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INTRODUCTION

The theory of domination was initiated by one of the authors (1955 a) independently of Landau (1953) who studied necessary and sufficient conditions for a score structure in tournaments. Dominance theory was developed by several authors, notably G. Kreweras (1965), and a more general definition and theory useful in a large variety of "lattice path" problems is now available. In his study of Smirnov tests, G.P. Steck (1974) has obtained a determinantal expression for rank dominance when the null hypothesis is not true. Uniting the work on dominance of combinatorists and statisticians, competitors to some standard non-parametric tests may be obtained following the suggestion of Narayana (1974) re. statistical inference. We thus obtain a refinement of the Smirnov test for the classical two-sample problem. Tables of dominance profiles of the refinement to the Smirnov test and a new dominance test for Lehmann alternatives are given for levels up to 10 % and $4 \leqslant m , n \leqslant 8$. Some examples and power calculations are also included.

1. THE DOMINANCE THEOREM AND ITS SPECIAL CASES.

The following theorem, which can be proved by the same combinatorial technique introduced by Narayana (1955 b) represents the most general form of the dominance theorem.

Définition

Let $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ be two vectors of non-decreasing, non-negative integers satisfying

$$b_i \leqslant a_i \quad (1, \ldots, n).$$

Then vector $\underline{a}$ dominates vector $\underline{b}$, written $\underline{b} \leqslant \underline{a}$

Theorem (G. Kreweras, 1965)

Let $0 \leqslant b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$ and $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ be two sets of integers satisfying $b_i \leqslant a_i$. Let $\underline{s}^{(j)} = (s_{1j} \ldots, s_{nj})$ $j = 1, 2, \ldots$ be a set of vectors satisfying the inequalities

$$0 \leqslant s_{1j} \ldots \leqslant s_{nj} \; ; \; b_i \leqslant s_{ij} \leqslant s_{i,j+1} \leqslant a_i \quad (j = 1, 2, \ldots ; i = 1, \ldots, n). \tag{1}$$

If $|(b, a; r)|$ denotes the number of $r \times n$ matrices $[s_{ij}]$ satisfying (1), then for $r = 1, 2, \ldots$

$$|(b, a; r)| = \| C_{ij}^{(r)} \| = \det_{n \times n} C_{ij}^{(r)} \tag{2}$$

where

$$C_{ij}^{(r)} = \binom{a_i - b_j + r}{r + j - i}_+ \quad \text{or} \quad C_{ij}^{(r)} = \binom{a_{n-j+1} - b_{n-i+1} + r}{r + j - i}_+ \tag{3}$$

and

$$\binom{Y}{Z}_+ = \begin{cases} \binom{Y}{Z} & \text{if } Y \geqslant Z \\ 0 & \text{if } Y < Z \text{ or } Z < 0. \end{cases}$$

Geometrically, if a particle moves in n-dimensions from the lattice point $\underline{b} = (b_1, \ldots, b_n)$ to $\underline{a} = (a_1, \ldots, a_n)$ stopping at r intermediate points $s^{(1)}, \ldots, s^{(r)}$ satisfying (1), we count in how many different ways this is possible.

Proof. The proof by induction follows essentially Narayana (1955) and is omitted.

Special Cases 1° The case $r = 1$ was rediscovered by Steck (1969) and an alternative form follows as a special case of Steck (1974).

2° When $b = 0$, $r = 1$, we have one-sided dominance and a formidable number of papers dealing with this case are available in the literature. The determinant (2) which reduces to

$$D_n = \begin{vmatrix} \binom{a_n+1}{2} & \binom{a_{n-1}+1}{2} & \dots & \binom{a_1+1}{n} \\ 1 & \binom{a_{n-1}+1}{1} & \dots & \binom{a_1+1}{n-1} \\ 0 & 1 & \dots & \binom{a_1+1}{n-2} \\ 0 & 0 & & \binom{a_1+1}{1} \end{vmatrix} \qquad (4)$$

is important for us in this paper.Given $(a_1, a_2, \dots, a_n)$ a program for evaluating $D_n(a_1, a_2, \dots, a_n)$ was kindly made available by S. W. Smillie.

