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On proximity and intersection inequalities for foliations on algebraic surfaces

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1 Introduction

In [11] Poincaré posed the problem of bounding the degree of a projective curve C which is invariant for a foliation F on $\mathbf{CP}^2$ in terms of the degree of the foliation. Recently, Carnicer (see, [4]) has shown that, if F is nondicritical then $deg(C) \leq deg(F) + 2$. In order to obtain some result of this type for the dicritical case, proximity equalities and inequalities for the orders of a foliation at infinitely near points are introduced in [2], showing that, in general, one has

$$deg(C) \leq deg(F) + 2 + a \tag{1}$$

where a can be computed from the singularities of C.

Proximity inequalities for linear systems (or complete ideals) of germs of functions at a nonsingular point of a surface come from Enriques (see [7]) and they appear recently in [5], [9] and [10]. Trying to study complete ideals in higher dimensions, proximity inequalities are developed for this case in [3] as intersection theoretical inequalities, i.e., by using intersection theory on a variety on which the lifting of the ideal becomes invertible.

The purpose of this paper is to give an approach to the problem stated by Poincaré (the relation between global invariants of C and F) on general complete smooth algebraic surfaces using intersection theory, i.e. intersection equalities and inequalities as an alternative to the proximity ones. As a reference on intersection theory on algebraic surfaces see, for instance, chapter II in [1]. For the case of $\mathbf{CP}^2$, we will revisit the results in [4] and [2] and

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we will obtain an explicit bound for the degree of C in terms of numerical invariants of the singularities of C.

2 Divisor classes associated to a foliation on a surface

Let S be an smooth algebraic surface on an algebraically closed field K of characteristic zero, and let F be a one-dimensional algebraic foliation on S. The foliation F is given by the data $(V_i, w_i)_{i\in I}$, where the V_i's are open sets, w_i is a nonzero regular differential 1-form on V_i, $S = \cup_{i\in I} V_i$ and $w_i = g_{ij} w_j$ on $V_i \cap V_j$ with $g_{ij} \in \mathcal{O}(V_i \cap V_j)^*$. Now, let $(f_i)_{i\in I}$ be a set of nonzero rational functions with $f_i = g_{ij} f_j$ on $V_i \cap V_j$ for each couple (i,j). The data $(f_i)_{i\in I}$ are local equations defining a Cartier divisor on S whose class modulo linear equivalence does not depend on the choice of $(V_i, w_i)_{i\in I}$, therefore this class only depends on F and it will be denoted by D_F. Moreover, since $f_i^{-1} w_i = f_j^{-1} w_j$ on $V_i \cap V_j$, the data $(f_i^{-1} w_i)_{i\in I}$ glue together in a rational differential form w_F which is determined by F up to multiplication by a rational function in $K(S)^*$.

The foliation F can be given, alternately, by the data $(V_i, v_i)_{i\in I}$, where v_i is a regular vector field on each open set V_i which is orthogonal to w_i and $v_i = g'_{ij} v_j$ on $V_i \cap V_j$ with $g'_{ij} \in \mathcal{O}(V_i \cap V_j)^*$. As above, the linear equivalence class of the Cartier divisor given by a set of rational functions $(f'_i)_{i\in I}$ with $f'_i = g'_{ij} f'_j$ on $V_i \cap V_j$ does not depend on the choice of the data and it will be denoted by D'_F. Again, the vector fields $(f_i'^{-1} v_i)_{i\in I}$ glue together in a rational vector field v_F on S which is determined by F up to multiplication by a rational function in $K(S)^*$. If K_S is the canonical (linear equivalence) class of S, then the tensor contraction of sections gives us the relation

$$D'_F = D_F + K_S.$$

Now, let C be a complete reduced algebraic curve on S. First, assume that all the components of C are not invariant by F. Then, w_F can be chosen such that its restriction to each component of C is a non zero rational differential form on the component. Let $n : \tilde{C} \to C \subset S$ be the normalization of C. For each $q \in \tilde{C}$, define the index $i_q(F, C)$ to be the order at q of $n^* w_j$ if $n(q) \in V_j$. It is clear that $i_q(F, C)$ does not depend on the choice of the open V_j nor on the data (V_j, w_j). By definition of the intersection number

$D_F \cdot C$ one gets

$$\sum_{q\in\tilde{C}} i_q(F,C) = D_F \cdot C + deg(n^* w_F).$$

On the other hand, by adjoinction formula, we obtain

$$deg(n^* w) = 2g - 2 = (K_S + C).C - deg\Delta,$$

where g is the geometrical genus of C (i.e. $2-2g$ is the topological Euler characteristic of $\tilde{C}$), and $\Delta = \sum_{p\in C}(2\delta_p)p$, with $\delta_p = length(\tilde{O}_{C,p}/O_{C,p})$, is the adjoinct zero cycle. Thus, if $I(C,F) = \sum_{q\in\tilde{C}} i_q(F,C)$, we get the formula

$$I(C,F) = (D_F + K_S + C)\cdot C - deg\Delta = (D'_F + C)\cdot C - deg\Delta. \quad (2)$$

