu^{*}(t)\right) + \left(|h|_\mu^{*}(t) - |h|_\mu^{*}(2t)\right)$$

Note that

$$\begin{aligned}(|f|-h)_\mu^{**}(t/2) &= \frac{2}{t}\int_0^t (|f|-h)_\mu^{*}(s)\,ds\\ &\le 2\frac{\||f|-h\|_X\,\phi_{X'}(t)}{t} \text{ (Hölder's inequality)}\\ &= 2\frac{\||f|-h\|_X}{\phi_X(t)} \text{ (since } \phi_{X'}(t)\phi_X(t) = t).\end{aligned}$$

On the other hand, by (7.3.1)

$$|h|_\mu^{**}(t) - |h|_\mu^{*}(t) \le C\,\|h\|_{BMO}.$$

While by (1.1.15)

$$\begin{aligned}|h|_\mu^{*}(t) - |h|_\mu^{*}(2t) &\le 2\left(|h|_\mu^{**}(2t) - |h|_\mu^{*}(2t)\right)\\ &\le C\,\|h\|_{BMO} \text{ (again by (7.3.1)).}\end{aligned}$$

Therefore, combining our findings we see that

$$\begin{aligned}
|f|_{\mu}^{**}(t/2) - |f|_{\mu}^{*}(t/2) &\leq C \inf_{0\leq h\in BMO}\left\{ \frac{\||f|-h\|_X}{\phi_X(t)} + \|h\|_{BMO}\right\} \\
&= \frac{C}{\phi_X(t)} \inf_{0\leq h\in BMO}\{\||f|-h\|_X + \phi_X(t)\|h\|_{BMO}\} \\
&\leq \frac{C}{\phi_X(t)} K(\phi_X(t), f; X(\Omega), BMO(\Omega)) \text{ (by (7.3.3))}. \qquad \square
\end{aligned}$$

CHAPTER 8

ESTIMATION OF GROWTH "ENVELOPES"

8.1. Summary

Triebel and his school, in particular we should mention here the extensive work of Haroske, have studied the concept of "envelopes" (cf. [**49**], a book mainly devoted to the computation of growth and continuity envelopes of function spaces defined on $\mathbb{R}^n$). On the other hand, as far as we are aware, the problem of estimating growth envelopes for Sobolev or Besov spaces based on general measure spaces has not been treated systematically in the literature. For a function space $Z(\Omega)$, which we should think as measuring smoothness, one attempts to find precise estimates of ("the growth envelope")

$$E^Z(t) = \sup_{\|f\|_{Z(\Omega)} \leq 1} |f|^*(t).$$

A related problem is the estimation of "continuity envelopes" (cf. [**49**]). For example, suppose that $Z := Z(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, then we let

$$E_C^Z(t) = \sup_{\|f\|_Z \leq 1} \frac{\omega_{L^\infty}(t,f)}{t},$$

and the problem at hand is to obtain precise estimates of $E_C^Z(t)$.

In this chapter we estimate growth envelopes of function spaces based on metric probability spaces using our symmetrization inequalities. Most of the results we shall obtain, including those for Gaussian function spaces, are apparently new. Our method, moreover, gives a unified approach.

In a somewhat unrelated earlier work [**64**], we proposed some abstract ideas on how to study certain convergence and compactness properties in the context of interpolation scales. In Section 8.6 we shall briefly show a connection with the estimation of envelopes.

8.2. Spaces defined on measure spaces with Euclidean type profile

To fix ideas, and for easier comparison, in this section we consider metric probability [1] spaces (Ω, d, μ) satisfying the standard assumptions and such that the corresponding profiles satisfy

$$t^{1-1/n} \preceq I_\Omega(t),\ t \in (0, 1/2). \tag{8.2.1}$$

Particular examples are the $\mathcal{J}_{1-\frac{1}{n}}$–Maz'ya domains on $\mathbb{R}^n$. By the Lévy-Gromov isoperimetric inequality, Riemannian manifolds with positive Ricci curvature also satisfy (8.2.1).

In this context the basic rearrangement inequalities (cf. (1.1.2), (1.1.9)) take the following form, if $f \in Lip(\Omega)$, then

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \preceq t^{1/n} |\nabla f|_\mu^{**}(t),\ t \in (0, 1/2), \tag{8.2.2}$$

and, if $f \in X + S_X$, then

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \preceq \frac{K(t^{1/n}, f, X, S_X)}{\phi_X(t)},\ t \in (0, 1/2). \tag{8.2.3}$$

Theorem 28. — *Let $X = X(\Omega)$ be a r.i. space on Ω, and let $\bar{S}_X(\Omega)$ be defined by*

$$\bar{S}_X(\Omega) = \left\{ f \in Lip(\Omega) : \|f\|_{\bar{S}_X(\Omega)} = \||\nabla f|\|_X + \|f\|_X < \infty \right\}.$$

Then,

$$E^{\bar{S}_X(\Omega)}(t) \preceq \int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s},\ t \in (0, 1/2). \tag{8.2.4}$$

In particular, if $X = L^p, 1 \leq p < n$, then (compare with [**49**] *and see also Remark 15 below)*

$$E^{\bar{S}_{L^p}(\Omega)}(t) \preceq t^{1/n-1/p},\ t \in (0, 1/2). \tag{8.2.5}$$

1. Note that, when we are dealing with domains Ω with finite measure, we can usually assume without loss that we are dealing with functions such that $|f|^{**}(\infty) = 0$. Indeed, we have

$$|f|^{**}(t) \leq \frac{1}{t} \|f\|_{L^1(\Omega)}.$$

For the usual function spaces on $\mathbb{R}^n$, we can usually work with functions in $C_0(\mathbb{R}^n)$, which again obviously satisfy $|f|^{**}(\infty) = 0$.

Proof. — Let f be such that $\|f\|_{\bar{S}_X(\Omega)} \leq 1$. Using the fundamental theorem of Calculus we can write

$$|f|_\mu^{**}(t) - |f|_\mu^{**}(1/2) = \int_t^{1/2} \left(|f|_\mu^{**}(s) - |f|_\mu^{*}(s)\right) \frac{ds}{s}. \tag{8.2.6}$$

This representation combined with (8.2.2) and Hölder's inequality [2], yields

$$\begin{aligned}
|f|_\mu^{**}(t) &\leq c_n \int_t^{1/2} s^{1/n} |\nabla f|_\mu^{**}(s) \frac{ds}{s} + |f|_\mu^{**}(1/2) \\
&\leq c_n \int_t^{1} s^{1/n-1} \left\||\nabla f| \chi_{(0,s)}\right\|_{\bar{X}} \phi_{\bar{X}'}(s) \frac{ds}{s} + 2\|f\|_{L^1} \\
&\leq \|f\|_{\bar{S}_X} c_n \int_t^{1} s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 2\|f\|_{L^1} \\
&\leq c_n \int_t^{1} s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 2c,
\end{aligned}$$

where in the last step we used the fact that $\|f\|_{\bar{S}_X(\Omega)} \leq 1$, and $\|f\|_{L^1} \leq c\|f\|_X$. Therefore,

$$\begin{aligned}
E^{\bar{S}_X(\Omega)}(t) &= \sup_{\|f\|_{\bar{S}_X(\Omega)} \leq 1} |f|_\mu^{*}(t) \\
&\leq \sup_{\|f\|_{\bar{S}_X(\Omega)} \leq 1} |f|_\mu^{**}(t) \\
&\leq c\left(\int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 1\right), \quad t \in (0, 1/2).
\end{aligned}$$

The second part of the result follows readily by computation since, if $X = L^p$, $1 \leq p < n$, then

$$\begin{aligned}
\int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} &= \int_t^1 s^{1/n-1} s^{1-1/p} \frac{ds}{s} \\
&\leq \int_t^\infty s^{1/n-1} s^{1-1/p} \frac{ds}{s} \\
&\simeq t^{1/n-1/p},
\end{aligned}$$

and

$$1 \leq c t^{1/n-1/p}, \text{ for } t \in (0, 1/2). \qquad \square$$

---

2. Write

$$\begin{aligned}
s|\nabla f|^{**}(s) &= \int_0^s |\nabla f|^{*}(u)\, du \\
&\leq \left\||\nabla f| \chi_{(0,s)}\right\|_{\bar{X}} \left\|\chi_{(0,s)}\right\|_{\bar{X}'} \\
&= \left\||\nabla f| \chi_{(0,s)}\right\|_{\bar{X}} \phi_{\bar{X}'}(s).
\end{aligned}$$

Remark 15. — *We can also deal in the same fashion with infinite measures. For comparison with* [**49**] *let us consider the case of* $\mathbb{R}^n$ *provided with Lebesgue measure. In this case* $I_{\mathbb{R}^n}(t) = c_n t^{1-1/n}$, *for* $t > 0$, *and (8.2.2) is known to hold for all* $t > 0$, *and for all functions in* $C_0(\mathbb{R}^n)$ *(cf.* [**69**]*). For functions in* $C_0(\mathbb{R}^n)$ *we can replace (8.2.6) by*

$$|f|^{**}(t) = \int_t^\infty \left(|f|^{**}(s) - |f|^*(s)\right) \frac{ds}{s}.$$

Suppose further that X *is such that*

$$\int_t^\infty s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} < \infty.$$

Then, proceeding with the argument given in the proof above, we see that there is no need to restrict the range of $t's$ *for the validity of (8.2.4), (8.2.5), etc. Therefore, for* $1 \leq p < n$, *we have (compare with* [**49**, Proposition 3.25]*)*

$$E^{S^1_X(\mathbb{R}^n)}(t) \leq c\left(\int_t^\infty s^{1/n-1}\phi_{\bar{X}'}(s)\frac{ds}{s} + 1\right), \quad t > 0. \tag{8.2.7}$$

The use of Hölder's inequality as effected in the previous theorem does not give the sharp result at the end point $p = n$. Indeed, following the previous method for $p = n$, we only obtain

$$\begin{aligned} E^{W^1_{L^n}(\Omega)}(t) &\preceq \int_t^1 s^{1/n-1} s^{1-1/n} \frac{ds}{s} \\ &\preceq \ln\frac{1}{t}, \quad t \in (0, 1/2). \end{aligned}$$

Our next result shows that using (8.2.2) in a slightly different form (applying Hölder's inequality on the left hand side) we can obtain the sharp estimate in the limiting cases (compare with [**49**, Proposition 3.27]).

Theorem 29. — $E^{\bar{S}_{L^n}(\Omega)}(t) \preceq \left(\ln\frac{1}{t}\right)^{1/n'}$, *for* $t \in (0, 1/2)$.