3° Many other special applications to lattice paths with diagonal steps, ballot problems (see for example [2]0 ara available but we do not discuss them in this paper, although they are relevant for ties in the two-sample problem.

2. A DOMINANCE APPROACH TO TESTING

The Smirnov two-sample problem in the continuous case was "discretized" by B.V. Gnedenko (see [1] pp. 70-71) connecting the refined Kolmogorov-Smirnov theory to the combinatorial theory of lattice paths. Given two independent random samples, say, $x_1, x_2, \dots, x_m$ from a population with distribution function (df) $F(x)$, and $y_1, y_2, \dots, y_n$ from a population with (df) $G(x)$, we rank the two samples together and obtain a Gnedenko-path. This is a lattice path from $(0, 0)$ to (m, n) whose kth step is one unit horizontally or one unit vertically according as the kth value in the combined ordering is a x or a y. Let h_j (v_i) be the horizontal (vertical) distance between the y-axis (x-axis) and the path (see Fig. 1). If $R_1 < R_2 < \dots < R_m$ and $S_1 < S_2 \dots < S_n$ denote the ranks (in the combined ordering) of $x's$ and $y's$ then (see also Fig. 1) it can be easily seen that

$$\left.\begin{array}{l} R_i = v_i + i \ , \ 1 \leqslant i \leqslant m \\ S_j = h_j + j \ , \ 1 \leqslant j \leqslant n \\ R_i \leqslant r_i \ , \ 1 \leqslant i \leqslant m \Leftrightarrow S_j \geqslant s_j \ , \ 1 \leqslant j \leqslant n, \end{array}\right\} \qquad (5)$$

where

$$\{r_1, \ldots r_m\} \text{ and } \{s_1, \ldots s_n\}$$

are complementary sets in $\{1, 2, \ldots m+n\}$. The proof of the last relation in (5) may be found in [12]. Now it is evident that a given path can be described in terms of any one of the integer sets of R_i's, S_j's, v_i's or h_j's. For convenience we describe a given path by the vector $(a_1, a_2, \ldots, a_n)$ where $a_j = m - h_j$, $1 \leqslant j \leqslant n$. Clearly $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant m$ and a path $\underline{a}$ lies above another path $\underline{b}$ if and only if $\underline{a}$ dominates $\underline{b}$. From (5) it is also clear that rank domination (in terms of $(R_1, \ldots R_m)$) is equivalent to the corresponding path domination. Also, it is worth noting that the set of all $\binom{m+n}{n}$ paths form a distributive lattice with the partial order introduced by the dominance relation. Various other interesting duality relations between lattice paths are available in the literature – the interested reader is referred to a forthcoming monograph by Narayana.

There are several nonparametric tests for testing the null hypothesis $H_o : F = G$ under the continuity assumption. We refer to Hájek [3] for an elementary exposition of these tests. Here we propose a new class of tests based on the dominance number D_n. Consider the set of all $\binom{m+n}{n}$ lattice paths from $(0, 0)$ to (m, n) as the sample space. Given any fixed path $\underline{a}$, the sets of paths of the form $\{\underline{b} : \underline{b} \leqslant \underline{a}\}$ and $\{\underline{b} : \underline{b} \geqslant \underline{a}\}$ with or without equalities are defined as *Gnedenko-Feller* (GF) regions. Any test procedure, which uses such regions or unions of such regions either as critical regions or as acceptance regions is defined as a *dominance test* (D-test). Note that all D-tests are nonparametric since all $\binom{m+n}{n}$ lattice paths are equally likely under H_o. Also, since path domination is equivalent to rank domination any D-test is also a rank test (not necessarily linear in ranks).

