CHRIS PETERS

Chapter 0. Introduction

Cours de l'institut Fourier, tome 23 (1995), p. 1-2
<http://www.numdam.org/item?id=CIF_1995__23__A2_0>

Chapter 0. Introduction

This course will mainly be an introduction into the techniques of complex algebraic geometry with a focus on surfaces. Some familiarity with curves is assumed (e.g. the material presented in [G]).

In this course a *surface* will be a *connected but not necessarily compact complex manifold of dimension* 2 and an *algebraic surface* will be a *submanifold of projective space of dimension* 2 *which is at the same time a projective variety.*

I will explain these concepts fully in section 2. For the moment let me just remark that by definition a surface is covered by open sets each of which is homeomorphic to an open set in $\mathbb{C}^2$ and that the transition functions are holomorphic maps from the open set in $\mathbb{C}^2$ where they are defined to $\mathbb{C}^2$ (for now a holomorphic map will be any C^∞-map whose coordinate functions are analytic in each variable separately). An algebraic surface in addition is a submanifold of complex projective space given as the zero locus of some polynomials.

Examples

1. Any connected open set in $\mathbb{C}^2$ is a surface.

2. If C and D are Riemann surfaces (or algebraic curves) their product $C \times D$ is a surface.

3. If $\gamma_1, \ldots, \gamma_k \in \mathbb{C}^2$ are k independent vectors (over the reals, so $k \leq 4$) the group $\Gamma = \mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_k$ acts on $\mathbb{C}^2$ and the quotient $\mathbb{C}^2/\Gamma$ is a complex manifold which is compact precisely when $k = 4$. In this case $\mathbb{C}^2/\Gamma$ is homeomorphic to the product of four circles or two real tori and is called a complex 2-torus.

These notes will be aiming at the so-called Enriques-Kodaira classification of surfaces which is the analogue in two dimensions of the (coarse) classification of Riemann surfaces by means of their genus. At this point it is not possible to formulate the main classification theorem. Several concepts and examples are needed which are gradually introduced. These concepts and examples in themselves are interesting and important, so stay with us!

For some of the technical details I refer to the litterature at the end the notes. Some brief comments will be given here. The reference [Beau] will be an important guide-line, which means that I mostly treat algebraic surfaces. I use [Beau] rather than [B-P-V] because results are often easier to prove in the algebraic setting. However the treatment of the classification will be based upon more modern ideas explained in [P].

Considering background the following remarks. A very general and useful book on complex algebraic geometry from the analytic point of view is [G-H] which will be used occasionally for some foundational material. For a more algebraic point of view I mention the books [Reid] (elementary, fun to read) and [Mu] (much less elementary, assumes a lot of algebra, but a very nice introduction indeed). Some background on commutative algebra is collected in Appendix A1 with [Reid] as a reference for the more elementary facts and [Ii] and [Ma] for the more advanced facts which are needed later in the course.

Sheaf theory, cohomology theories and Hodge theory will be mainly done from [Wa], a unique reference in that it collects all you ever want to know (and much more) about differentiable varieties and their cohomology theory. I will certainly not treat all proofs but formulate what is needed. Another useful reference is [Go] to which I occasionally refer.

Some background in algebraic topology is assumed such as singular (co)-homology, cap and cup products and Poincaré duality. I have given an overview of the results needed from algebraic topology in Appendix A2. Full details then can be gathered from [Gr] and [Sp]. More advanced algebraic topology will be taken from [Mu] and [Mi].

Finally, background details from complex analysis can be found in [Gu-Ro], a real classic on this subject. For another more modern treatment see [Gr-Re].

About the history of surface theory

Around 1850 an extensive study had been carried out of low degree surfaces in three dimensional projective space. It was shown that on a smooth cubic there were 27 lines. Names such as the Cayley cubic, the Kummer surface and the Steiner quartic are reminders of that period. The first generation of Italian geometers (Bertini, C. Segre, Veronese) started to look at surfaces embedded in higher dimensional projective spaces and their projections. The Veronese surface and the Del Pezzo surfaces originate from that period (1880-1890). Max Noether in Germany, using projections, established (1870-'75) an important formula for surfaces, nowadays called "Noether's Formula". The proof was not complete. Enriques, using a result of Castelnuovo, gave a correct proof in 1896. Castelnuovo and Enriques belong to the second generation of Italian geometers. From roughly 1890 to 1910 they really developed the theory of algebraic surfaces from a birational point of view, culminating in the Castelnuovo-Enriques surface classification. See the monograph [En].

The foundations of algebraic geometry were lacking in that period, many results were not clearly formulated and proofs were not always complete. These foundations were laid in the thirties and fourties by van der Waerden, Zariski and Weil. Zariski wrote a monograph [Za] about surfaces incorporating these new techniques.

The transcendental tools were developed by de Rham, Hodge and Lefschetz in the fourties and fifties. But decisive progress only came after sheaf theory had been developed and applied to algebraic geometry by Serre, Hirzebruch and Grothendieck (1955-1965). On this base Kodaira did his fundamental work on classification theory, including the non-algebraic surfaces (1960-1970). He completed the "Kodaira-Enriques classification" of surfaces. In the sixties in Moscow the Russian school of algebraic geometers (a.o. Manin, Shafarevich, Tjurin, Tjurina) did important work on the classification, see the monograph [Sh].

The Castelnuovo-Enriques classification relied on existing detailed knowledge of some classes of surfaces (rational and ruled surfaces, bi-elliptic surfaces, Enriques surfaces), but other classes were extensively studied for the purpose of this classification (K3-surfaces and elliptic surfaces) thereby gaining more detailed insight in these special classes.

Finer classification of surfaces went on in the seventies and eighties, but also some important new techniques and viewpoints from higher dimensional classification theory began to permeate surface theory. See [P] for recent developments. These new insights are incorporated in the presentation of the classification I give here.