Proof. — Suppose that $\|f\|_{\bar{S}_{L^n}(\Omega)} \leq 1$. First we rewrite (8.2.2) as

$$\left(\frac{|f|^{**}_\mu(s) - |f|^*_\mu(s)}{s^{1/n}}\right)^n \leq c_n \left(|\nabla f|^{**}_\mu(s)\right)^n, \quad s \in (0, 1/2).$$

Integrating, we thus find,

$$\begin{aligned} \int_t^{1/2} \left(|f|^{**}_\mu(s) - |f|^*_\mu(s)\right)^n \frac{ds}{s} &= \int_t^{1/2} \left(\frac{|f|^{**}_\mu(s) - |f|^*_\mu(s)}{s^{1/n}}\right)^n ds \\ &\leq c_n \int_t^1 \left(|\nabla f|^{**}_\mu(s)\right)^n ds \\ &\leq C_n \||\nabla f|\|^n_{L^n} \text{ (by Hardy's inequality)} \\ &\leq C_n. \end{aligned}$$

Now, for $t \in (0,1/2)$,

$$\begin{aligned}
|f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &= \int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s)\right) \frac{ds}{s} \\
&\leq \left(\int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s)\right)^n \frac{ds}{s}\right)^{1/n} \left(\int_t^1 \frac{ds}{s}\right)^{1/n'} \\
&\quad \text{(Hölder's inequality)} \\
&\leq C_n^{1/n} \left(\ln \frac{1}{t}\right)^{1/n'}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|f|_{\mu}^{**}(t) &\leq C_n^{1/n} \left(\ln \frac{1}{t}\right)^{1/n'} + \|f\|_{L^1} \\
&\leq C_n^{1/n} \left(\ln \frac{1}{t}\right)^{1/n'} + \|f\|_{L^n} \\
&\leq C_n^{1/n} \left(\ln \frac{1}{t}\right)^{1/n'}, \text{ for } t \in (0,1/2).
\end{aligned}$$

Consequently,

$$E^{\bar{S}_{L^n}(\Omega)}(t) \preceq \left(\ln \frac{1}{t}\right)^{1/n'}, \text{ for } t \in (0,1/2). \tag{8.2.8}$$ □

The case $p > n$, is somewhat less interesting for the computation of growth envelopes since we should have $E^{\bar{S}_{L^p}(\Omega)}(t) \leq c$. We now give a direct proof of this fact just to show that our method unifies all the cases.

Proposition 4. — *Let $p > n$, then there exists a constant $c = c(n,p)$ such that*

$$E^{\bar{S}_{L^p}(\Omega)}(t) \leq c, \ t \in (0,1/2).$$

Proof. — Suppose that $\|f\|_{\bar{S}_{L^p}(\Omega)} \leq 1$, and let $s \in (0,1/2)$. We estimate as follows

$$\begin{aligned}
|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) &\leq ct^{1/n-1} \int_0^t |\nabla f|_{\mu}^{*}(s)\, ds \\
&\leq ct^{1/n-1} \left(\int_0^1 |\nabla f|_{\mu}^{*}(s)^p ds\right)^{1/p} t^{1/p'} \\
&\leq ct^{1/n-1/p}.
\end{aligned}$$

Thus, using a familiar argument, we see that for $t \in (0,1/2)$,

$$\begin{aligned}
|f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &\leq c \int_t^1 s^{1/n-1/p-1} ds \\
&\leq c \frac{1 - t^{1/n-1/p}}{(1/n - 1/p)} \\
&\leq \frac{c}{(1/n - 1/p)}.
\end{aligned}$$

It follows that

$$\begin{aligned} |f|_{\mu}^{*}(t) \leq |f|_{\mu}^{**}(t) &\leq \frac{c}{(1/n-1/p)} + |f|_{\mu}^{**}(1/2) \\ &\leq \frac{c}{(1/n-1/p)}, \end{aligned}$$

and we obtain

$$E^{\bar{S}_{L^p}(\Omega)}(t) \leq c, \ t \in (0,1/2). \qquad \square$$

The methods discussed above apply to Sobolev spaces based on general r.i. spaces. As another illustration we now consider in detail the case of the Sobolev spaces based on the Lorentz spaces $L^{n,q}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded *Lip* domain of measure 1. The interest here lies in the fact that in the critical case $p = n$, the second index q plays an important role. Indeed, for $q = 1$, as is well known, we have $W^1_{L^{n,1}} \subset L^{\infty}$, while this is no longer true for $W^1_{L^{n,q}}$ if $q > 1$. In particular, for the space $W^1_{L^{n,n}} = W^1_{L^n}$. The next result thus extends Theorem 29 and provides an explanation of the situation we have just described through the use of growth envelopes.

Theorem 30. — *Let* $1 \leq q \leq \infty$, *then* $E^{W^1_{L^{n,q}}(\Omega)}(t) \preceq \left(\ln \frac{1}{t}\right)^{1/q'}$, *for* $t \in (0,1/2)$.

Proof. — Consider first the case $1 \leq q < \infty$. Suppose that $\|f\|_{W^1_{L^{n,q}}(\Omega)} \leq 1$. From (8.2.2) we get

$$\left(|f|^{**}(s) - |f|^{*}(s)\right)^q \leq c_n \left(|\nabla f|^{**}(s) s^{1/n}\right)^q, \ s \in (0,1/2).$$

Then,

$$\begin{aligned} |f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &= \int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s)\right) \frac{ds}{s} \\ &\leq \left(\int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s)\right)^q \frac{ds}{s}\right)^{1/q} \left(\int_t^{1/2} \frac{ds}{s}\right)^{1/q'} \\ &\leq c \left(\int_t^{1/2} \left(|\nabla f|_{\mu}^{**}(s) s^{1/n}\right)^q \frac{ds}{s}\right)^{1/q} \left(\int_t^{1/2} \frac{ds}{s}\right)^{1/q'} \\ &\leq c \, \||\nabla f|\|_{L^{n,q}} \left(\ln \frac{1}{t}\right)^{1/q'}. \end{aligned}$$

Therefore, as before

$$\begin{aligned} |f|_{\mu}^{**}(t) &\leq c \, \||\nabla f|\|_{L^{n,q}} \left(\ln \frac{1}{t}\right)^{1/q'} + |f|_{\mu}^{**}(1/2) \\ &\leq c \left(\ln \frac{1}{t}\right)^{1/q'}, \ t \in (0,1/2), \end{aligned}$$

and the desired estimate for $E^{W^1_{L^{n,q}}(\Omega)}(t)$ follows.

When $q=\infty$, and $\|f\|_{W^1_{L^{n,\infty}}(\Omega)} \leq 1$, we estimate

$$\begin{aligned} |f|_\mu^{**}(s) - |f|_\mu^{*}(s) &\leq c_n \, |\nabla f|^{**}(s) s^{1/n} \\ &\leq c_n \, \||\nabla f|\|_{L^{n,\infty}} \\ &\leq c_n . \end{aligned}$$

Consequently,

$$|f|_\mu^{**}(t) - |f|_\mu^{**}(1/2) \leq c_n \int_t^1 \frac{ds}{s}$$

and we readily get

$$E^{W^1_{L^{n,\infty}}(\Omega)}(t) \leq c\left(\ln \frac{1}{t}\right), \quad t \in (0,1/2). \qquad \square$$

Remark 16. — *Note that, in particular,*

$$E^{W^1_{L^{n,1}}(\Omega)}(t) \leq c,$$

which again reflects the fact that $W^1_{L^{n,1}}(\Omega) \subset L^\infty(\Omega)$.

Remark 17. — *As before, all the previous results hold for the* $W^1_{L^{p,q}}(\mathbb{R}^n)$ *spaces.*

We now show that a similar method, replacing the use of (8.2.2) by (8.2.3), allow us to obtain sharp estimates for growth envelopes of Besov spaces (see (7.1.9)).

Theorem 31. — *Let* $p > n > 1$, $1 \leq q \leq \infty$. *Then*

$$E^{B_p^{n/p,q}([0,1]^n)}(t) \preceq \left(\log \frac{1}{t}\right)^{1/q'}, \quad t \in (0,1/2).$$

Proof. — The starting point is

$$|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \leq c \frac{\omega_{L^p}(t^{1/n}, f)}{t^{1/p}}, \quad t \in (0,1/2).$$

Then,

$$\begin{aligned} |f|_\mu^{**}(t) - |f|_\mu^{**}(1/2) &= \int_t^{1/2} \left(|f|_\mu^{**}(s) - |f|_\mu^{*}(s)\right) \frac{ds}{s} \\ &\leq c \int_t^{1/2} \frac{\omega_{L^p}(s^{1/n}, f)}{s^{1/p}} \frac{ds}{s} \\ &\leq c \left(\int_t^1 \left(\frac{\omega_{L^p}(s^{1/n}, f)}{s^{1/p}}\right)^q \frac{ds}{s}\right)^{1/q} \left(\int_t^1 \frac{ds}{s}\right)^{1/q'} \\ &\quad \text{(Hölder's inequality)} \\ &\leq c \, \|f\|_{B_p^{n/p,q}([0,1]^n)} \left(\log \frac{1}{t}\right)^{1/q'} . \end{aligned}$$

Thus, a familiar argument now gives (compare with [**49**, (1.9)])

$$E^{B_p^{n/p,q}([0,1]^n)}(t) \preceq \left(\log \frac{1}{t}\right)^{1/q'}, \quad t \in (0, 1/2). \qquad \square$$

8.3. Continuity Envelopes

Suppose that $Z := Z(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, then (cf. [**49**] and the references therein) one defines the continuity envelope by

$$E_C^Z(t) = \sup_{\|f\|_{Z(\mathbb{R}^n)} \leq 1} \frac{\omega_{L^\infty}(t, f)}{t}.$$

At this point it is instructive to recall some known interpolation inequalities. Let $\|f\|_{\dot{W}^1_{L^{n,1}}} = \int_0^\infty |\nabla f|^*(s) s^{1/n} \frac{ds}{s}$. We interpolate the following known embeddings (cf. [**89**]): for $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|f\|_{L^\infty(\mathbb{R}^n)} \preceq \|f\|_{\dot{W}^1_{L^{n,1}(\mathbb{R}^n)}}.$$

Consequently,

$$K(t, f; L^\infty(\mathbb{R}^n), \dot{W}^1_{L^\infty}(\mathbb{R}^n)) \preceq K(t, f, \dot{W}^1_{L^{n,1}}(\mathbb{R}^n), \dot{W}^1_{L^\infty}(\mathbb{R}^n)). \tag{8.3.1}$$

It is well-known that for continuous functions we have (cf. [**13**])

$$K(t, f; L^\infty(\mathbb{R}^n), \dot{W}^1_{L^\infty}(\mathbb{R}^n)) = \omega_{L^\infty}(t, f) \simeq \sup_{|h| \leq t} \|f(\cdot + h) - f(\cdot)\|_{L^\infty}.$$

On the other hand using [**67**, Theorem 2] and Holmstedt's Lemma (see [**14**, Theorem 3.6.1]) we find

$$K(t, f, \dot{W}^1_{L^{n,1}}(\mathbb{R}^n), \dot{W}^1_{L^\infty}(\mathbb{R}^n)) \simeq \int_0^{t^n} |\nabla f|^*(s) s^{1/n} \frac{ds}{s}.$$

Inserting this information back to (8.3.1) we find

$$\omega_{L^\infty}(t, f) \preceq \int_0^{t^n} |\nabla f|^*(s) s^{1/n} \frac{ds}{s}.$$

Therefore,

$$\begin{aligned}
\frac{\omega_{L^\infty}(t, f)}{t} &\preceq \frac{1}{t} \int_0^{t^n} |\nabla f|^*(s) s^{1/n} \frac{ds}{s} \\
&\preceq \frac{1}{t} \||\nabla f|\|_{L^{p,q}} \left(\int_0^{t^n} s^{(1/n - 1/p)q'} \frac{ds}{s}\right)^{1/q'} \\
&\preceq \frac{1}{t} \||\nabla f|\|_{L^{p,q}} t^{1-n/p}.
\end{aligned}$$

Thus, we have (compare with [**49**, (1.15)]) that for $1 \leq q < \infty$,

$$E_C^{W^1_{L^{p,q}}(\mathbb{R}^n)}(t) \preceq t^{-n/p}. \tag{8.3.2}$$

Remark 18. — *It is actually fairly straightforward at this point to derive a general relation between* E^Z *and* E_C^Z. *In* [**65**] *we have shown for* $f \in C_0^\infty(\mathbb{R}^n)$ *we have*

$$|f|^{**}(t) - |f|^*(t) \leq c_n \omega_{L^\infty}(t^{1/n}, f), \ t > 0. \tag{8.3.3}$$

We now proceed formally, although the details can be easily filled-in by the interested reader. From (8.3.3) we find

$$\frac{|f|^{**}(t) - |f|^*(t)}{t} \leq c_n \frac{\omega_{L^\infty}(t^{1/n}, f)}{t}, \ t > 0.$$

Then

$$\begin{aligned} |f|^*(t) &\leq |f|^{**}(t) \\ &= \int_t^\infty \frac{|f|^{**}(s) - |f|^*(s)}{s} ds \ (\textit{since } f^{**}(\infty) = 0) \\ &\leq c_n \int_t^\infty \frac{\omega_{L^\infty}(s^{1/n}, f)}{s} ds. \end{aligned}$$

Taking supremum over the unit ball of $Z(\mathbb{R}^n)$ *we obtain*

$$E^{Z(\mathbb{R}^n)}(t) \preceq \int_{t^{1/n}}^\infty E_C^{Z(\mathbb{R}^n)}(s)\, ds.$$

Thus, for example, from (8.3.2) we find that for $p < n$

$$\begin{aligned} E^{W^1_{L^{p,q}}(\mathbb{R}^n)}(t) &\preceq \int_{t^{1/n}}^\infty s^{-n/p}\, ds \\ &\simeq t^{1/n - 1/p}, \end{aligned}$$

which should be compared with (8.2.5).