We illustrate a D-test procedure involving one GF region only. For testing $H_o : F = G$ against $H_1 : F > G$, an appropriate D-test consists of the *GF* critical region defined by $\{\underline{b} : \underline{b} \leqslant \underline{c}\}$. The critical path c will depend on the level – if α is the level, then the number of paths dominated by $\underline{c}$ should not exceed $\left[\binom{m+n}{n}\alpha\right]$; of course, for any lattice path $\underline{a}$, the number of paths dominated by $\underline{a}$ is equal to D_n given by (4). In Table 1 we present 5 % critical paths $(5 \leqslant m, n \leqslant 10)$ for the above test procedure. Also presented in Table 1 are the critical values and number of paths for the

Wilcoxon-rank-sum test (U-test) for comparison. It is interesting to note that $U = \sum_{i=1}^{m} R_i - \frac{m(m+1)}{2}$ is equal to the area under the path, namely , $a_1 + a_2 + \ldots + a_n$.

Example 1 (Hájek (1969) p. 70). Two independent random samples of $m = n = 10$ are drawn from two normal populations with $\mu_1 = 500, \sigma = 100$ and $\mu_2 = 580, \sigma = 100$ respectively. The data is given below (the values are rounded off to the nearest integers).

$x's$: 458, 620, 552, 327, 406, 733, 430, 498, 505, 558
$y's$: 746, 599, 690, 502, 425, 556, 491, 642, 622, 533

we shall illustrate both the U-test and D-test for testing $H_o : \mu_1 = \mu_2$ vs $H_1 : \mu_1 < \mu_2$. The combined ranking of $x's$ and $y's$ is as follows :

x	x	y	x	x	y	x	y	x	y
327	406	425	430	458	491	498	502	505	533
x	y	x	y	x	y	y	y	x	y
552	556	558	599	620	622	642	690	733	746

U-test : $U = 86 - \frac{m(m+1)}{2} = 31$. Since the 5 % point A_{10} (from Table 1) is 27, we accept H_0 at 5 % level of significance. In fact it is easy to find that $p(U \leqslant 31) = 0.083$ and $p(U < 31) = 0.072$. Consequently the U-test accepts H_0 at 5 % level of significance irrespective of randomization.

D-test : From Table 1, the 5 % critical vector is 122347788 (shown in Fig. 2) with corresponding D_n value 9193. Since the observed Gnedenko path, 111234568, is dominated by the critical vector (see Fig. 2), the null hypothesis is rejected at 5 % level. The actual level attained is 9193/184756 = 0.0497.

Note : It should be pointed out that critical paths given in Table 1 are not necessarily the best – they are given only for illustrative purpose. However, D-tests attain better (or more) levels than the Wilcoxon test since the partial ordering of paths by D_n is finer than the partial order introduced by area under the path. The problem of finding the best dominance test for a given problem is open. However, in the following Section we obtain some dominance tests which possess good properties in the case of Lehmann alternatives.