Now, assume that all the components of C are invariant by F. Then, the vector field v_F can be chosen such that it is defined at the generic point of each component of C and, by assumption, tangent to the component. Therefore, its restriction to C can be lifted to a rational vector field $\tilde{v}_{F,C}$ on $\tilde{C}$. For each $q \in \tilde{C}$, we define the index $j_q(F,C)$ as follows: If $n(q) \in V_i$ and F is given at V_i by the vector field v_i, then v_i is tangent to $C \cap V_i$, so it defines a vector field on a neighborhood of $n(q)$ at C which can be lifted to the complement of q at a neighborhood of q in $\tilde{C}$ and extended to q. Then $j_q(F,C)$ is the order at q of this extension. It is clear that the index $j_q(F,C)$ only depends on F, C and q. Setting $J(C,F) = \sum_{q\in\tilde{C}} j_q(F,C)$ and, taking into acount the definition of the intersection number $D'_F \cdot C$, as above one has

$$J(C,F) = D'_F \cdot C + deg(\tilde{v}_{F,C}) = D'_F \cdot C + 2 - 2g$$

and by adjoinction,

$$J(C,F) = (D'_F - K_S - C)\cdot C - deg\Delta = (D_F - C)\cdot C + deg\Delta. \quad (3)$$

Note that $deg(\tilde{v}_{F,C}) = 2-2g$ is nothing but the Poincaré-Hopf formula and, for this reason, the last equality is also known as the Poincaré-Hopf formula for F and C (see [6]).

If S is the projective plane $\mathbf{P}^2$, then $D_F = aL$ for some integer a, L being the hyperplane class. Since $K_{\mathbf{P2}} = -3L$ one has $D'_F = (a-3)L$. Thus, if C is a reduced projective plane curve of degree m whose components are not invariant by F, the formula (2) gives us

$$I(C,F) = (a - 3 + m)m - deg\Delta$$

and, in particular, if C is smooth

$$I(C,F) = (a-3+m)m.$$

Thus, the integer $(a-3+m)m$ can be viewed as the number of points of C such that F is tangent to C. If $C = L$ is a line, we get

$$I(C,F) = a-2,$$

so that the integer $d = a-2$ is just the number of points at which the foliation is tangent to a general line and, for this reason, it is called the degree of F and denoted by $deg(F)$.

If all the components of $C \subset \mathbf{P}^2$ are invariant by F, then the formula (3) can be written as

$$J(C,F) = (a-m)m + deg\Delta = (d+2-m)m + deg\Delta.$$

Remark 1. If we take homogeneous coordinates $(X_0 : X_1 : X_2)$ on $\mathbf{P}^2$, then F can be given by a 1-form

$$\Omega = A(X_0,X_1,X_2)dX_0 + B(X_0,X_1,X_2)dX_1 + C(X_0,X_1,X_2)dX_2$$

where A, B and C are homogeneous polynomials of the same degree, say k, which are free from common factors and satisfy the Euler's equality

$$X_0A + X_1B + X_2C = 0.$$

In local affine coordinates, for instance in the chart V_0 with coordinates $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}$, F is given by the form $w_0 = B(1,x,y)dx + C(1,x,y)dy$. Comparing with the form $w_1 = A(v,1,u)dv + C(v,1,u)du$ in $V_0 \cap V_1$, where V_1 is the affine chart with coordinates $v = \frac{X_0}{X_1}, u = \frac{X_2}{X_1}$, we get $w_1 = v^{k+1}w_0$. This means that $D_F = (k+1)L$ and we have $deg(F) = k-1$.

Assume that we know that F is given on an affine plane by the form $w = b(x,y)dx + c(x,y)dy$, where b and c are polynomials and $l = max(deg(b), deg(c))$. Then, from the above discussion, it is easy to see that $deg(F) = l$ (resp. $deg(F) = l+1$) if the infinity line is (resp. is not) invariant by F. Thus, the degree of a foliation in the plane can be also described directly from affine data.

3 The local situation. Proximity and intersection inequalities

Let us consider a closed point p on an smooth algebraic surface S and let $\pi : \tilde{S} \to S$ be a morphism which is obtained by blowing-up a constellation $\mathcal{C}$ of points infinitely near p. More precisely, one has $\pi = \pi_N \circ \pi_{N-1} \circ \cdots \circ \pi_1 \circ \pi_0$, where $\pi_0 : S_1 \to S_0 = S$ is the blowing-up with center at p, π_1 is the blowing-up of S_1 with center at a finite set of closed points $\Sigma_1 \subset \pi_0^{-1}(p)$, and, in general, $\pi_i : S_{i+1} \to S_i$ is the blowing-up with center at a finite set of closed points Σ_i in $\pi_{i-1}^{-1}(\Sigma_{i-1})$. The constellation $\mathcal{C}$ is nothing but the set of points (infinitely near p) $\mathcal{C} = \cup_{i=0}^{N} \Sigma_i$, where $\Sigma_0 = \{p\}$.