8.4. General isoperimetric profiles

In the previous sections we have focused mainly on function spaces on domains with isoperimetric profiles of Euclidean type; but our inequalities also provide a unified setting to study estimates for general profiles. For a metric measure space (Ω, d, μ) of finite measure we consider r.i. spaces $X(\Omega)$. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$, the homogeneous Besov space $\dot{b}^\theta_{X,q}(\Omega)$ is defined by

$$\dot{b}_X^{\theta,q}(\Omega) = \left\{ f \in X + S_X : \|f\|_{\dot{b}_X^{\theta,q}(\Omega)} = \left(\int_0^{\mu(\Omega)} \left(K(s, f; X, S_X)\, s^{-\theta} \right)^q \frac{ds}{s} \right)^{1/q} < \infty \right\},$$

with the usual modifications when $q = \infty$. The Besov space $b^\theta_{X,q}(\Omega, \mu)$ is defined by

$$\|f\|_{b_X^{\theta,q}(\Omega,\mu)} = \|f\|_X + \|f\|_{\dot{b}_X^{\theta,q}(\Omega)}.$$

Notice that if $X = L^p$, then $\dot{b}^{\theta,q}_{L^p}(\Omega) = \dot{b}^{\theta,q}_p(\Omega)$ (resp. $b^{\theta,q}_{L^p}(\Omega) = b^{\theta,q}_p(\Omega)$) (see (7.1.1)).

Theorem 32. — *Let X be a r.i. space on Ω. Let $g(s)=\frac{s}{I_{\Omega}(s)}$ where I_{Ω} denotes the isoperimetric profile of (Ω,d,μ). Let $b_X^{\theta,q}(\Omega)$ be a Besov space $(0<\theta<1,\ 1<q<\infty)$, then for $t\in(0,\mu(\Omega)/2)$ we have that*

$$E^{b_X^{\theta,q}(\Omega)}(t)\leq c\left(1+\left(\int_t^{\mu(\Omega)/2}\left(g(s)^{\theta}\left(\frac{g(s)}{g'(s)}\right)^{1/q}\right)^{q'}\frac{ds}{(s\phi_X(s))^{q'}}\right)\right),$$

where, as usual $1/q'+1/q=1$.

Proof. — Let $f\in X+S_X$, and let us write $K(t,f;X,S_X):=K(t,f)$. By Theorem 7 we know that

$$|f|_{\mu}^{**}(t)-|f|_{\mu}^{*}(t)\leq c\frac{K(g(t),f)}{\phi_X(t)},\ 0<t<\mu(\Omega).$$

Taking into account that, $\left(-|f|_{\mu}^{**}\right)'(t)=\left(|f|_{\mu}^{**}(t)-|f|_{\mu}^{*}(t)\right)/t$, we get

$$|f|_{\mu}^{**}(t)-|f|_{\mu}^{**}(\mu(\Omega)/2)=\int_t^{\mu(\Omega)/2}\left(-|f|_{\mu}^{**}\right)'(s)\,ds\leq c\int_t^{\mu(\Omega)/2}\frac{K(g(s),f)}{\phi_X(s)}\frac{ds}{s}.$$

Since $I_{\Omega}(s)$ is a concave continuous increasing function on $(0,\mu(\Omega)/2)$, $g(s)$ is differentiable on $(0,\mu(\Omega)/2)$. Then, by Hölder's inequality we have

$$\begin{aligned}R(t)&=\int_t^{\mu(\Omega)/2}\frac{K(g(s),f)}{\phi_X(s)}\frac{ds}{s}\\&=\int_t^{\mu(\Omega)/2}K(g(s),f)\left(\frac{g(s)}{g(s)}\right)^{\theta}\left(\frac{g'(s)}{g(s)}\right)^{1/q}\left(\frac{g(s)}{g'(s)}\right)^{1/q}\frac{ds}{s\phi_X(s)}\\&\leq R_1(t)R_2(t),\end{aligned}$$

where

$$R_1(t)=\left(\int_t^{\mu(\Omega)/2}\left(K(g(s),f)\,g(s)^{-\theta}\right)^q\left(\frac{g'(s)}{g(s)}\right)ds\right)^{1/q},$$

and

$$R_2(t)=\left(\int_t^{\mu(\Omega)/2}\left(g(s)^{\theta}\left(\frac{g(s)}{g'(s)}\right)^{1/q}\right)^{q'}\frac{ds}{(s\phi_X(s))^{q'}}\right)^{1/q'}.\tag{8.4.1}$$

By a change of variables

$$R_1(t)=\left(\int_{g^{-1}(t)}^{g^{-1}(\mu(\Omega)/2)}\left(K(z,f)\,z^{-\theta}\right)^q\frac{dz}{z}\right)^{1/q}\leq\|f\|_{\dot{b}^{\theta}_{X,q}(\Omega)}\tag{8.4.2}$$

Combining (8.4.1) and (8.4.2) we obtain

$$\begin{aligned}|f|_{\mu}^{**}(t) &\leq c\,\|f\|_{b^{\theta}_{X,q}(\Omega)}\,R_2(t)+2\,\|f\|_X\\ &\leq 2c\,(1+R_2(t))\,\|f\|_{b^{\theta}_{X,q}(\Omega)}\,.\end{aligned}$$

Therefore, taking sup over all f such that $\|f\|_{b^{\theta}_{X,q}(\Omega)}\leq 1$ we see that

$$E^{b^{\theta,q}_{X}(\Omega)}(t)\leq 2c\,(1+R_2(t))\,,\ t\in(0,\mu(\Omega)/2).$$

□

Example 3. — *Consider the Gaussian measure* $(\mathbb{R}^n,\gamma_n)$. *Then (cf.* [**16**]*) we can take as isoperimetric estimator*

$$I_{\gamma_n}(t)=t\left(\log\frac{1}{t}\right)^{1/2},\qquad t\in(0,1/2).$$

Thus,

$$g(t)=\frac{1}{\left(\log\frac{1}{t}\right)^{1/2}}\ \ \textit{and}\ \ g'(s)=\frac{1}{2\left(\log\frac{1}{s}\right)^{\frac{3}{2}}s},$$

and

$$\begin{aligned}&\left(1+\left(\int_t^{1/2}\left(g(s)^{\theta}\left(\frac{g(s)}{g'(s)}\right)^{1/q}\right)^{q'}\frac{ds}{(s\phi_X(s))^{q'}}\right)^{1/q'}\right)\\ &=\left(\int_t^{1/2}\left(\log\frac{1}{s}\right)^{q'\left(1-\frac{\theta}{2}\right)-1}\frac{ds}{s\,(\phi_X(s))^{q'}}\right)^{1/q'}\\ &\leq\frac{1}{\phi_X(t)}\left(\int_t^{1/2}\left(\log\frac{1}{s}\right)^{q'\left(1-\frac{\theta}{2}\right)-1}\frac{ds}{s}\right)^{1/q'}\\ &\preceq\frac{1}{\phi_X(t)}\left(\log\frac{1}{t}\right)^{\left(1-\frac{\theta}{2}\right)}.\end{aligned}$$

Therefore we find that

$$E^{B^{\theta,q}_{X}(\mathbb{R}^n,\gamma_n)}(t)\preceq\frac{1}{\phi_X(t)}\left(\log\frac{1}{t}\right)^{\left(1-\frac{\theta}{2}\right)},\qquad t\in(0,1/2).$$

8.5. Envelopes for higher order spaces

In general it is not clear how to define higher order Sobolev and Besov spaces in metric spaces. On the other hand for classical domains (Euclidean, Riemannian manifolds, etc.) there is a well developed theory of embeddings that one can use to estimate growth envelopes. The underlying general principle is very simple. Suppose that the function space $Z=Z\,(\Omega)$ is continuously embedded in $Y=Y\,(\Omega)$ and Y is a rearrangement invariant space, then, since (cf. (2.2.6)) $Y\subset M\,(Y)$, where $M(Y)$

is the Marcinkiewicz space associated with Y (cf. Section 2.2), we have (cf. (2.2.5)) for all $f \in Z$,

$$\sup_t |f|^*(t)\phi_Y(t) \leq \|f\|_Y \leq c\|f\|_Z,$$

where $\phi_Y(t)$ is the fundamental function of Y, and c is the norm of the embedding $Z \subset Y$. Consequently, for all $f \in Z$, with $\|f\|_Z \leq 1$, for all $t > 0$,

$$|f|^*(t) \leq \frac{c}{\phi_Y(t)}.$$

Therefore,

$$E^Z(t) \preceq \frac{1}{\phi_Y(t)}.$$

For example, suppose that $p < \frac{n}{k}$, then from

$$W_p^k(\mathbb{R}^n) \subset L^{q,p}, \text{ with } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

and

$$\phi_{L^{q,p}}(t) = t^{1/q} = t^{1/p-k/n}$$

we get (compare with (8.2.5) above and [**49**, (1.7)])

$$E^{W_p^k(\mathbb{R}^n)}(t) \preceq t^{k/n-1/p}.$$

In the limiting case we have (cf. [**9**], [**78**])

$$W_{\frac{n}{k}}^k(\mathbb{R}^n) \subset L^{[\infty,\frac{n}{k}]}.$$

For comparison consider $W_{\frac{n}{k}}^k(\Omega)$, where Ω is a domain on $\mathbb{R}^n$, with $|\Omega| = 1$. One can readily estimate the decay of functions in $L^{[\infty,\frac{n}{k}]}$ as follows:

$$\begin{aligned}
|f|^{**}(t) - |f|^{**}(1) &= \int_t^1 \left(|f|^{**}(s) - |f|^*(s)\right)\frac{ds}{s} \\
&\leq \left(\int_t^1 \left(|f|^{**}(s) - |f|^*(s)\right)^{\frac{n}{k}}\frac{ds}{s}\right)^{1/(\frac{n}{k})}\left(\int_t^1 \frac{ds}{s}\right)^{1/(\frac{n}{k})'} \\
&\leq \|f\|_{L^{[\infty,\frac{n}{k}]}}\left(\log\frac{1}{t}\right)^{1-\frac{k}{n}}.
\end{aligned}$$

Combining these observations we see that for functions in the unit ball of $W_{\frac{n}{k}}^k(\Omega)$ we have

$$|f|^{**}(t) \preceq c\left(\log\frac{1}{t}\right)^{1-\frac{k}{n}}, \text{ for } t \in (0, 1/2).$$

Consequently

$$E^{W_{\frac{n}{k}}^k(\Omega)}(t) \preceq \left(\log\frac{1}{t}\right)^{1-\frac{k}{n}}.$$

In particular, when $k = 1$ then $1 - \frac{k}{n} = \frac{1}{n'}$, and the result coincides with Theorem 29 above.