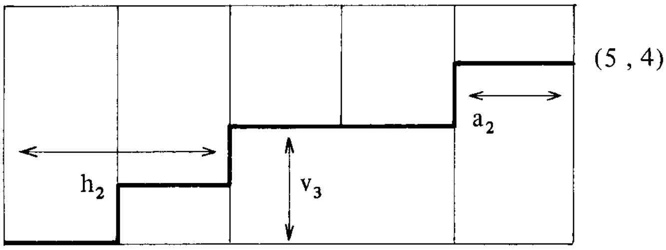

Figure 1

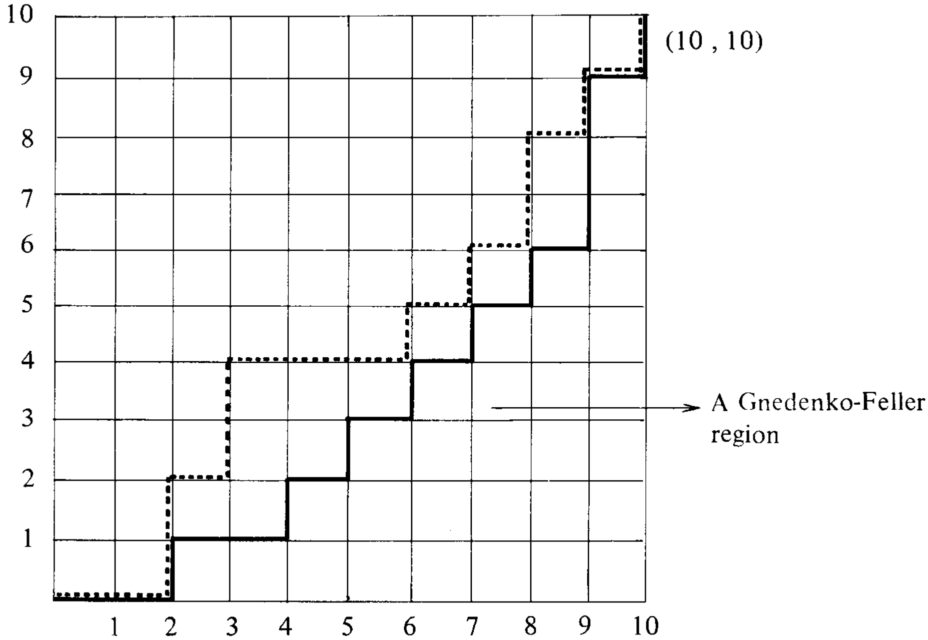

Figure 2

Note : 5 % critical path corresponding to the path 122347788 is shown in Fig. 2 by broken lines – the path represented by solid lines is the observed path 111234568 (initial zeros omitted).

(m, n)	5 % of $\binom{m+n}{n}$	Area of U-test : A_n	# of paths with Area A_n	Critical path for D-test : $\underset{\sim}{c}$	# of paths dominated by $\underset{\sim}{c}$
(5,5)	12	4	12	1122	12
(6,5)	23	5	19	1223	23
(7,5)	39	6	29	136	37
(8,5)	64	8	60	11244	63
(9,5)	100	9	83	1266	100
(10,5)	150	11	149	11159	150
(6,6)	46	7	43	22233	46
(7,6)	85	8	63	112234	85
(8,6)	150	10	122	11178	149
(9,6)	250	12	220	24446	250
(10,6)	400	14	374	114466	395
(7,7)	171	11	167	124444	170
(8,7)	321	13	302	23348	320
(9,7)	572	15	519	1111578	569
(10,7)	972	17	854	34688	965
(8,8)	643	15	534	334466	642
(9,8)	1215	18	1127	123788	1211
(10,8)	2187	20	1819	1135688	2175
(9,9)	2431	21	2283	13566666	2429
(10,9)	4618	24	4375	12444778	4604
(10,10)	9237	27	8241	122347788	9193

Note : Initial zeros of paths are omitted for convenience, for example $1122 \equiv 01122$ since the number of elements in the path vector is exactly $n = 5$.

3. A REFINEMENT OF THE SMIRNOV D^+ (m, n)-test.

Let F_m and G_n be the sample distribution functions of $x's$ and $y's$ respectively. The Smirnov statistics are defined as

$$D^+(m, n) = \sup_x [F_m(x) - G_n(x)], \; D^-(m, n) = \sup_x [G_m(x) - F_n(x)],$$
$$D(m, n) = \max \{D^+(m, n), D^-(m, n)\}.$$

In what follows we confine our attention to D^+ (m, n) which is appropriate for testing $H_o : F = G$ against $H_1 : F(x) > G(x)$ for at least one x. The rejection region for H_o against H_1 is defined by

$$\left\{ \frac{mn}{d} D^+ (m, n) > s_\alpha \right\} \tag{6}$$

where s_α depends on the level and $d = (m, n)$, the greatest common divisor of m and n. As is well known (see for example [11])

$$mn\, D^+ (m, n) = \sup_{1 \leqslant i \leqslant m} \{(m + n)\, i - mR_i\}$$

so that, we have

$$\left\{ \frac{mn\, D^+(m, n)}{d} \leqslant s_\alpha \right\} = \left\{ R_i \geqslant \frac{i\,(m + n) - s_\alpha d}{m}, \ 1 \leqslant i \leqslant m \right\} \tag{7}$$

Recalling our remarks that rank domination is equivalent to the corresponding path domination, we see from (6) and (7) that actually the acceptance regions of D^+ (m, n) test are indeed GF regions and hence D^+ is a dominance test.