For each $q \in \mathcal{C}$, the integer i such that $q \in \Sigma_i$ is called the level of q and is denoted by $l(q)$. We also denote by $B_q = \pi_i^{-1}(q)$ the exceptional divisor of the blowing-up of q at S_{i+1}, by E_q the strict transform of B_q at $\tilde{S} = S_{N+1}$ and by E_q^* the total transform of B_q at $\tilde{S}$ (the transforms are with respect to the morphism $\pi_N \circ \pi_{N-1} \circ \cdots \circ \pi_{i+1}$). The abelian group of relative divisors $\mathbf{E} = Div(\tilde{S}/S)$ is free, being both $\{E_q\}_{q \in \mathcal{C}}$ and $\{E_q^*\}_{q \in \mathcal{C}}$ basis of it. The base change between the above basis is given by the so-called "proximity matrix", i.e. the incidence matrix with respect to the proximity relation (see [3] and [9]). The proximity relation is defined as follows: If $r, q \in \mathcal{C}$, then r is said to be proximate to q, and denoted by $r \to q$ if either $r \in B_q$ or $l(r) \geq l(q) + 2$ and r is on the strict transform of B_q at $S_{l(r)}$. It is obvious that the change of basis is given by

$$E_q = E_q^* - \sum_{r \to q, r \in \mathcal{C}} E_r^*.$$

On $\mathbf{E}$ one has the intersection form $\mathbf{E} \times \mathbf{E} \longrightarrow \mathbf{Z}$, where for $E_1, E_2 \in \mathbf{E}$, the integer $E_1 \cdot E_2$ means the degree of the intersection class (on the surface). The following proposition will provide us some useful calculus with the intersection numbers.

Proposition 1 . *With assumptions and notations as above, one has*

1. $E_q^* \cdot E_r^* = 0$ *if* $q \neq r$ *and* $E_q^* \cdot E_q^* = -1$, *for* $q, r \in \mathcal{C}$.
2. $K_{\tilde{S}/S} = \sum_{q \in \mathcal{C}} E_q^*$, *where* $K_{\tilde{S}/S}$ *is the relative canonical divisor for* π.

Proof: Assertion 1 follows from the projection formula and the fact that $B_q \cdot B_q = -1$ on $X_{l(q)}$, meanwhile 2 is a consequence of the obvious fact that $K_{S_{i+1}/S_i} = \sum_{q \in \Sigma_i} B_q$ for any i.

Now, consider a foliation F on S whose germ at $p \in S$ is given by the germ of 1-form $w \in \Omega^1_{S,p}$. The order of F at p, $\nu_p(F)$, is defined to be the largest integer ν such that $w \in m^{\nu}_{S,p}\Omega^1_{S,p}$. If C is a constellation of points infinitely near p then, for any i, $0 \leq i \leq N+1$, we denote by F_i the strict transform of F on the surface S_i (i.e., the extension to S_i of the inverse image of $F|_{S-\{p\}}$ on the complement of $(\pi_i \circ \cdots \circ \pi_1 \circ \pi_0)^{-1}(p)$). Set $\tilde{F} = F_{N+1}$. For any $q \in C$ we define the order of F at q, $\nu_q(F)$, to be the order at q of $F_{l(q)}$. The point q is said to be nondicritical (resp. dicritical) for F if B_q is (resp. is not) invariant by $F_{l(q)+1}$, or equivalently, if E_q is (resp. is not) invariant by $\tilde{F}$. Set $\epsilon_q(F) = 0$ (resp. $\epsilon_q(F) = 1$) if q is nondicritical (resp. dicritical) for F, and consider the divisor class on $\tilde{S}$ given by

$$E_{\tilde{F}} = \sum_{q \in C} (\nu_q(F) + \epsilon_q(F)) E^*_q.$$

(More precisely, we should write $E_{\tilde{F}/F}$ instead $E_{\tilde{F}}$, but we use this notation since no confusion arises in the paper).

The following proposition gives us the computation of the divisor class $D_{\tilde{F}}$ associated to the foliation $\tilde{F}$.

Proposition 2 : *With the assumptions and notations given above, the following equality holds:*

$$D_{\tilde{F}} = \pi^* D_F - E_{\tilde{F}}$$

Proof: From the definition of dicriticalness, it is easy to see that if w_q defines locally $F_{l(q)}$ at a neighbourhood U of q, then one has an equality

$$(\pi^*_{l(q)} w_q)\mathcal{O}_{S_{l(q)+1}} = \mathcal{O}_{S_{l(q)+1}}(-(\nu_q(F) + \epsilon_q(F))B_q)F_{q+1} \;|_{\pi^{-1}_{l(q)}(U)}$$

Indeed, writing $\nu = \nu_q(F)$ and taking formal coordinates x, y around q, we get a Taylor expansion

$$w_q = (a_\nu(x,y) + a_{\nu+1}(x,y) + \cdots)dx + (b_\nu(x,y) + b_{\nu+1}(x,y) + \cdots)dy,$$

where a_j and b_j are homogeneous polynomials of degree j. Then, q is nondicritical (resp. dicritical) if and only if $xa_\nu + yb_\nu \neq 0$ (resp. $xa_\nu + yb_\nu = 0$). Therefore, at a general point of B_q one has $\pi^*_{l(q)} w_q = x^\nu w'$ (resp. $\pi^*_{l(q)} w_q = x^{\nu+1} w'$), where the local equation x of the exceptional divisor B_q does not divide w'.