Likewise we can deal with the case of general isoperimetric profiles but we shall leave the discussion for another occasion.

8.6. K and E functionals for families

It is of interest to point out a connection between the different "envelopes" discussed above and a more general concept introduced somewhat earlier in [**64**], but in a different context. One of the tools introduced in [**64**] was to consider the K and E functionals for families, rather than single elements.

Given a compatible pair of spaces (X, Y) (cf. [**14**]), and a family of elements, $F \subset X + Y$, we can define the $K-$functional and $E-$functional [3] of the family F by (cf. [**64**])

$$K(t, F; X, Y) = \sup_{f \in F} K(t, f; X, Y).$$
$$E(t, F; X, Y) = \sup_{f \in F} E(t, f; X, Y).$$

The connection with the Triebel-Haroske envelopes can be seen from the following known computations. If we let $\|f\|_{L^0} = \mu\{\text{supp} f\}$, then

$$|f|^* (t) = E(t, f; L^0, L^\infty).$$

Therefore,

$$E^{Z(\Omega)}(t) = E(t, \text{unit ball of } Z(\Omega); L^0(\Omega), L^\infty(\Omega)).$$

Moreover, since on Euclidean space we have

$$\omega_{L^\infty}(t, f) \simeq K(t, f; L^\infty(\mathbb{R}^n), \dot{W}^1_{L^\infty}(\mathbb{R}^n))$$

we therefore see that

$$E_C^{Z(\mathbb{R}^n)}(t) = K(t, f; \text{unit ball of } Z(\mathbb{R}^n); L^\infty(\mathbb{R}^n), \dot{W}^1_{L^\infty}(\mathbb{R}^n)).$$

This suggests the general definition for metric spaces

$$E_C^{Z(\Omega)}(t) = K(t, f; \text{unit ball of } Z(\Omega); L^\infty(\Omega), S_{L^\infty}(\Omega)).$$

This provides a method to expand the known results to the metric setting using the methods discussed in this paper. Another interesting aspect of the connection we have established here lies in the fact, established in [**64**], that one can reformulate classical convergence and compactness criteria for function spaces (*e.g.*, Kolmogorov's compactness criteria for sets contained in L^p) in terms of conditions on these (new)

3. Recall that (cf. [**14**], [**64**]),

$$K(t, f; X, Y) = \inf\{\|f - g\|_X + t\|g\|_Y : g \in Y\}$$
$$E(t, f; X, Y) = \inf\{\|f - g\|_Y : \|g\|_X \le t\}.$$

functionals. For example, according to the Kolmogorov criteria, for a set of functions F to be compact on $L^p(\mathbb{R}^n)$ one needs the uniform continuity on F at zero of $\omega_{L^p}(t, F)$. In our formulation we replace this condition by demanding the continuity at zero of

$$K(t, F; L^p, W^1_{L^p}).$$

Again, to develop this material in detail is a long paper on its own, however, let us note in passing that the failure of compactness of the embedding $W^1_{L^p}(\Omega) \subset L^{\bar{p}}(\Omega)$, for $p = n$, is consistent with the blow up at zero predicted by the fact that the converse of (8.2.8) also holds. One should compare this with the estimate (8.2.7) which is consistent with the Relich compactness criteria for Sobolev embeddings on bounded domains, when $p < n$.

CHAPTER 9

LORENTZ SPACES WITH NEGATIVE INDICES

9.1. Introduction and Summary

As we have shown elsewhere (cf. [**9**], [**66**]), the basic Euclidean inequality

$$f^{**}(t) - f^{*}(t) \leq c_n t^{1/n} \left|\nabla f\right|^{**}(t)$$

leads to the optimal Sobolev inequality

$$\|f\|_{L^{[\bar{p},p]}} = \left\{ \int_0^\infty \left(\left(|f|^{**}(t) - |f|^{*}(t) \right) t^{1/\bar{p}} \right)^p \frac{dt}{t} \right\}^{1/p} \leq c_n \left\| |\nabla f| \right\|_{L^p}, \tag{9.1.1}$$

where $1 < p \leq n$, $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n}$. The use of the $L^{[\bar{p},p]}$ conditions makes it possible to consider the limiting case $p = n$ in a unified way. Now (9.1.1) is also meaningful when $p > n$, albeit the only reason for the restriction $p \leq n$, is that, if we don't impose it, then $\bar{p} < 0$, and thus the condition defined by $\|f\|_{L^{[\bar{p},p]}} < \infty$ is not well understood. It is was shown in [**78**] that these conditions are meaningful. In this chapter we show a connection between the Lorentz $L^{[\bar{p},p]}$ spaces with negative indices and Morrey's theorem.

9.1.1. Lorentz conditions.

— Let (Ω, d, μ) be a metric measure space. Let $0 < q \leq \infty$, $s \in \mathbb{R}$. We define

$$\begin{aligned} L^{[s,q]} &= L^{[s,q]}(0, \mu(\Omega)) \\ &= \left\{ f \in L^1(\Omega) : \left\{ \int_0^{\mu(\Omega)} \left(\left(|f|_\mu^{**}(t) - |f|_\mu^{*}(t) \right) t^{1/s} \right)^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}. \end{aligned}$$

For $0 < q \leq \infty$, $s \in [1, \infty]$, these spaces were defined in Chapter 7. They coincide with the usual $L^{s,q}$ spaces when $0 < q \leq \infty$, $s \in [1, \infty)$ (cf. [**71**]).

Our first observation is that $L^{[s,q]} \neq \varnothing$. Indeed, for $s < 0$, we have

$$\begin{aligned} 0 < \|\chi_A\|_{L^{[s,q]}} &= \frac{\mu(A)}{(q - q/s)^{1/q}} [\mu(A)^{q/s-q} - 1]^{1/q} \\ &\leq \frac{\mu(A)^{1/s}}{(q - q/s)^{1/q}}. \end{aligned}$$

It is important to remark that the cancellation at zero afforded by $|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t)$ is crucial here. Indeed, if we attempt to extend the usual definition of Lorentz spaces by letting $s < 0$, then we find that $\|\chi_A\|_{L^{(s,q)}} = \int_0^{\mu(A)} t^{q/s} \frac{dt}{t} < \infty$ iff $\mu(A) = 0$.

9.2. The role of the $L^{[\bar{p},p]}$ spaces in Morrey's theorem

For definiteness we work on $\mathbb{R}^n$ with Lebesgue measure m. We show that many arguments we have discussed in this paper are available in the context of Lorentz spaces with negative index.

Let $f \in L^{[\bar{p},p]}$ where $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n} < 0$. Then, for $0 < t_1 < t_2$, we can write

$$\begin{aligned} f^{**}(t_1) - f^{**}(t_2) &= \int_{t_1}^{t_2} (f^{**}(t) - f^{*}(t))\, t^{1/\bar{p}} t^{-1/\bar{p}} \frac{dt}{t} \\ &\leq \left(\int_{t_1}^{t_2} \left((f^{**}(t) - f^{*}(t))\, t^{1/\bar{p}} \right)^p \frac{dt}{t} \right)^{1/p} \left(\int_{t_1}^{t_2} t^{-p'/\bar{p}} \frac{dt}{t} \right)^{1/p'} \\ &= \|f\|_{L^{[\bar{p},p]}} \left(\int_{t_1}^{t_2} t^{-p'/\bar{p}} \frac{dt}{t} \right)^{1/p'}. \end{aligned}$$

Note that since $\frac{-p'}{\bar{p}} - 1 = \frac{p}{p-1}(\frac{n-p}{np}) - 1 = \frac{1}{p-1}[\frac{n-p-n}{n}] < 0$, the function $t^{-p'/\bar{p}-1}$ is decreasing and therefore,

$$\int_{t_1}^{t_2} t^{-p'/\bar{p}} \frac{dt}{t} \leq \int_0^{t_2 - t_1} t^{-p'/\bar{p}} \frac{dt}{t} = \frac{-\bar{p}}{p'} |t_2 - t_1|^{\frac{-p'}{\bar{p}}}.$$

Thus,

$$f^{**}(t_1) - f^{**}(t_2) \leq \left(\frac{-\bar{p}}{p'} \right)^{1/p'} \|f\|_{L^{[\bar{p},p]}} |t_2 - t_1|^{\frac{-1}{\bar{p}}}. \tag{9.2.1}$$

The localization property in this context takes the following form. Suppose that $f \in L^{[\bar{p},p]}$ is such that there exists a constant $C > 0$, such that $\forall B$ open ball, it follows that $f\chi_B \in L^{[\bar{p},p]}(0,m(B))$, with $\|f\chi_B\|_{L^{[\bar{p},p]}} \leq C\|f\|_{L^{[\bar{p},p]}}$. Then, from (9.2.1) we get

$$(f\chi_B)^{**}(t_1) - (f\chi_B)^{**}(t_2) \leq C\left(\frac{-\bar{p}}{p'}\right)^{1/p'} \|f\|_{L^{[\bar{p},p]}} |t_2 - t_1|^{\frac{\alpha}{n}},$$

where $\alpha = 1 - \frac{n}{p}$. Applying this inequality replacing t_i by $t_i m(B)$, $i = 1, 2$; we get

$$(f\chi_B)^{**}(t_1 m(B)) - (f\chi_B)^{**}(t_2 m(B)) \leq C\left(\frac{-\bar{p}}{p'}\right)^{1/p'} \|f\|_{L^{[\bar{p},p]}} |t_2 - t_1|^{\frac{\alpha}{n}} m(B)^{\frac{\alpha}{n}}.$$

Letting $t_1 \to 0, t_2 \to 1$, we then find

$$\mathit{ess}\sup_B (f) - \frac{1}{m(B)}\int_B f \leq C\left(\frac{-\bar{p}}{p'}\right)^{1/p'} \|f\|_{L^{[\bar{p},p]}} m(B)^{a/n}.$$

Applying this inequality to $-f$ and adding we arrive at

$$\mathit{ess}\sup_B (f) - \mathit{ess}\inf_B f \leq 2C\left(\frac{-\bar{p}}{p'}\right)^{1/p'} \|f\|_{L^{[\bar{p},p]}} m(B)^{a/n}.$$

Let $x, y \in \mathbb{R}^n$, and consider $B = B(x, 3|x-y|)$ (*i.e.*, the ball centered at x, with radius $3|x-y|$), then

$$\begin{aligned}|f(x) - f(y)| &\leq \mathit{ess}\sup_B f - \mathit{ess}\inf_B f \\ &\leq c_n 2\left(\frac{-\bar{p}}{p'}\right)^{1/p'} C\|f\|_{L^{[\bar{p},p]}} |x-y|^{\alpha}.\end{aligned}$$

At this point we could appeal to (9.1.1) to conclude that

$$|f(x) - f(y)| \leq c_n 2\left(\frac{-\bar{p}}{p'}\right)^{1/p'} C\,\||\nabla f|\|_p |x-y|^{\alpha}.$$

Similar arguments apply when dealing with Besov spaces. In this case the point of departure is the corresponding replacement for (9.1.1) that is provided by the Besov embedding

$$\int \left[\left(|f|_\mu^{**}(t) - |f|_\mu^{*}(t)\right) t^{\frac{1}{p} - \frac{\theta}{n}}\right]^q \frac{dt}{t} \leq c\int \left[t^{-\frac{\theta}{n}} K(t^{1/n}, f; L^p, \dot{W}^1_{L^p})\right]^q \frac{dt}{t},$$

where $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{\theta}{n}$. $\theta \in (0,1)$, $1 \leq q \leq \infty$. Notice that we don't assume anymore that $\theta p \leq n$.