Note that for fixed m, n there are only finite number of levels attained by D^+ (m, n) test and not every path can be a possible boundary path (of the acceptance region) of D^+ (m, n)-test because of the restriction (7). If we can interpose a path in between two Smirnov boundary paths given by (7), then we can define a new dominance test. Let N^+ (m, n) be the critical region (in the set of all lattice paths) defined by the acceptance regions of $D^+(m,n)$ plus the acceptance regions defined by the new boundaries interposed between Smirnov boundaries as above. Suppose two adjacent Smirnov boundaries (m, n fixed) are given by $\underline{a}' = (a'_1, \ldots, a'_n) \leqslant \underline{a} = (a_1, \ldots, a_n)$ with $\Sigma\, a_i > \Sigma\, a'_i + 1$. It immediately follows that there is at least one path sandwiched between a and a'. Among such intermediate paths we pick say $a^{(1)}, \ldots, a^{(p)}$ which satisfy the relations

$$\left.\begin{aligned} \underline{a}^{(o)} < \underline{a}^{(1)} < \ldots < \underline{a}^{(p)} < \underline{a}^{(p+1)} \\ \sum_{i=1}^{n} a_i^{(j)} = \sum_{i=1}^{n} a_i^{(j-1)} + 1 \ \text{ for } 1 \leqslant j \leqslant p + 1 \end{aligned}\right\} \tag{8}$$

where we have set $\underline{a}^{(o)} = \underline{a}'$, $\underline{a}^{(p+1)} = \underline{a}$. The interposed paths satisfying (8) together with the Smirnov boundary paths constitute N^+-boundaries defining a $N^+(m, n)$ test. The reason why we insist on (8) is to get nested sets of critical regions. The following proposition shows that the N^+ test always refines the D^+ test when $m = n \geqslant 2$.

Proposition 1 (i) For $m \neq n$, $N^+(m, n)$ can be obtained from N^+ (n, m) by duality. (ii) For $n \geqslant 2$, N^+ (n, n) is a true refinement of D^+ (n, n). (We conjecture $N^+ (m, n)$ is a true refinement of $D^+ (m, n)$ if and only if $d = (m, n) > 1$, otherwise D^+, N^+ coincide.)

The duality referred to in the proposition is as follows: If $(a_1, \ldots, a_n)$ is in $N^+ (m, n)$ then the dual path in $N^+ (n, m)$, for $m \neq n$, consists of $n - a_n$ $0's$, $a_n - a_{n-1}$ $1's$, $\ldots$, $a_1 - 0$ $m's$. For example $0011 \in N^+ (4, 3)$; the dual path in $N^+ (3, 4)$ is (0 0 2). As seen from part (ii), any N^+-test refines the D^+-test ; however such a refinement is not generally unique – the choice of refinement will depend on the alternative. We present in Table 2 a N^+-test procedure suitable for Lehmann alternatives. It should also be pointed out that similar refinements for $D^- (m, n)$ and $D^+ (m, n)$ procedures can be constructed as above – but we will not do this here. Power considerations are postponed to Section 4.