Now, looking at the strict transform $\tilde{F}$ as a composite of successive transforms by blowing-ups, and taking into account the definitions of E^*_q

and of $E_{\tilde{F}}$, we get, locally at p, an equality of $\mathcal{O}_{\tilde{S}}$-submodules of $\Omega^1_{\tilde{S}}$ as follows

$$\pi^* w\, \mathcal{O}_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(-E_{\tilde{F}})\, \tilde{F}.$$

Thus, by definition of $D_{\tilde{F}}$, we conclude $D_{\tilde{F}} = \pi^* D_F - E_{\tilde{F}}$ as required.

Corollary 1 . *With the above assumptions one has*

$$D'_{\tilde{F}} = \pi^* D'_F - E'_{\tilde{F}}$$

where $E'_{\tilde{F}} = E_{\tilde{F}} - K_{\tilde{S}/S} = \sum_{q\in \mathcal{C}} (\nu_q(F) + \epsilon_q(F) - 1) E^*_q$.

Proof: It follows from the equalities $D'_{\tilde{F}} = D_{\tilde{F}} + K_{\tilde{S}}$, $D'_F = D_F + K_S$, $K_{\tilde{S}} = \pi^* K_S + K_{\tilde{S}/S}$ and 2 in proposition 1.

Now, we will derive the intersection equalities or inequalities for any constellation $\mathcal{C}$ of points infinitely near p and a germ of a foliation F at p. First assume that $q \in \mathcal{C}$ is nondicritical for F. By applying formula (3) to the invariant curve E_q of $\tilde{F}$ one gets

$$J(E_q, \tilde{F}) = (D_{\tilde{F}} - E_q) \cdot E_q = (-E_{\tilde{F}} - E_q) \cdot E_q.$$

Moreover, if $\mathcal{N} \subset \mathcal{C}$ is the subset of nondicritical points and $E_{\mathcal{N}} = \sum_{r\in\mathcal{N}} E_r$, then for any $r \in \mathcal{N}$ such that $E_q \cap E_r \neq \emptyset$ the j-index of $\tilde{F}$ with respect to E_q at the intersection point $E_q \cap E_r$ is at least 1 and therefore we have $J(E_q, \tilde{F}) \geq (E_{\mathcal{N}} - E_q) \cdot E_q$. Thus, we get the inequality

$$(-\bar{E}_{\tilde{F}}) \cdot E_q \geq 0 \tag{4}$$

for any $q \in \mathcal{N}$, where $\bar{E}_{\tilde{F}} = E_{\tilde{F}} + E_{\mathcal{N}}$.

If q is dicritical, equality (2) for E_q and $\tilde{F}$ gives us

$$I(E_q, \tilde{F}) = (D'_F + E_q) \cdot E_q = D_{\tilde{F}} \cdot E_q - 2$$

since the curve E_q has genus 0. By proposition 2, we obtain the equality

$$(-E_{\tilde{F}}) \cdot E_q = 2 + I(E_q, \tilde{F}), \tag{5}$$

for $q \notin \mathcal{N}$, or equivalently,

$$(-\bar{E}_{\tilde{F}}) \cdot E_q = 2 - E_{\mathcal{N}} \cdot E_q + I(E_q, \tilde{F}).$$

The expressions (4) and (5) will be referred as *intersection equality and inequality for F and C*.

Finally, we will consider the particular case in which the constellation gives rise to a reduction of singularities of the foliation. More precisely, since the characteristic of K is zero, for $p \in S$ and F as above, there always exists a constellation $\mathcal{C}$ of points infinitely near p with the following properties (see [12] and [4] for details):

(a) Any point of $\pi^{-1}(p)$ is regular or a singular simple point for $\tilde{F}$. For a regular point we mean a point of order 0, and for a singular simple one, a point of order 1 such that, if the foliation is given around it by the germ of vector field v then, the linear endomorphism induced by v on the Zariski tangent space has the eigenvalues λ, μ with $\lambda \neq \mu \neq 0$ and $\frac{\lambda}{\mu} \notin \mathbf{Q}_+ = \{x \in \mathbf{Q} | x > 0\}$.

(b) If $q \in \mathcal{C}, q \notin \mathcal{N}$, then all the points of E_q are regular and $\tilde{F}$ is transversal to E_q at them.

(c) If q and r are both dicritical, then $E_q \cap E_r = \emptyset$.