Remark 19. — *In the usual argument the use of the Lorentz spaces with negative indices was implicit. The idea being that we can estimate* $\left(\int_{t_1}^{t_2}\left((f^{**}(t) - f^*(t))\,t^{1/\bar{p}}\right)^p \frac{dt}{t}\right)^{1/p}$ *through the use of*

$$f^{**}(t) - f^*(t) \leq c_n t^{1/n} |\nabla f|^{**}(t).$$

Namely,

$$\begin{aligned}
f^{**}(t_1) - f^{**}(t_2) &= \int_{t_1}^{t_2} \left(f^{**}(t) - f^{*}(t)\right) \frac{dt}{t} \\
&\leq \int_{t_1}^{t_2} t^{1/n-1} \left|\nabla f\right|^{**}(t)\, dt \ \textit{(basic inequality)} \\
&\leq \left(\int_{t_1}^{t_2} \left|\nabla f\right|^{**}(t)^p\, dt\right)^{1/p} \left(\int_{t_1}^{t_2} t^{p'(1/n-1)}\, dt\right)^{1/p'} \\
&\quad \textit{(Hölder's inequality)} \\
&\leq c_p \left\|\left|\nabla f\right|\right\|_p \left(\int_{t_1}^{t_2} t^{p'(1/n-1)}\, dt\right)^{1/p'} \quad \textit{(Hardy's inequality)} \\
&\leq c_p \left\|\left|\nabla f\right|\right\|_p \left(\int_{0}^{|t_2-t_1|} t^{p'(1/n-1)}\, dt\right)^{1/p'} \quad \textit{(since } t^{p'(1/n-1)} \textit{ decreases)} \\
&= c_{p,n} \left\|\left|\nabla f\right|\right\|_p \left|t_2 - t_1\right|^{1/n-1/p} \\
&= c_{p,n} \left\|\left|\nabla f\right|\right\|_p \left|t_2 - t_1\right|^{\alpha/n}.
\end{aligned}$$

At this point it is not difficult to reformulate many of the results in this paper using the notion of Lorentz spaces with negative index. As an example we simply state the following result and safely leave the details to the reader.

Theorem 33. — *Let (Ω, d, μ) be a probability metric space that satisfies the relative isoperimetric property and such that*

$$t^{1-1/n} \preceq I_\Omega(t),\ t \in (0, 1/2).$$

Then, if $p > n$

$$b_p^{n/p,1}(\Omega) \subset L^{\bar{p},1}$$

where $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n}$. Moreover, if $f \in b_p^{n/p,1}(\Omega)$, then $\forall B \subset \Omega$, $f\chi_B \in L^{\bar{p},1}$, and $\|f\chi_B\|_{L^{\bar{p},1}} \preceq \|f\|_{b_p^{n/p,1}(\Omega)}$, with constants independent of B. In particular, it follows that $f \in C(\Omega)$.

9.3. An interpolation inequality

In this section we formulate the basic argument of this chapter as in interpolation inequality.

Lemma 6. — *Suppose that (Ω, d, μ) is a probability measure. Let $s < 0, 1 \leq q \leq \infty$, and suppose that $-q' > s$. Then for all $f \in L^1(\Omega)$ we have,*

$$\|f\|_{L^\infty} \leq \left(\frac{-s}{q'}\right)^{1/q'} \|f\|_{L^{[s,q]}} + \|f\|_{L^1}.$$

Proof. — We use the argument of the previous section verbatim. Let $0 < t_1 < t_2 < 1$. By the fundamental theorem of calculus, we have

$$\begin{aligned}
\left|f_\mu^{**}\right|(t_1) - |f|_\mu^{**}(t_2) &= \int_{t_1}^{t_2} \left(|f|_\mu^{**}(t) - |f|_\mu^{*}(t)\right) \frac{dt}{t} \\
&= \int_{t_1}^{t_2} \left(|f|_\mu^{**}(t) - |f|_\mu^{*}(t)\right) t^{1/s} t^{-1/s} \frac{dt}{t} \\
&\leq \left\{ \int_{t_1}^{t_2} \left\{ \left(|f|_\mu^{**}(t) - |f|_\mu^{*}(t)\right) t^{1/s} \right\}^q \frac{dt}{t} \right\}^{1/q} \left\{ \int_{t_1}^{t_2} t^{-q'/s} \frac{dt}{t} \right\}^{1/q'} \\
&\leq \left(\frac{-s}{q'}\right)^{1/q'} \|f\|_{L^{[s,q]}} \left|t_1 - t_2\right|^{-q'/s}.
\end{aligned}$$

Therefore letting $t_1 \to 0^+$, $t_2 \to 1^-$, we find

$$\|f\|_{L^\infty} - \|f\|_{L^1} \leq \left(\frac{-s}{q'}\right)^{1/q'} \|f\|_{L^{[s,q]}}. \qquad \square$$

9.4. Further remarks

Good portions of the preceding discussion can be extended to the context of real interpolation spaces. In this framework one can consider spaces that are defined in terms of conditions on $\frac{K(t,f;\bar{X})}{t} - K'(t,f;\bar{X})$, where $\bar{X}$ is a compatible pair of Banach spaces. An example of such construction are the modified Lions-Peetre spaces defined, for example, in [**50**], [**51**] and the references therein. The usual conditions defining these spaces are of the form

$$\|f\|_{[X_0,X_1]_{\theta,q}} = \left\{ \int_0^\infty \left(t^{-\theta}(K(t,f;X_0,X_1) - tK'(t,f;X_0,X_1))\right)^q \frac{dt}{t} \right\}^{1/q} < \infty,$$

where $\theta \in (0,1), q \in (0,\infty]$. Adding the end points $\theta = 0,1$, produces conditions that still make sense and are useful in analysis (cf. [**66**] and the references therein). Observe that when $\bar{X} = (L^1, L^\infty)$, we have

$$\frac{K(t,f;\bar{X})}{t} - K'(t,f;\bar{X}) = |f|^{**}(t) - |f|^{*}(t),$$

and therefore

$$[X_0, X_1]_{\theta,q} = L^{[\frac{1}{1-\theta},q]}.$$

Therefore the discussion in this chapter suggests that it is of interest to consider, more generally, the spaces $[X_0, X_1]_{\theta,q}$, for $\theta \in \mathbb{R}$. In particular this may allow, in some cases, to treat L^p and *Lip* conditions in a unified manner. For example, in [**29**] and [**71**] results are given that imply that for certain operators T, that include gradients, an inequality of the form

$$\|f\|_{[Y_0,Y_1]_{\theta_0,q_0}} \leq c\,\|Tf\|_{[X_0,X_1]_{\theta_1,q_1}}$$

can be extrapolated to a family of inequalities that involve the $[Y_0, Y_1]_{\theta_0,q_0}$ spaces defined here. In particular

$$\|f\|_{L^{n'}} \leq c\,\|\nabla f\|_{L^1}\,, f \in C_0^1(R^n)$$

implies

$$K(t, f; L^1, L^\infty) - tK'(t, f; L^1, L^\infty) \leq ct^{1/n} K(t, \nabla f; L^1, L^\infty).$$

Thus from one inequality we can extrapolate "all" the classical Sobolev inequalities through the use of the $[Y_0, Y_1]_{\theta,q}$ spaces with θ possibly negative. To pursue this point further would take us too far away from our main concerns in this paper, so we must leave more details and applications for another occasion.

CHAPTER 10

CONNECTION WITH THE WORK OF GARSIA AND HIS COLLABORATORS

In this section we shall discuss the connection of our results with the work of Garsia and his collaborators (cf. [**41**], [**40**], [**42**], [**44**], [**43**], [**81**], [**31**]...). We argue that our results can be seen as an extension the work by Garsia [**43**], [**44**], and some [1] of the work by Garsia-Rodemich [**40**], to the metric setting. Indeed, [**44**], [**43**] were one of the original motivations behind [**65**] and some of our earlier writings.

In [**44**] it is shown that for functions on $[0, 1]$, if [2] $p \geq 1$,

$$\left.\begin{array}{c} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array}\right\} \leq \frac{4^{1/p}}{\log \frac{3}{2}} \int_x^1 Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}, \tag{10.1.1}$$

(see Section 4.1.1 above), and where

$$Q_p(\delta, f) = \left\{ \frac{1}{\delta} \int \int_{|x-y|<\delta} |f(x) - f(y)|^p \, dx \, dy \right\}^{1/p}.$$

In particular, if $\int_0^1 Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}} < \infty$, then f is essentially continuous, and in fact, a.e. $x, y \in [0, 1]$

$$|f(x) - f(y)| \leq 2 \frac{4^{1/p}}{\log \frac{3}{2}} \int_0^{|x-y|} Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}. \tag{10.1.2}$$

Moreover, in [**44**] more general moduli of continuity based on Orlicz spaces are considered: for a Young's function A, normalized so that $A(1) = 1$, let

$$Q_A(\delta, f) = \inf \left\{ \lambda > 0 : \frac{1}{\delta} \int \int_{|x-y|<\delta} A\left(\frac{|f(x) - f(y)|}{\lambda} \right) dx \, dy \leq 1 \right\}.$$

---

1. The results of [**40**], while very similar, are formulated in terms of moduli of continuity that in some cases cannot be readily identified with the ones we consider in this paper.
2. The case $p = 1$ is also trivially true since

$$\int_0^1 Q_1(\delta, f) \frac{d\delta}{\delta^2} < \infty,$$

readily implies that f is constant.