4. DOMINANCE TESTS AGAINST LEHMANN ALTERNATIVES.

Alternatives of the type $G(x) = F^k(x)$ with $k \geqslant 2$ integral are known as Lehmann alternatives (following Lehmann (1953)) and they are found to be appropriate in some biological and other problems as noted by Shorack (1967). For any fixed value of k, Lehmann has in principle described uniformly most powerful (UMP) tests for testing $F = G$ against $G = F^k$, but they are tedious from the application point of view even for $k = 2$. Here we remark that by using Lehmann (1953) equation (4.5) it is easily seen that if $\underline{a}$ dominates b, $L_k(b) \geqslant L_k(a)$, where L_k is the probability of a path when $G = F^k$ is true. *It immediately follows that every* UMP *critical region is a union of a finite number of* GF *regions.* Hence it is possible to approximate UMP tests with GF critical regions. As it is fairly easy to make power calculations when a small number of *GF* regions is involved, we propose a test DL_k (2) for the case $k = 2$ and compare its power with the powers of Wilcoxon and other tests. The feature of DL_k (2) which is attractive is that at most two *GF* regions are involved in the critical region and it can be considered as an approximation to the UMP test. We describe below the method of construction of DL_2 (2) ; this method can be extended for $k > 2$ as well, but we shall not give the details here as asymptotic properties are being studied.

To define DL_2 (2) we start off with the two nested sequences of paths, where each path consists of n elements ;

Table 2
N^{+}-boundaries (of acceptance region) with number of paths in the critical region

Boundaries of $N^{+}(n, n)$ | n	3	4	5	6	7	8	9	10
	# $[N^{+}(n, n)]$							
1	1	1	1	1	1	1	1	1
2*	4	5	6	7	8	9	10	11
12	6	8	10	12	14	16	18	20
13*	11	17	24	32	41	51	62	74
23*	13	23	36	52	71	93	118	146
123	15	28	45	66	91	120	153	190
124*				114	166	230	307	398
134*				156	246	365	517	706
234*				192	316	485	706	986
1234				220	364	560	816	1140
1235*						835	1245	1777
1245*						1135	1795	2701
1345*						1415	2335	3646
2345*						1655	2785	4416
12345						1820	3060	4845
12346*							4061	6483
12356*							5161	8628
12456*							6211	10828
13456*							7171	12853
23456*							7996	14503
123456							8568	15504
123457*								19144
123467*								23148
123567*								26998
124567*								30598
134567*								33898
234567*								36758
1234567								38760
Total paths	20	70	252	924	3432	12870	48620	184756

Note : Starred paths are the interposed boundaries.

Table 2. (contd.)

Boundaries of $N^+(m, n)$	$\# N^+(m, n)$ $m = 5$ $n = 4$	$m = 6$ $n = 5$	$m = 7$ $n = 4$	$m = 7$ $n = 5$	$m = 7$ $n = 6$	$m = 8$ $n = 5$	$m = 8$ $n = 7$
1	1	1	1	1	1	1	1
2	5	6	5	6	7	6	8
12	9	11	11	12	13	13	15
13	18	25	20	26	33	27	42
23	27	41	35	46	58	51	78
24		–	51	–	–	81	–
123		55	–	66	78	–	105
124		83	69	94	126	106	180
134				130	182	151	280
135					–	204	–
234					238		380
235							–
1234							455
1235							620
1245							812
1345							1008
Total paths	126	462	330	792	1716	1287	6435

$m = 6$ $n = 4$		$m = 6$ $n = 8$		$m = 8$ $n = 4$	
Boundaries	# paths	Boundaries	# paths	Boundaries	# paths
1	1	1	1	1	1
2	5	2	7	2	5
12	10	12	14	3*	15
13	19	13	34	13	21
23	31	23	64	14*	40
24*	47	123	91	24	55
		124	139	25*	86
		134	209		
		234	289		
		235*	384		
		1235	454		
Total paths	210	Total paths	3003	Total paths	495

$$0\ldots01\,,0\ldots011\,,\ldots\ldots,01\ldots1\,,1\ldots1 \qquad (P_1)$$
$$0\ldots02\,,0\ldots012\,,\ldots\ldots,01\ldots12\,,01\ldots13 \quad (Q_1)$$