Thus, the simple points lie on the components E_q with $q \in \mathcal{N}$ and, among them, we have those of type $E_r \cap E_q$ with $r \in \mathcal{N}$ and $E_r \cap E_q \neq \emptyset$, these last ones being called simple corners. It is well known that, for a simple point, there exist two and only two invariant formal germs of curve which are both smooth and meet transversely. Thus, for a simple corner, the germs of E_q and E_r are the invariant ones, and for a simple not corner point on E_q, the germ of E_q is invariant and there exists another one (may be formal) which is smooth and transversal to E_q. From the above observations and from the properties (a), (b) and (c), it follows that for any reduced germ C at p invariant by F, the constellation $\mathcal{C}$ also gives the embedded resolution of singularities of C. A constellation $\mathcal{C}$ with the above properties (a), (b) and (c) will be called a *reduction* (of singularities) *constellation for F at p*. Next theorem is the consequence of the results in this section.

Theorem 1 . *If $\mathcal{C}$ is a reduction constellation for a germ of foliation F at the point p of the smooth surface S, then, with the notations as above, the following equalities and inequalities hold:*

$$(-\tilde{E}_{\tilde{F}}) \cdot E_q \geq 0, \textit{for } q \in \mathcal{N}. \tag{6}$$

$$(-E_{\tilde{F}}) \cdot E_q = 2, \textit{for } q \notin \mathcal{N}. \tag{7}$$

Proof: (6) is just (4) for this particular case and (7) follows from (5) and property (b), since this property implies that $I(E_q, \tilde{F}) = 0$.

Remark 2. Intersection inequalities (6) and equalities (7) can be shown to be equivalent to the proximity inequalities and equalities introduced in [2]. In fact, if $q \notin \mathcal{N}$, then the intersection $(-E_{\tilde{F}}) \cdot E_q$ can be computed by proposition 1 giving us, by substitution in (7), the equivalent equality

$$\nu_q(F) - 1 = \sum_{r \to q} (\nu_r(F) + \epsilon_r(F)). \tag{8}$$

In the same way, if $q \in \mathcal{N}$, writing (4) in the form

$$(-E_{\tilde{F}} - E_q) \cdot E_q \geq t_q = \#\{r \in \mathcal{N} | E_q \cap E_r \neq \emptyset \ and \ r \neq q\} \ . .$$

and since

$$E_{\tilde{F}} + E_q = (\nu_q(F)+1)E_q^* + \sum_{r \to q} (\nu_r(F)+\epsilon_r(F)-1)E_r^* + \sum_{r \in \mathcal{C}_q} (\nu_r(F)+\epsilon_r(F)-1)E_r^*$$

where $\mathcal{C}_q = \{r \in \mathcal{C} | r \neq q \ and \ r \not\to q\}$, and $E^* \cdot E_q = 0$ for $r \in \mathcal{C}_q$, we get

$$\nu_q(F) + 1 \geq \sum_{r \to q} (\nu_r(F) + \epsilon_r(F) - 1) + t_q.$$

Now, there exists a bijection between the points in $\mathcal{C} \cap B_q$ and the points $r \in \mathcal{C}$ such that r is infinitely near q and $E_r \cap E_q \neq \emptyset$ and $r \neq q$. Moreover, if we blow-up those r satisfying the above condition and which are dicritical and we add to $\mathcal{C}$ the points proximate to q in the exceptional divisors, then we get a new reduction constellation $\mathcal{C}'$ for which every r infinitely near q with $E_r \cap E_q \neq \emptyset$ is nondicritical. Thus, for $\mathcal{C}'$ we have $t_q \geq \#(\mathcal{C}' \cap B_q)$ and therefore, the above inequality can be written as follows

$$\nu_q(F)+1 \geq \sum_{r \to q, l(r)=l(q)+1} (\nu_r(F)+\epsilon_r(F)) + \sum_{r \to q, l(r) \geq l(q)+2} (\nu_r(F)+\epsilon_r(F)-1). \tag{9}$$

To compare (8) and (9) with the proximity equalities in [2], note that these last ones involve summations extended to all the points infinitely near q (i.e., divisorial primes whose valuations are centered in q) together with the points in the constellation. Now, if $\mathcal{C}'$ is replaced by another constellation $\mathcal{C}$", obtained by adding to $\mathcal{C}'$ all the simple points in B_q which are not in $\mathcal{C}'$, then we obtain that (9) for the reduction constellation $\mathcal{C}$" includes all the possible nonzero sums in its right hand member. Thus, (9) for $\mathcal{C}$" is the proximity inequality for q and, in the same way, (8) for $\mathcal{C}$ is the proximity equality for q. Therefore, intersection equalities and inequalities for reduction constellation imply the proximity ones. The converse is also true, since for $\mathcal{C}$" as above we have $t_q = \#(B_q \cap \mathcal{C}")$.

4 Behavior of invariant curves. Globalization

Assume that S, p and F are as in section 2. Take a reduction constellation $\mathcal{C}$ for F at p, $\pi : \tilde{S} \to S$ being the associated morphism. Consider a reduced curve $C \subset S$ with $p \in C$, such that all its components are invariant by F and set $\pi^*C = E_C + \tilde{C}$, where $\tilde{C}$ is the strict transform of C in $\tilde{S}$ and $E_C = \sum_{q \in \mathcal{C}} e_q(C) E_q^*$, $e_q(C)$ being the multiplicity of the strict transform of C at q.

