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TOURNAMENT CONFIGURATION AND WEIGHTED VOTING

by

B.E. WYNNE and T.V. NARAYANA

1. INTRODUCTION

Integer sequences can serve as powerful explicit descriptions of fundamental processes underlying organized activities. Seemingly unrelated decision processes may be perceived as mere variants of one another, once defined in terms of their essential sequences. As an example, two diverse sporting and/or business activities will be shown to involve the same recursively describable integer sequence.

2. RANDOM TOURNAMENTS

Let us define $T_2 = T_3 = 1$

(1) $$T_{2n+1} = 2T_{2n} - T_n \, , \; T_{2n} = 2T_{2n-1} \quad \text{for} \quad n \geqslant 2 \, .$$

Then T_n is the total number of ways in which a knock-out tournament among n entrants can be structured in order to determine a champion. Were no byes allowed, the number of entrants must be $n = 2^t$, where t is the number of match rounds in the tournament. To satisfy the practical general conditions we must allow any reasonable number of entrants, yet provide at least one match per round of the tournament.

As Capell and Narayana [1] showed, T_n is the number of tournaments or possible match play configurations for n entrants and is given by Table 1..

TABLE 1.

Match Tournament Entrants and Configurations

Entrants,	n	2	3	4	5	6	7	8	9	10	11
Configurations,	T_n	1	1	2	3	6	11	22	42	84	165

Great care must be taken not to confuse T_n with the number of "match-tree" configurations as defined by Maurer [2].

3. *WEIGHTED VOTING PROCEDURE*

A "Board of Directors" problem was posed in [4]. There, the challenge was to define an algorithm which would generate the minimal-sum set of weights for aggregating the votes of up to m participants such that under any behavior other than total abstention, (1) no tally of weighted votes could result in a tie, and (2) any group decision would always reflect the will of the majority of actual voters. An incomplete computational proof and an accompanying generation method satisfying the challenge have been submitted [6].

The essence of that method was to recursively compute a vector element increment, I_m , as follows :

$$I_m = 2(I_{m-1}) - \text{mod}_2(m-1)(I_{[m/2]-1}) \tag{2}$$

where $m \geqslant 3$ and $I_0 = 0$, $I_1 = I_2 = 1$.

Then, given the integer weighting elements, $W_{m-1,j}$, of the weights vector, W_{m-1} , the elements of the vector W_m are given by :

$$W_{m,j} = W_{m-1} + I_m \tag{3}$$

where $j = 1,\ldots,m-1$ and $W_{m,m} = I_m$.

Finally, denoting the sum of the weights vector elements as S_m , the data of Table 2 is generated for illustration. Comparison of the T_n values of Table 1 with the I_m values of Table 2 discloses the same sequence of numbers. That sequence is # 297, p. 53 of [5] . This equivalence is more readily seen by restating T_{n+1} as in (2) :

(4) $$T_{n+1} = 2T_n - \mathrm{mod}_2(n-1)\, T_{[n/2]}$$

with $T_0 = 1$, $T_2 = T_3 = 1$ and $n \geqslant 3$.

TABLE 2.

Non-distorting, Tie-avoiding Integer Vote Weights

members, m	1	2	3	4	5	6	7	8	9	10	11	12	13
totals, S_m	1	3	9	21	51	117	271	607	1363	3013	6643	14491	31495
column vectors of vote weights, $[W_m]$	1	2	4	7	13	24	46	88	172	337	667	1321	2629
		1	3	6	12	23	45	87	171	336	666	1320	2628
			2	5	11	22	44	86	170	335	665	1319	2627
				3	9	20	42	84	168	333	663	1317	2625
					6	17	39	81	165	330	660	1314	2622
						11	33	75	159	324	654	1308	2616
							22	64	148	313	643	1297	2605
								42	126	291	621	1275	2583
									84	249	579	1233	2541
										165	495	1149	2457
											330	984	2292
												654	1962
													1308

NOTE

The underlinded values along the diagonal of vector elements are the I_m values.

4. *CONCLUSION*

It is easy to verify that subsets of W_n are tie-avoiding and non-distorting i.e. any y smallest weights sum exceeds any y-1 exclusive larger weights sum taken W_n . Indeed, Kreweras (personal communication) has indicated to us an elegant proof of these facts based on "triangular sums" for non-decreasing positive integer sequences i.e. sums of the type

$$\theta(x_1,x_2,\ldots,x_m) = \sum_{i=1}^{m} x_i \cdot \min(i,m+1-i).$$

While this proof shows that T_n is in this sense a minimal triangular sequence, (T_{n+2} being equal to $1 + \theta(T_1,T_2,\ldots,T_n)$ where $T_1 = T_2 = 1$) the question whether $[W_m]$ are minimal-sum or even minimal-dominance in the sense of [3] is an open question. We propose thus the following conjecture : prove or disprove that the weight vectors W_m of Table 2 are minimal-sum vectors.

REFERENCES

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