For $i \geqslant 1$ define P_{i+1}, Q_{i+1} recursively by

$$P_{i+1} = P_i + 1\ldots1\,, \quad Q_{i+1} = Q_i + 1\ldots1$$

and let $$P = (P_1\,, P_2\,, \ldots) = (p_1\,, p_2\,, \ldots),$$

where $p_1\,, p_2\,, \ldots,$ are the individual paths in $P_1\,, P_2\,, \ldots,$ in that order ; similarly $Q = (Q_1\,, Q_2\,, \ldots) = (q_1\,, q_2 \ldots)$. Then the *GF* regions of DL_2 (2) are defined by

$$0\ldots.0\,,0\ldots01\,,0\ldots011\,, q_i \cup p_{i_j} \qquad \text{for} \qquad i \geqslant 1,$$

where p_{i_j} is the element of P_i which satisfies

$$L_2\ (p_{ij}) \geqslant L_2\ (q_i) \text{ and } L_2\ (p_{ij+1}) < L_2\ (q_i).$$

In Table 3 the dominance profile of DL_2 (2) only for $m = n = 6$ is presented ; complete tables for $4 \leqslant m\,, n \leqslant 8$ are available with the authors. We close this paper with some power comparisons.

Power Comparisons : For Lehmann alternatives the exact powers of *GF* regions can be computed by using a closed formula, recently obtained by Steck [12]. Using Shorack [10] we have made randomized power comparisons between the Wilcoxon and DL_2 (2) tests (under Lehmann alternatives) for $4 \leqslant m\,, n \leqslant 8$. It is found that DL_2 (2) is slightly uniformly better than the Wilcoxon test and the power increase ranged from 1 % to 5 %. Some examples are given for illustration :

	Wilcoxon	DL_2 (2)	
$m = n = 6$	.3796	.3999	
$m = n = 8$	.4451	.4664	($\alpha = .1$ in all cases)
$m = 8,\ n = 5$	.3816	.3911	

In the case of refined Smirnov test power comparisons showed that the N^+-test is uniformly better than the Smirnov test (D^+-test) at the usual levels of significance for $3 \leqslant m \leqslant n \leqslant 10$. In Table 4 the randomized powers of D^+ and N^+ tests are presented for $3 \leqslant m \leqslant n \leqslant 10$. The power increase is about 10 %.

Table 3
Dominance profile of DL_2 (2) for $m = n = 6$

GF Region	*Level*
0	1
1	2
11	3
11 ∪ 2	4
1111 ∪ 12	7
11111 ∪ 112	9
111111 ∪ 1112	11
111111 ∪ 11112	12
111122 ∪ 11113	23
111122 ∪ 111113	24
112222 ∪ 111123	36
122222 ∪ 111223	42
222222 ∪ 112223	46
222222 ∪ 122223	48
222233 ∪ 122224	84
222333 ∪ 222224	95

Table 4

Randomized power of Smirnov and refined Smirnov tests against Lehmann alternatives $G = F^k$, $k = 2, 3$.

Level of Significance	$M = 3$ $N = 3$				$M = 4$ $N = 4$			
	$k = 2$		$k = 3$		$k = 2$		$k = 3$	
	D^+	N^+	D^+	N^+	D^+	N^+	D^+	N^+
.01	–	–	–	–	–	–	–	–
.05	–	–	–	–	.15834	.17708	.26029	.29969
.10	.23382	.2500	.3409	.36817	.28664	.29663	.43779	.4588
	$M = 5$ $N = 5$				$M = 6$ $N = 6$			
.01	.05262	.05758	.11103	.12402	.06763	.07222	.15353	.16833
.05	.19409	.20250	.34082	.35604	.21404	.23620	.37901	.42681
.10	.30213	.34578	.46642	.54501	.33996	.3672	.53739	.57943
	$M = 7$ $N = 7$				$M = 8$ $N = 8$			
.01	.07397	.08693	.17261	.20965	.09191	.09381	.22650	.23126
.05	.23359	.27002	.42219	.49382	.26759	.27946	.49170	.51196
.10	.38174	.38952	.60548	.61869	.39471	.43255	.62499	.70166
	$M = 9$ $N = 9$				$M = 10$ $N = 10$			
.01	.09776	.11275	.24141	.28464	.10970	.12848	.27796	.32975
.05	.28622	.30805	.52405	.56455	.30277	.34057	.55592	.61982
.10	.42054	.46543	.66701	.72818	.45478	.48454	.71714	.75444

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