Note that, if F_f is the foliation defined locally at p by the form df, where $f = 0$ is a local equation of C, then $E_C = \bar{E}_{\tilde{F}_f}$. Indeed, if s_q denotes the number of points $r \in \mathcal{C}$ such that $q \to r$ then, by computing $\bar{E}_{\tilde{F}_f}$ one gets

$$\bar{E}_{\tilde{F}_f} = \sum_{q \in \mathcal{C}} (e_q(C) - 1 + s_q) E_q^* + \sum_{q \in \mathcal{C}} E_q = \sum_{q \in \mathcal{C}} e_q(C) E_q^* = E_C$$

since $\nu_q(F_f) = e_q(C) - 1 + s_q$.

The morphism π provides an embedded resolution for C at p, so $\tilde{C}$ can be viewed, at a neighborhood of p, as the normalization of C. Thus, one can introduce the total j-index at p by

$$j_p(C, F) = \sum_{q \in \pi^{-1}(p) \cap \tilde{C}} j_q(C, F).$$

Note that the components of C are invariant by F_f as above and that the total j-index $j_p(C, F_f)$ is the degree of the adjoinction divisor Δ at p.

Proposition 3 : *With notations and assumptions as above one has*

$$j_p(C, F) \geq \bar{E}'_{\tilde{F}} \cdot \tilde{C} = -\bar{E}'_{\tilde{F}} \cdot E_C$$

where $\bar{E}'_{\tilde{F}} = \bar{E}_{\tilde{F}} - K_{\tilde{S}/S}$ *as in corollary 1. The equality holds if and only if* $j_q(\tilde{C}, \tilde{F}) = 1$ *for every* $q \in \pi^{-1}(p)$ *at an invariant component of the exceptional divisor.*

Proof: If v is a vector field defining F at p, then, for each $q \in \pi^{-1}(p)$, v can be lifted, locally at q and outside $\pi^{-1}(p)$, to a vector field $h\tilde{v}$, where h is a local equation for $E'_{\tilde{F}}$ and $\tilde{v}$ defines $\tilde{F}$ at q. Thus, if m_q denotes the multiplicity intersection, the definition of j-index gives us

$$j_q(C, F) = m_q(\tilde{C}, E'_{\tilde{F}}) + j_q(\tilde{C}, \tilde{F}).$$

Now, if E_r is the component of the exceptional divisor of π containing q, then $j_q(\tilde{C}, \tilde{F}) = 0$ if $r \notin \mathcal{N}$, and $j_q(\tilde{C}, \tilde{F}) \geq 1$ if $r \in \mathcal{N}$, since q is a regular point in the first case and a singular simple one in the second case. Summing up on q, one gets

$$j_q(C, F) \geq (E'_F + E_{\mathcal{N}}) \cdot \tilde{C} = \bar{E}'_F \cdot \tilde{C},$$

and the equality holds if and only if $j_q(\tilde{C}, \tilde{F}) = 1$ for every q on the divisors $E_r, r \in \mathcal{N}$.

Remark 3. In particular, for the foliation F_f as above, we get the well-known formula

$$deg_p \Delta = -(E_C - K_{\tilde{S}/S}) \cdot E_C = \sum_{q \in C} (e_q(C) - 1) e_q(C).$$

Now, we will globalize preserving the notations of the local case. Assume that the curve C is complete and denote by Γ the set of singular points of F which are on C. For each $p \in \Gamma$ take a reduction constellation $\mathcal{C}_p$ for F, denote by $\mathcal{C}$ the union of the constellations $\mathcal{C}_p$ and by $\pi : \tilde{S} \to S$ the morphism obtained by blowing-up, successively by levels, the points of $\mathcal{C}$. As above, keep the notation $\tilde{C}$ for the strict transform of C in $\tilde{S}$, and $E_C, E_{\tilde{F}}, \bar{E}_{\tilde{F}}, \ldots$ for the sum over Γ of the corresponding divisors $E_{C,p}, E_{\tilde{F},p}, \bar{E}_{\tilde{F},p}, \ldots$ introduced before for each constellation $\mathcal{C}_p$.

Theorem 2 : *With the above assumptions, set $R = E_C - \bar{E}_{\tilde{F}}$. Let H be a divisor on S such that $\pi^* H \geq R$. Then*

$$(D_F + H - C) \cdot C_1 \geq 0.$$

where C_1 is the reduced curve consisting of the component of C not contained in $Supp(H)$.

Proof: Applying formula (3) to F and C, one gets

$$J(C, F) - deg\Delta = (D_F - C) \cdot C.$$

On the other hand, by proposition 3 and remark 3, one has

$$J(C, F) - deg\Delta \geq -(\bar{E}_{\tilde{F}} - E_C) \cdot E_C = R \cdot E_C = -R \cdot \tilde{C}.$$

Thus, formula (3) leads to $-R \cdot \tilde{C} \leq (D_F - C) \cdot C$.