In [**44**] and Deland [**31**, (1.1), (1.3)] the following analogues of (10.1.1) and (10.1.2) are shown to hold:

$$\left.\begin{array}{r} f^*(x)-f^*(1/2) \\ f^*(1/2)-f^*(1-x) \end{array}\right\} \leq \frac{2}{\log\frac{3}{2}} \int_x^1 Q_A(\delta,f)A^{-1}(\frac{4}{\delta})\frac{d\delta}{\delta} \tag{10.1.3}$$

and

$$|f(x)-f(y)| \leq c \int_0^{|x-y|} Q_A(\delta,f)A^{-1}(\frac{4}{\delta})\frac{d\delta}{\delta}. \tag{10.1.4}$$

We will show in a moment that our inequalities readily give the following version of (10.1.3) for all r.i. spaces $X[0,1]$:

$$\left.\begin{array}{r} f^*(x)-f^*(1/2) \\ f^*(1/2)-f^*(1-x) \end{array}\right\} \leq c \int_x^1 \frac{K(\delta,f;X,\dot{W}^1_X)}{\phi_X(\delta)}\frac{d\delta}{\delta}. \tag{10.1.5}$$

To relate this inequality to Garsia's results we compare the modulus of continuity to $K-$functionals. Thus, we let

$$\omega_A(\delta,f) = \inf\left\{\lambda>0 : \sup_{h\leq\delta}\int_0^{1-\delta} A\left(\frac{|f(x+h)-f(x)|}{\lambda}\right)dx \leq 1\right\},\quad \delta\in(0,1).$$

Then, as is well known (cf. [**13**], [**65**]),

$$K(\delta,f;L_A,\dot{W}^1_{L_A}) \simeq \omega_A(\delta,f), \tag{10.1.6}$$

and we have

***Lemma* 7.** — $\sup_{0<\sigma<\delta} Q_A(\sigma,f) \preceq K(\delta,f;L_A,\dot{W}^1_{L_A})$.

Proof. — To see this note that, for all $\lambda>0$, $\delta\in(0,1)$, we have

$$\begin{aligned}
\frac{1}{\delta}\iint_{\{(x,y)\in[0,1]^2:|x-y|<\delta\}} & A\left(\frac{|f(x)-f(y)|}{2\lambda}\right)dx\,dy \\
&\leq \frac{1}{\delta}\int_0^{\delta}\int_0^{1-\delta} A\left(\frac{|f(x+h)-f(x)|}{\lambda}\right)dx\,dh \\
&\leq \sup_{h\leq\delta}\int_0^{1-\delta} A\left(\frac{|f(x+h)-f(x)|}{\lambda}\right)dx.
\end{aligned}$$

Therefore, if we let $\lambda=\omega_A(\delta,f)$, by the definitions,

$$\frac{1}{\delta}\iint_{\{(x,y)\in[0,1]^2:|x-y|<\delta\}} A\left(\frac{|f(x)-f(y)|}{2\lambda}\right)dx\,dy \leq 1,$$

and consequently

$$\begin{aligned}
Q_A(\delta,f) &\leq 2\omega_A(\delta,f) \\
&\preceq K(\delta,f;L_A,\dot{W}^1_{L_A}).
\end{aligned}$$

□

To complete the picture let us also note that

$$\phi_{L_A}(t) = \frac{1}{A^{-1}(\frac{1}{t})}.$$

We now show in detail (10.1.5). One technical problem we have to overcome is that the results of this paper do not apply directly for functions on $[0,1]$, since the isoperimetric profile of $[0,1]$ is $I(t) \equiv 1$, and therefore I does not satisfy the required hypotheses to apply our general machinery (cf. Condition 1 in Chapter 2, and [**70**], [**71**]). Therefore while the inequalities (1.1.8), and their corresponding signed rearrangement variants are valid (cf. Chapter 4), our results cannot be applied directly. However, we will now show that our methods can be readily adapted to yield the one dimensional result as well.

To prove (1.1.8) for $n = 1$, we need to establish the following inequality (compare with (1.1.2), letting formally $I(t) = 1$)

$$f^{**}(t) - f^{*}(t) \leq t\left(|f'|\right)^{**}(t), t \in (0,1).$$

While [**70**] formally does not cover this case, it turns out that we can easily prove this inequality directly using the method of "truncation by symmetrization", which was apparently introduced in [**72**]. Indeed, a known elementary result of Duff [**35**] states that

$$\|(f^{*})'\|_{L^{p}[0,1]} \leq \|f'\|_{L^{p}[0,1]}.$$

The truncation method of [**72**] (cf. also [**36**, discussion before Corollaire 2.4]), as it is developed in detail in [**60**], when applied to the case $p = 1$, yields the corresponding Pólya-Szegö inequality (as formulated in [**72**])

$$t\left((f^{*})'\right)^{**}(t) \leq t\left(|f'|\right)^{**}(t),\ t \in (0,1).$$

We can (and will) assume without loss that f is bounded, then (cf. [**60**]),

$$t\left((f^{*})'\right)^{**}(t) = \int_{0}^{t} |(f^{*})'|\, ds = f^{*}(0) - f^{*}(t) < \infty.$$

Now, since $f^{**}(0) = f^{*}(0)$, and f^{**} is decreasing, we have

$$f^{**}(t) - f^{*}(t) \leq f^{**}(0) - f^{*}(t) = f^{*}(0) - f^{*}(t).$$

Therefore, combining these estimates we arrive at

$$f^{**}(t) - f^{*}(t) \leq t\left(|f'|\right)^{**}(t),\ t \in (0,1),$$

as required. At this point the proof of Theorem 7 applies without changes to yield for $0 < t \leq 1/2$,

$$f^{**}(t) - f^{*}(t) \leq c\frac{K(t, f; X, \dot{W}_X^1)}{\phi_X(t)}.$$

Moreover, using [**9**, (4.1)] we have

$$f^{*}(t/2) - f^{*}(t) \leq 2\left(f^{**}(t) - f^{*}(t)\right).$$

Thus,

$$
\begin{aligned}
f^*(t/2) - f^*(t) &\le c\frac{K(t,f;X,\dot W_X^1)}{\phi_X(t)} \\
&\le 2c\frac{K(\frac{t}{2},f;X,\dot W_X^1)}{\phi_X(t)} \left(\text{since } \frac{K(t,f;X,\dot W_X^1)}{t} \text{ and } \frac{1}{\phi_X(t)} \text{ decrease}\right) \\
&\le \frac{2c}{\ln 2}\int_{\frac{t}{2}}^{t} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s}. \qquad (10.1.7)
\end{aligned}
$$

Given $t \in (0,1/2)$, let $N = N(t)$ be such that $\frac{t}{2} \le 2^{-(N+1)} < t < 2^{-N} \le \frac{1}{2}$, then

$$
\begin{aligned}
f^*(t) - f^*(1/2) &\le f^*(2^{-(N+1)}) - f^*(1/2) \\
&= \sum_{j=1}^{N}\left(f^*(2^{-(j+1)}) - f^*(2^{-j})\right) \\
&\le C\sum_{j=1}^{N}\int_{2^{-(j+1)}}^{2^{-j}} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s} \\
&\le C\int_{2^{-(N+1)}}^{1/2} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s} \\
&= C\int_{2^{-(N+1)}}^{2^{-N}} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s} + C\int_{2^{-N}}^{1/2} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s}.
\end{aligned}
$$

Now,

$$
\int_{2^{-N}}^{1/2} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s} \le \int_{t}^{1/2} \frac{K(s,f;X,\dot W_X^1)}{\phi_X(s)}\frac{ds}{s}.
$$

Moreover, we will show in a moment that

(10.1.8)
$$
\int_{2^{-(N+1)}}^{2^{-N}} \frac{K(s,f;X,\dot W_X^1)}{s}\frac{1}{\phi_X(s)}ds \le \frac{4}{\ln(1/2)}\int_t^{1/2} K(s,f;X,\dot W_X^1)\frac{1}{\phi_X(s)}\frac{ds}{s}.
$$

Collecting these results we see that there exists a universal constant $c > 0$ such that,

$$
f^*(t) - f^*(1/2) \le c\int_t^{1/2} K(s,f;X,\dot W_X^1)\frac{1}{\phi_X(s)}\frac{ds}{s},\ t \in (0,1/2).
$$

The previous inequality applied to $-f$ yields the second half of Garsia's inequality

$$
f^*(1/2) - f^*(1-t) \le c\int_t^{1/2} K(s,f;X,\dot W_X^1)\frac{1}{\phi_X(s)}\frac{ds}{s},\ t \in (0,1/2).
$$

We complete the details of the proof of (10.1.8) using various monotonicity properties of the functions involved and the position of t in the interval:

$$\begin{aligned}
&\int_{2^{-(N+1)}}^{2^{-N}} \frac{K(s,f;X,\dot{W}_X^1)}{s}\frac{1}{\phi_X(s)}\,ds \\
&\leq \frac{K(2^{-(N+1)},f;X,\dot{W}_X^1)}{2^{-(N+1)}}\frac{1}{\phi_X(2^{-(N+1)})}2^{-(N+1)} \quad \Big(\frac{K(r)}{r}\downarrow, \frac{1}{\phi_X(r)}\downarrow\Big) \\
&= K(2^{-(N+1)},f;X,\dot{W}_X^1)\frac{1}{\phi_X(2^{-(N+1)})}\frac{1}{\ln(2^{-N}/t)}\int_t^{2^{-N}}\frac{ds}{s} \\
&\leq \frac{4}{\ln(1/2)}\int_t^{2^{-N}} K(s,f;X,\dot{W}_X^1)\frac{1}{\phi_X(s)}\frac{ds}{s} \quad \Big(K\uparrow, \frac{r}{\phi_X(r)}\uparrow\Big) \\
&\leq \frac{4}{\ln(1/2)}\int_t^{1/2} K(s,f;X,\dot{W}_X^1)\frac{1}{\phi_X(s)}\frac{ds}{s}.
\end{aligned}$$

In particular, our results thus give versions of (10.1.1), (10.1.2), (10.1.3), but replacing $Q_A(\delta,f)$ with the usual modulus of continuity $K(\delta,f;L_A,\dot{W}_{L_A}^1)$. We also note that Deland [**31**] found the following improvement to (10.1.3)

$$\left.\begin{matrix} f^*(x)-f^*(1/2) \\ f^*(1/2)-f^*(1-x) \end{matrix}\right\} \preceq \int_x^1 Q_A(\delta,f)\,dA^{-1}(\tfrac{c}{\delta}),\ 0<x<1/2.$$

This is of particular interest when dealing with the space $X=e^{L^2}$. Indeed, in this case $A(t)=e^{t^2}-1$, and therefore

$$\phi_X(t)=\frac{1}{\left(\ln\frac{e}{t}\right)^{1/2}}.$$

Consequently, from (10.1.4) (or (10.1.5)) one finds that a sufficient condition for continuity can be formulated as: there exists $0<a<1$, $c>0$, such that

$$\int_0^a Q_A(\delta,f)\left(\ln\frac{c}{\delta}\right)^{1/2}\frac{d\delta}{\delta}<\infty. \tag{10.1.9}$$

On the other hand, Deland's improved condition for continuity replaces (10.1.9) by

$$\int_0^a Q_A(\delta,f)\frac{d\delta}{\left(\ln\frac{c}{\delta}\right)^{1/2}\delta}<\infty. \tag{10.1.10}$$

In our formulation (10.1.10) corresponds to a condition of the form

$$\int_0^a K(\delta,f;L_A,\dot{W}_{L_A}^1)\,d\left(\frac{1}{\phi_A(t)}\right)<\infty.$$

While we don't have any new insight to add to Deland's improvement we should point out here that Deland's improvement is automatic for spaces far away from L^{∞}, in the sense that $\underline{\alpha}_{\Lambda(X)} > 0$. Indeed, we have

Lemma 8. — *Suppose that $X = X[0,1]$ is a r.i. space such that $\underline{\alpha}_{\Lambda(X)} > 0$. Then there exists a re-norming of X, that we shall call $\bar{X}$, such that*

$$\int_0^1 K(\delta, f; \bar{X}, \dot{W}^1_{\bar{X}})\, d\left(\frac{1}{\phi_{\bar{X}}(\delta)}\right) < \infty \Longleftrightarrow \int_0^1 \frac{K(\delta, f; X, \dot{W}^1_X)}{\phi_X(\delta)} \frac{d\delta}{\delta} < \infty. \tag{10.1.11}$$

Proof. — Let $\bar{\phi}(t) = \int_0^t \phi_X(s) \frac{ds}{s}$, then, since $\frac{\phi_X(s)}{s}$ decreases, we have $\bar{\phi}(t) \geq \phi_X(t)$, and

$$([-\bar{\phi}(t)]^{-1})' = \frac{1}{\bar{\phi}(t)^2} \frac{\phi_X(t)}{t} \leq \frac{1}{\bar{\phi}(t)t} \leq \frac{1}{t\phi_X(t)}.$$

Moreover, since $\underline{\alpha}_{\Lambda(X)} > 0$, we have (cf. [**88**, Lemma 2.1])

$$\bar{\phi}(t) \preceq \phi_X(t).$$

Therefore there exists an equivalent re-norming of X, which we shall call $\bar{X}$, such that

$$\phi_X(t) \simeq \phi_{\bar{X}}(t) = \bar{\phi}(t).$$

Moreover, we clearly have

$$K(\delta, f; X, \dot{W}^1_X) \simeq K(\delta, f; \bar{X}, \dot{W}^1_{\bar{X}}).$$

We can also see that,

$$\begin{aligned}
\left([\phi_{\bar{X}}(t)]^{-1}\right)' &= ([-\bar{\phi}(t)]^{-1})' \\
&= \bar{\phi}(t)^{-2} \frac{\phi_X(t)}{t} \\
&\simeq \frac{1}{(\phi_X(t))^2} \frac{\phi_X(t)}{t} \\
&\simeq \frac{1}{\phi_X(t)t}.
\end{aligned}$$

Consequently (10.1.11) holds when $\underline{\alpha}_{\Lambda(X)} > 0$. □

On the other hand Deland's improvement does not follow from the previous Lemma, since from the point of view of the theory of indices $\underline{\alpha}_{\Lambda(e^{L^2})} = 0$. For more details on how to overcome this difficulty for spaces close to L^{∞} we must refer to Deland's thesis [**31**].