Now, set $\pi^*H = R + H'$ where H' is an effective divisor. First assume that H contains no component of C or, equivalently, $\tilde{C}$ is not the support of H'. Then, by the projection formula one has

$$-R \cdot \tilde{C} = -\pi^*H \cdot \tilde{C} + H' \cdot \tilde{C} \geq -H \cdot C,$$

and hence, by substituting in the above inequality, one finds

$$(D_F + H - C) \cdot C \geq 0.$$

If H and C have some common components, then by subtracting those common components, we can write $H - C = H_1 - C_1$, where H_1 and C_1 have no common components. Then $\pi^*H_1 \geq R_1$ where $R_1 = E_{C_1} - \bar{E}_{\tilde{F}}$, so the conclusion is as in the first case.

Remark 4. An alternative proof of theorem 2 can be given by applying formula (3) to the foliation $\tilde{F}$ on $\tilde{S}$ and to the curve $\tilde{C} + E_{\mathcal{N}}$. Indeed, from the computation of $D_{\tilde{F}}$ in proposition 2, (3) can be written as follows

$$J(\tilde{C} + E_{\mathcal{N}}, \tilde{F}) - deg\Delta(\tilde{C} + E_{\mathcal{N}}) = (\pi^*D_F - \bar{E}_{\tilde{F}} - \pi^*C + E_C) \cdot (\tilde{C} + E_{\mathcal{N}}) =$$

$$= (D_F - C) \cdot C + R \cdot \tilde{C} + R \cdot E_{\mathcal{N}}$$

Now, note that the left hand side member of the above equality is greater or equal than the sum of the j-indices over the simple points of $\tilde{F}$ in the support of $E_{\mathcal{N}}$ which are neither simple corners nor points on $\tilde{C}$. On the other hand, by the equality preceding (4), this summation is equal to the intersection number $R \cdot E_{\mathcal{N}}$. Thus, we conclude

$$0 \leq (D_F - C) \cdot C + R \cdot \tilde{C} \leq (D_F - C + H) \cdot C.$$

if $Supp(H)$ contains no component of C, and the end of the proof is as above.

Remark 5. Looking at the expression of $\bar{E}_{\tilde{F}}$ in the basis $\{E_q^*\}$, we get

$$\bar{E}_{\tilde{F}} = \sum_{q \in C} (\nu_q(F) + 1 - s_q(F)) E_q^*$$

where $s_q(F)$ is the number of points $r \in \mathcal{N}$ such that $q \rightarrow r$. Thus, the divisor R is given by $R = \sum_{q \in C} l_q E_q^*$ where $l_q = e_q(C) - \nu_q(F) - 1 + s_q(F)$. Condition $\pi^*H \geq R$ means that H passes through the constellations in C with virtual multiplicities $\{l_q\}$ (see [5], [2] for instance). Thus theorem 2 is an intersection theoretically reformulation of theorem 1 in [2], which holds for arbitrary smooth surfaces.

Also note that, for those points q for which the embedded resolution of C is achieved, we have $l_q \leq 0$, and that the condition $\pi^*H \geq R$ implies that H is effective. Therefore, to find H such that $\pi^*H \geq R$, it is only necessary to impose conditions of passage virtual through the points needed to desingularize C as embedded curve. For instance, if C has only ordinary singularities, then H should be an effective divisor going through the singularities p of C with multiplicities al least $e_p(C) - \nu_p(F) - 1$. If C has only nodes as singularities, then it is clear that H can be taking to be 0 and, therefore, theorem 2 gives us the inequality

$$(D_F - C) \cdot C \geq 0.$$

Writing $D_F = D'_F - K_S$, the above inequality is just the one given in [8] also for arbitrary surfaces (this result being first established for $S = \mathbf{P}^2$ by Cerveau and Lins in [6]). Next theorem says that above inequality also holds if F has only nondicritical singularities.

Theorem 3 : *Let F be a foliation on the smooth surface S and $C \subset S$ a complete reduced curve whose components are invariant by F. Assume that all the points on the constellation giving rise to an embedded resolution of C are nondicritical for F. Then*

$$(D_F - C) \cdot C \geq 0.$$

Proof: With notations as in theorem 2 and remark 4, we only need to see that $l_q \leq 0$ for every point q necessary in the embedded resolution of C, because this implies that H can be taken to be 0. Now, since such a point q is nondicritical, from the computation preceding formula (4) and the last observation, one shows that

$$J(E_q, \tilde{F}) \geq (E_{\mathcal{N}} - E_q) \cdot E_q + (\tilde{C} \cdot E_q).$$

Therefore, instead of (4) one obtains $(-\bar{E}_{\tilde{F}}) \cdot E_q \geq \tilde{C} \cdot E_q$, and hence,

$$(-R) \cdot E_q = (-E_C) \cdot E_q - (-\bar{E}_{\tilde{F}}) \cdot E_q \leq (-E_C) \cdot E_q - (\tilde{C}) \cdot E_q =$$

$$= (-E_C - \tilde{C}) \cdot E_q - \pi^*C \cdot E_q = 0.$$

Thus, by inverse induction on the level of q, it follows that $l_q \leq 0$ using the recurrence equality

$$l_q = (-R) \cdot E_q^* = (-R) \cdot E_q + \sum_{r \to q} (-R) \cdot E_r^*.$$

Remark 6. Theorem 3 is a intersection theoretical reformulation of theorem in [4] which holds for arbitrary smooth surfaces. In fact, if S is the projective plane $\mathbf{P}^2$, one has $D_F = (degF + 2)L$ and theorem 3 gives us the inequality,

$$(degF + 2 - degC) degC \geq 0$$

whence $degC \leq degF + 2$. In the same way if $a = degH$, theorem 2 gives the following bound (see [2, theorem 1])

$$degC \leq degF + 2 + a.$$

Finally, note that theorem 2 particularized for the case in which $H = 0$ is a little bit weaker than the argument leading to prove theorem 3 (i.e., $l_q \leq 0$ for every q). Indeed, if $R = \sum_{q \in C} k_q E_q = \sum_{q \in C} l_q E_q^*$, then $l_q \leq 0$ for all q implies $k_q \leq 0$ for all q, but not conversely. This follows from the base change expressions $k_q = l_q + \sum_{q \to r} k_r$.

Remark 7. Assume that, for each dicritical point q, we fix a nonnegative integer n_q and we consider reduced complete invariant curves with $\tilde{C} \cdot E_q = n_q$ for each $q \notin \mathcal{N}$. If $h_q = R \cdot E_q$, then, from the intersection inequalities and equalities (6) and (7), we get $h_q \geq 0$ for $q \in \mathcal{N}$ and $h_q = 2 - t_q - n_q$ for $q \notin \mathcal{N}$. Thus, $l_q = (-R) \cdot E_q^* = -h_q + \sum_{r \to q} l_r$ and the l's and k's can be computed from the h's, i.e. from invariants of F and the n's. A direct computation of the k's in terms of the h's can be done directly in terms of the dual graph of π, since, obviously,

$$h_q = \sum_{r \in C} k_r (E_r \cdot E_q)$$

for every q.

Remark 8. With notations as above, assume that C' is a set of constellations at the points of S which contain all the dicritical points for F along C (the constellations in C' need not be the reduction ones). Denote by $\pi' : S' \rightarrow S$ the morphism obtained by composition of the successive blowing-ups of the points in $\mathcal{C}'$ and by F', $E_{F'}$, C', and $E_{C'}$ the corresponding $\tilde{F}$, $E_{\tilde{F}}$, $\tilde{C}$, and E_C for this particular constellation $\mathcal{C}'$. Also denote by $\mathcal{N}'$ the set of nondicritical points in $\mathcal{C}'$. Since F' on S' has no dicritical divisor, theorem 3 can be applied for each curve $C' + E_{\mathcal{N}''}$ where $\mathcal{N}''$ is a subset of $\mathcal{N}'$ and $E_{\mathcal{N}''} = \sum_{q \in \mathcal{N}'} E_q$. Thus, one has

$$0 \leq (\pi'^* D_F - E_{F'} - C' - E_{\mathcal{N}''}) \cdot (C' + E_{\mathcal{N}''}) =$$

$$= (D_F - C) \cdot C + (E_{C'} - E_{F'} - E_{\mathcal{N}''}) \cdot (C' + E_{\mathcal{N}''}).$$

This inequality can be viewed as a formula giving us the limitations of the divisor class C in terms of the foliation F and the singularities of C. In fact, the second sum in the right hand side member only depends on the combinatorics and the multiplicities of C and F at the points of $\mathcal{C}'$, whereas the first sum depends on the class C on S to be controlled. For instance, if $S = \mathbf{P}^2$, then the above inequality can be written as follows

$$(degC)^2 - (degF + 2)degC - (E_{C'} - E_{F'} - E_{\mathcal{N}''}) \cdot (C' + E_{\mathcal{N}''}) \leq 0$$

which gives us the inequality
$degC \leq$

$$\leq \frac{1}{2}[(degF + 2) + \sqrt{(degF + 2)^2 + 4(E_{C'} - E_{F'} - E_{\mathcal{N}''}) \cdot (C' + E_{\mathcal{N}''})}].$$

Finally, note that the intersection product $(E_{C'} - E_{F'} - E_{\mathcal{N}''}) \cdot (C' + E_{\mathcal{N}''})$ can be computed explicitly from proposition 1 giving an expression in terms of the combinatorics of $\mathcal{C}'$ (i.e. proximity relationship) and the local orders at the infinitely near points of $\mathcal{C}'$ for F and C. Thus, for instance, if $\mathcal{N}''$ is empty, then

$$(E_C - E_{F'}) \cdot C' = (E_C - E_{F'}) \cdot (-E_C) = \sum_{q \in \mathcal{C}'} (e_q(C) - \nu_q(F) + \epsilon_q(F)) e_q(C).$$

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