For applications to Fourier series, the appropriate moduli of continuity defined for periodic functions on, say, $[0, 2\pi]$, are defined by (cf. [**44**], [**31**, (1.1), (1.3)])

$$W_A(h, f) = \inf\left\{\lambda > 0 : \int_0^{2\pi} A\left(\frac{|f(x+h) - f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Then, we also have

$$\begin{aligned}\sup_{\sigma<\delta} Q_A(\sigma, f) &\preceq K(\delta, f; L_A[0,2\pi], \dot{W}^1_{L_A}[0,2\pi])\\ &\simeq \sup_{h\leq\delta} W_A(h, f).\end{aligned}$$

It follows from our work that the results of [**44**] can be now extended to r.i. spaces. In this connection we note that (just like in [**44**] for L^p spaces) one could also use the boundedness of the Hilbert transform on r.i. spaces where one has control of the Boyd indices (cf. [**21**], [**13**]). However, to continue with this topic will take us too far away from our main concerns here so we must leave the discussion for another occasion.

For further applications to: the path continuity of stochastic processes, Fourier series, random Fourier series and embeddings we refer to [**40**], [**44**], [**43**], [**31**] and the references therein. Moreover, under suitable assumptions on the connection between the isoperimetric profile and the measure of balls (cf. [**92**]) one can also formulate the Besov conditions as entropy conditions as it is customarily done in probability (cf. the discussion in Pisier [**84**, Remarque, p. 14]).

APPENDIX A

SOME REMARKS ON THE CALCULATION OF K–FUNCTIONALS

A.1. Introduction

It seemed to us useful to collect for our reader some known computations of $K-$functionals of the form $K(t,f;X(\Omega),\dot{W}_X^1(\Omega))$, where X is a r.i. space. We don't claim any originality, but we provide detailed proofs when we could not find suitable references.

In the Euclidean case, for smooth (*Lip*) domains, these estimates are well known for L^p spaces (cf. [**13**], [**54**], [**94**]), and can be readily extended to r.i. spaces (cf. [**65**]):

$$\begin{aligned} K(t,f;X(\Omega),\dot{W}_X^1(\Omega)) &\simeq \omega_X(t,f) \\ &= \sup_{|h|\leq t}\left\|\left(f(.+h)-f(.)\right)\chi_{\Omega(h)}\right\|_X \end{aligned}$$

where

$$\Omega(h)=\{x\in\Omega : x+th\in\Omega, 0\leq t\leq 1\}.$$

Consider $(\mathbb{R}^n,|\cdot|,d\gamma_n)$, *i.e.*, $\mathbb{R}^n$ with Gaussian measure. The fact that this measure is not translation invariant makes the computation of the K-functional somewhat more complicated. We discuss the necessary modifications in some detail for $n=1$.

We consider spaces on $(\mathbb{R},|\cdot|,d\gamma_1)$. Let $p\in[1,\infty]$, and let

$$K_\gamma(t,f,L^p,\dot{W}_p^1)=\inf\left\{\|f-g\|_{L^p(\mathbb{R},d\gamma_1)}+t\,\|g'\|_{L^p(d\gamma_1)}\right\}.$$

This functional was studied by the approximation theory community (cf. Ditzian-Totik [**34**], Ditzian-Lubinsky [**33**] and the references therein). For example, from [**34**, p. 183], we have

$$\text{(A.1.1)}\quad \begin{aligned} K_\gamma(t,f,L^p,\dot{W}_p^1) \simeq \sup_{0<h\leq t}&\|f(.+h)-f(.)\|_{L^p\left((-\frac{1}{2h},\frac{1}{2h}),d\gamma_1\right)}+ \\ &\inf_c\|f-c\|_{L^p\left((\frac{1}{2t},\infty),d\gamma_1\right)}+\inf_c\|f-c\|_{L^p\left((-\infty,\frac{-1}{2t}),d\gamma_1\right)} \end{aligned}$$

The main part of the right hand side of (A.1.1) is the modulus

$$\Omega_\gamma(t,f) = \sup_{0<h\leq t} \|f(.+h)-f(.)\|_{L^p\left((-\frac{1}{2h},\frac{1}{2h}),d\gamma_1\right)}.$$

Indeed, $\Omega_\gamma(t,f)$ controls the characterization of the corresponding interpolation spaces. For example, it follows (cf. [**34**, Theorem 11.2.5]) that for $\theta\in(0,1)$,

$$K_\gamma(t,f,L^p,\dot{W}_p^1) = O(t^\theta) \Longleftrightarrow \Omega_\gamma(t,f) = O(t^\theta). \tag{A.1.2}$$

More generally, a similar result holds for $(\mathbb{R}, d\gamma_\lambda)$, where for $\lambda>1$, $d\gamma_\lambda(x) = e^{-x^\lambda}\,dx$. Indeed, in this case (A.1.1) holds replacing $\frac{1}{2h}$ throughout by $\frac{1}{\lambda h^{1/(1-\lambda)}}$:

$$\begin{aligned} K_{\gamma_\lambda}(t,f,L^p,\dot{W}_p^1) \simeq \sup_{0<h\leq t} &\|f(.+h)-f(.)\|_{L^p\left((-\frac{1}{\lambda h^{1/(1-\lambda)}},\frac{1}{\lambda h^{1/(1-\lambda)}}),d\gamma_\lambda\right)} + \\ &\inf_c \|f-c\|_{L^p\left((\frac{1}{\lambda t^{1/(1-\lambda)}},\infty),d\gamma_\lambda\right)} + \\ &\inf_c \|f-c\|_{L^p\left((-\infty,\frac{-1}{\lambda t^{1/(1-\lambda)}}),d\gamma_\lambda\right)}. \end{aligned} \tag{A.1.3}$$

Again the main part of the right hand side is the modulus of continuity

$$\Omega_{\gamma_\lambda}(t,f) = \sup_{0<h\leq t} \|f(.+h)-f(.)\|_{L^p((-\frac{1}{\lambda h^{1/(1-\lambda)}},\frac{1}{\lambda h^{1/(1-\lambda)}}),d\gamma_\lambda)}.$$

Likewise the analogue of (A.1.2) holds.

More generally, the estimates above have been extended to the class of the so called "Freud weights" of the form $w(x) = e^{Q(x)}$. Here we assume that Q is a given function in $C^1(\mathbb{R})$ such that Q is even, $\lim_{x\to\infty} Q'(x)=\infty$, and such that there exists $A>0$, such that $Q'(x+1)\leq AQ'(x)$, for all $x>0$. For complete details we refer again to [**34**].

Although one would expect that the $n-$dimensional extensions of the $K-$functional estimates above should not be very difficult, we have not been able find references, even after consultation with many experts. On the other hand, as is well known, one can avoid this difficulty through the use of an alternate characterization of $K-$functionals for Gaussian measure using appropriate semigroups. We provide some details in the next sections.

For the last section of this chapter, connecting semigroups and Gaussian Besov spaces, we are grateful to Stefan Geiss and Alessandra Lunardi for precious information, in particular, for pointing out the relevant literature. In this last regard we also refer to the recent paper by Geiss-Toivola [**46**]. In connection with this last section we should mention the recent formulation of fractional Poincaré inequalities in [**80**].

A.2. Semigroups and Interpolation

A family $\{G(t)\}_{t>0}$ of operators on a Banach space A is called an equibounded, strongly continuous semigroup if the following conditions are satisfied:

(i) $$G(t+s) = G(t)G(s)$$

(ii) $$\text{There exists } M > 0 \text{ such that } \sup_{t>0} \|G(t)\|_{A\to A} \leq M$$

(iii) $$\lim_{t\to 0} \|G(t)\, a - a\|_A = 0, \text{ for } a \in A.$$

The infinitesimal generator Λ is defined on

$$D(\Lambda) = \left\{ a \in A : \lim_{t\to 0} \frac{G(t)\, a - a}{t} \text{ exists} \right\}$$

by

$$\Lambda a = \lim_{t\to 0} \frac{G(t)\, a - a}{t}.$$

We consider

$$K(t, a; A, D(\Lambda)) = \inf \left\{ \|a_0\|_A + t\, \|\Lambda a_1\|_A : a = a_0 + a_1 \right\}.$$

For equibounded strongly continuous semigroups we have the well known estimate, apparently going back to Peetre [**82**] (cf. [**14**], [**32**], [**83**])

$$K(t, a; A, D(\Lambda)) \simeq \sup_{0<s\leq t} \|(G(s) - I)\, a\|_A \,, \tag{A.2.1}$$

where I =identity operator on A. The proof can be accomplished using the decomposition

$$a = \underbrace{\left(a - \frac{1}{t} \int_0^t G(s)\, a\, ds \right)}_{a_0 \epsilon A} + \underbrace{\frac{1}{t} \int_0^t G(s)\, a\, ds.}_{a_1 \epsilon D(\Lambda)}$$

Note that the right hand side of (A.2.1) should be thought as a generalized modulus of continuity which in the classical case corresponds to the semigroup of translations $G(s)f = f(s + \cdot)$.

In [**32**, Corollary 7.2] the following alternate estimates were pointed out

$$\begin{aligned} K(t, a; A, D(\Lambda)) &\simeq \frac{1}{t} \int_0^t \|(G(s) - I)\, a\|_A\, ds \\ &\simeq \frac{1}{t} \left\| \int_0^t (G(s) - I)\, a\, ds \right\|_A \\ &\simeq \frac{1}{t} \left\| \int_{t/2}^t (G(s) - I)\, a\, ds \right\|_A \\ &\simeq \frac{1}{t} \int_{t/2}^t \|(G(s) - I)\, a\, ds\|_A \,. \end{aligned}$$

The preceding estimates can be further improved under more restrictions on the semigroups. Recall that a semigroup is said to be holomorphic if:

(i) $$G(t)a \in D(\Lambda) \quad \text{for all } a \in A,$$

and

(ii) There exists a constant $C > 0$ such that $\|\Lambda G(t)a\|_A \leq C\frac{\|a\|_A}{t}, \forall a \in A, t > 0.$

In [**32**] it is shown that for holomorphic semigroups we have the following improvement of (A.2.1)

$$K(t, a; A, D(\Lambda)) \simeq \|(G(t) - I)a\|_A . \tag{A.2.2}$$

Peetre [**83**, p. 33] pointed out, without proof, that for holomorphic semigroups we also have

$$K(t, a; A, D(\Lambda)) \simeq \sup_{s \leq t} \|\Lambda G(s)a\|_A .$$

However, we can only prove a somewhat weaker result here.

Lemma 9. — *Suppose that $\{G(t)\}_{t>0}$ is an holomorphic semigroup on a Banach space A. Let $c_1 > 1$, be such that for all $t > 0$, and for all $a \in A$ (cf. (A.2.2) above),*

$$\frac{1}{c_1} \|(G(t) - I)a\|_A \leq K(t, a; A, D(\Lambda)) \leq c_1 \|(G(t) - I)a\|_A .$$

Then, there exist absolute constants $c_2(m), c_3(m)$ such that for all $t > 0$, for all $a \in A$, for all $m \geq 2$,

$$\begin{aligned} K(t, a; A, D(\Lambda)) - c_1^2 K\left(\frac{t}{m}, a; A, D(\Lambda)\right) \\ \leq c_2(m) \sup_{s \leq t} \|\Lambda G(s)a\|_A \leq c_3(m) K(t, a; A, D(\Lambda)). \end{aligned} \tag{A.2.3}$$

Proof. — It is easy to show that there exists an absolute constant $C > 0$ such that

$$\sup_{s \leq t} \|\Lambda G(s)a\|_A \leq CK(t, a; A, D(\Lambda)). \tag{A.2.4}$$

Indeed, let $a = a_0 + a_1$, be any decomposition with $a_0 \in A$, $a_1 \in D(\Lambda)$. Then, using the properties of holomorphic semigroups, we have

$$\begin{aligned} \sup_{s \leq t} s \|\Lambda G(s)a\|_A &\leq \sup_{s \leq t} s \|G(s)\Lambda a_0\|_A + \sup_{s \leq t} s \|G(s)\Lambda a_1\|_A \\ &\leq C (\|a_0\|_A + t \|\Lambda a_1\|_A) . \end{aligned}$$

Consequently, (A.2.4) follows by taking infimum over all such decompositions.

We now prove the left hand side of (A.2.3). Observe that, for $t > 0$ we have $G(t)a \in D(\Lambda)$, therefore we can write $\frac{d}{dt}(G(t)a) = \Lambda G(t)a$. Consequently, for all $m \geq 2$,

$$\begin{aligned}
K(t,a;A,D(\Lambda)) &\leq c_1 \left\|(G(t)-I)a\right\|_A \\
&\leq c_1 \left\| \int_0^{t/m} \Lambda G(s)a\,ds \right\|_A + c_1 \left\| \int_{t/m}^{t} \Lambda G(s)a\,ds \right\|_A \\
&= c_1 \left\|(G(t/m)-I)a\right\|_A + c_1 \left\| \int_{t/m}^{t} \frac{s}{s} \Lambda G(s)a\,ds \right\|_A \\
&\leq c_1 \left\|(G(t/m)-I)a\right\|_A + c_1 \int_{t/m}^{t} \frac{1}{s} \left\|s\Lambda G(s)a\right\|_A ds \\
&\leq c_1 \left\|(G(t/m)-I)a\right\|_A + c_1 \frac{m}{t} \sup_{s\leq t} \left\|s\Lambda G(s)a\right\|_A \frac{(m-1)}{m} t \\
&\leq c_1^2 K(\frac{t}{m}, a; A, D(\Lambda)) + c_1(m-1) \sup_{s\leq t} \left\|s\Lambda G(s)a\right\|_A,
\end{aligned}$$

as we wished to show. □

Recall the definition of real interpolation spaces. Let $\theta \in (0,1)$, $q \in (0,\infty)$,

$$(A,D(\Lambda))_{\theta,q} = \left\{ a \in A : \|a\|^q_{(A,D(\Lambda))_{\theta,q}} = \int_0^\infty \left(t^{-\theta} K(s,a;A,D(\Lambda))\right)^q \frac{dt}{t} < \infty \right\},$$

and

$$(A,D(\Lambda))_{\theta,\infty} = \left\{ a \in A : \|a\|_{(A,D(\Lambda))_{\theta,\infty}} = \sup_{t>0} \left\{ t^{-\theta} K(s,a;A,D(\Lambda)) \right\} < \infty \right\}.$$

From the previous Lemma we see that

Proposition 5. — *Suppose that $\{G(t)\}_{t>0}$ is an holomorphic semigroup on a Banach space A. Then $(A,D(\Lambda))_{\theta,q}$ can be equivalently described by*

$$(A,D(\Lambda))_{\theta,q} = \left\{ a : \left\{ \int_0^\infty \left(t^{-\theta} \sup_{s\leq t} \|\Lambda G(s)a\|_A \right)^q \frac{dt}{t} \right\}^{1/q} < \infty \right\},$$

with the obvious modification if $q = \infty$, and where the constants of the underlying norm equivalences depend only on θ.

Proof. — One part follows readily from (A.2.4). For the less trivial inclusion we proceed as follows. Given $\theta \in (0,1)$, select m such that $m^{-\theta} c_1^2 < 1$. Then from Lemma 9, there exists an absolute $c_2(m) > 0$ such that

$$K(t,a;A,D(\Lambda)) \leq c_2(m) \sup_{s\leq t} \|\Lambda G(s)a\|_A + c_1^2 K(\frac{t}{m}, a; A, D(\Lambda)).$$

Thus

$$\left(\int_0^\infty \left(t^{-\theta}K(t,a;A,D(\Lambda))\right)^q \frac{dt}{t}\right)^{1/q} \leq c_2(m)\left(\int_0^\infty \left(t^{-\theta}\sup_{s\leq t}\|\Lambda G(s)a\|_A\right)^q \frac{dt}{t}\right)^{1/q} +$$
$$c_1^2\left(\int_0^\infty (t^{-\theta}K(\frac{t}{m},a;A,D(\Lambda)))^q \frac{dt}{t}\right)^{1/q}$$
$$= c_2(m)\left(\int_0^\infty \left(t^{-\theta}\sup_{s\leq t}\|\Lambda G(s)a\|_A\right)^q \frac{dt}{t}\right)^{1/q} +$$
$$c_1^2 m^{-\theta}\left(\int_0^\infty \left(t^{-\theta}K(t,a;A,D(\Lambda))\right)^q \frac{dt}{t}\right)^{1/q}.$$

Hence,

$$\left(\int_0^\infty \left(t^{-\theta}K(t,a;A,D(\Lambda))\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\leq (1-c_1^2 m^{-\theta})^{-1}c_2(m)\left(\int_0^\infty \left(t^{-\theta}\sup_{s\leq t}\|\Lambda G(s)a\|_A\right)^q \frac{dt}{t}\right)^{1/q}. \qquad \square$$

We have the following well known result (cf. [**22**], [**14**])

Theorem 34. — *Let $\{G(t)\}_{t>0}$ be an equibounded, strongly continuous semigroup on the Banach space A. Let $\theta \in (0,1), q \in (0,\infty]$; then (with the usual modifications when $q=\infty$)*

$$(i) \qquad (A,D(\Lambda))_{\theta,q} = \left\{a : \left\{\int_0^\infty \left(t^{-\theta}\sup_{0<s\leq t}\|G(s)a-a\|_A\right)^q \frac{dt}{t}\right\}^{1/q} < \infty\right\}.$$

(ii) Moreover, if the semigroup is analytic then we also have the following characterizations (with the usual modifications when $q=\infty$)

$$(ii_1) \qquad (A,D(\Lambda))_{\theta,q} = \left\{a : \left\{\int_0^\infty \left(t^{-\theta}\|G(t)a-a\|_A\right)^q \frac{dt}{t}\right\}^{1/q} < \infty\right\},$$

$$(ii_2) \qquad (A,D(\Lambda))_{\theta,q} = \left\{a\in A : \int_0^\infty \left(t^{-\theta}\sup_{s\leq t} s\|\Lambda G(s)a\|_A\right)^q \frac{dt}{t} < \infty\right\},$$

$$(ii_3) \qquad (A,D(\Lambda))_{\theta,q} = \left\{a\in A : \int_0^\infty \left(t^{1-\theta}\|\Lambda G(t)a\|_A\right)^q \frac{dt}{t} < \infty\right\}.$$

Proof. — The characterizations *(i)*, *(ii_1)* and *(ii_2)* follow (respectively) from (A.2.1), (A.2.2) and Proposition 5. To prove *(ii_3)* we remark that, on the one hand,

$$\int_0^\infty \left(t^{1-\theta}\|\Lambda G(t)a\|_A\right)^q \frac{dt}{t} \leq \int_0^\infty \left(t^{-\theta}\sup_{s\leq t} s\|\Lambda G(s)a\|_A\right)^q \frac{dt}{t}.$$

On the other hand, since $\frac{d}{dt}(G(t)a) = \Lambda G(t)a$,

$$\begin{aligned}\int_0^\infty \left(t^{-\theta}\left\|G(t)a-a\right\|_A\right)^q \frac{dt}{t} &= \int_0^\infty \left(t^{-\theta}\left\|\int_0^t \Lambda G(s)a\,ds\right\|_A\right)^q \frac{dt}{t}\\ &\leq \int_0^\infty \left(t^{-\theta}\int_0^t \left\|\Lambda G(s)a\,ds\right\|_A\right)^q \frac{dt}{t}\\ &\leq c_{\theta,q}\int_0^\infty \left(t^{1-\theta}\left\|\Lambda G(t)a\right\|_A\right)^q \frac{dt}{t},\end{aligned}$$

where the last step follows from Hardy's inequality. □

Remark 20. — *Related interpolation spaces (obtained by the "complex method") can be characterized, under suitable conditions, using functional calculus. By the known relations between these different interpolation methods one can obtain further characterizations and embedding theorems for the real method (cf.* [**96**]*). In this setting fractional powers of the infinitesimal generator* Λ*, play the role of fractional derivatives. We must refer to* [**94**] *and* [**83**] *for a complete treatment.*

A.3. Specific Semigroups

Two basic examples of semigroups on $L^p((\mathbb{R}^n), d\gamma_n)$, which are relevant for this paper are given by

1. Ornstein-Uhlenbeck semigroup, defined by

$$G(t)f(x) = (1-e^{-2t})^{-n/2}\int e^{-\frac{e^{-2t}(|x|^2+|y|^2-2\langle x,y\rangle)}{1-e^{-2t}}} f(y)\,d\gamma_n(y),$$

with generator

$$\Lambda = \frac{1}{2}\Delta_x - \langle x, \nabla_x\rangle.$$

2. Poisson-Hermite semigroup

$$P_t f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}} G\left(\frac{t^2}{4s}\right) f(x)\,ds,$$

with generator

$$\Lambda_{1/2} = -(-\Lambda)^{1/2}.$$

For example, P_t on $L^\infty(\mathbb{R}^n)$ is analytic although not strongly continuous. Restricting P_t to $\widetilde{L^\infty(\mathbb{R}^n)}$, the subspace of elements of $L^\infty(\mathbb{R}^n)$ such that $\lim \|P_t f - f\|_\infty = 0$, remedies this deficiency and we have (cf. [**94**])

$$(L^\infty((\mathbb{R}^n), d\gamma_n), D(\Lambda_{1/2}))_{\theta,\infty} = (\widetilde{L^\infty((\mathbb{R}^n)}, d\gamma_n), D(\Lambda_{1/2}))_{\theta,\infty} = Lip_\theta(\mathbb{R}^n).$$

In particular, it follows from Theorem 34 that $f \in Lip_\theta(\mathbb{R}^n)$, iff

$$\|P_t f - f\|_\infty = 0(t^\theta).$$

For other characterizations of Besov spaces we must refer to [**83**], [**94**] and the references therein. For a treatment of fractional derivatives in Gaussian Lipschitz spaces using semigroups and classical analysis we refer to [**45**].

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