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# Chapter 4. More Advanced Tools from Algebraic Geometry

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# Chapter 4. More advanced tools from algebraic geometry

## 8. Normalisation and Stein factorisation

Normalisation is a cruder process than desingularisation and is easy to describe algebraically. Zariski's main theorem is proved and, finally, the Stein factorisation theorem for projective morphisms. If the target space is normal much more can be said and this will be used later on.

In Appendix A1.3 I have gathered some fundamental properties of normal rings which are used freely in what follows.

A variety $X$ is *normal* at $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is normal. $X$ is called normal if it is normal at every point.

**Example 1.**

1. Any smooth point is normal. If one uses the fact that the local ring is a unique factorisation domain (in the analytic case, see [Gr-Re,Chapter 2§2], in the algebraic case see [Mu, §1B. ]) this is easy. Suppose that a rational function $P/Q$ satisfies an equation
$$(P/Q)^n + A_{n-1}(P/Q)^{n-1} + \ldots + A_0 = 0, \quad A_i \in \mathcal{O}_{X,x}$$
you multiply with $P^n$ to see that $Q$ divides $P^n$. Since you may assume that $P$ and $Q$ have no common factor in $\mathcal{O}_{X,x}$ it follows that $Q$ in fact must be a unit in $\mathcal{O}_{X,x}$ and so $P/Q \in \mathcal{O}_{X,x}$.

2. If $X$ is a reducible hypersurface at $x$, the point $x$ is not normal. Indeed, if $f$ and $g$ are local equations of two distinct hypersurfaces at $x$ making up $X$, one can introduce the function $h := f/(f+g)$. It cannot be holomorphic along $X$ since it would be identically zero along one component and identically 1 along the other component. By assumption $fg = 0$ along $X$ and so
$$h^2 - h = \frac{f^2 - f(f+g)}{(f+g)^2} = 0.$$
This shows that the meromorphic function $h$ is integral over $\mathcal{O}_{X,x}$.

3. A curve is normal if and only if it is smooth. In the algebraic setting a short proof can be found in [Ii, §2.7]. In the complex-analytic setting this follows from a more general fact, namely that the set of normal points form a subvariety of codimension two or more [Gr-Re, Chapter 6 §5].

**Definition 2.** *Let $X$ be a variety. A pair $(X', f)$ consisting of a normal variety $X'$ and a morphism $f : X' \to X$ is called a normalisation of $X$ if $f$ is finite and birational.*

**Theorem 3.** *For any affine resp. projective algebraic variety $X$ the normalisation exists as an affine resp. projective algebraic variety. It is unique in the following sence. If $f'' : X'' \to X$ is another normalisation, there exists an isomorphism $\iota : X' \to X''$ with $f'' \circ \iota = f'$.*

**Proof:**
Step 1. The affine case.

Since the coordinate ring $R(X)$ of $X$ is Noetherian its integral closure $R'$ in the field of rational functions on $X$ is a finitely generated $R(X)$-module, and so $R' = \mathbb{C}[X_1, \ldots, X_n]/I'$. The ideal $I'$ defines a variety $X'$ and the embedding $R[X] \hookrightarrow R'$ defines a morphism $X' \to X$. The map is finite by definition and clearly birational since $R[X]$ and $R'$ have the same quotient field.

The uniqueness is obvious: both $X'$ and $X''$ correspond to the integral closure of $R[X]$ in the function field $\mathbb{C}[X]$. Here you use that $X''$ is normal if and only if its coordinate ring is normal. This folows from the fact that an integral domain is normal if and only if the localisations in all maximal ideals are normal.

Step 2. $X$ is a projective subvariety of $\mathbb{P}^n$.

Now one uses the homogeneous coordinate ring $R[X]$ of $X$ and forms its integral closure $R'$ in $\mathbb{C}(X)$ which is of the form $\mathbb{C}[X_0, \ldots, X_n]/I'$ where $I'$ is a homogeneous ideal which defines a normal projective variety $X'$. Again, the inclusion $R[X] \hookrightarrow R'$ defines a morphism $X' \to X$. I claim that in the standard affine pieces $U_i = \{ (X_0, \ldots, X_n) \mid X_i \neq 0 \}$ this variety is just the normalisation constructed in Step 1. To see this, let $X_i := X \cap U_i$ with coordinate ring $R_i$ and similarly we define $X_i' := X' \cap U_i$ with ring $R_i'$. The localisation of $R[X]$ in $X_i$ is isomorphic to $R_i$ ('making inhomogeneous'). Since localisation and normalisation commute, the ring $R_i'$ is just the normalisation of the ring $R_i$. By uniqueness of the normalisation in the affine case it then follows that $X_i'$ is the normalisation of $X_i$. In particular, the map $X' \to X$ is finite as in the affine case. The uniqueness also follows from the fact that $X'$ is a union of affine normalisations for which one has uniqueness. ∎

Next, I want to give a simple proof of Zariski's Main Theorem, using however the complex topology. First I need

**Proposition 4.** *Let $f : X \to Y$ be a morphism between projective varieties such that the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Then the fibres of $f$ are connected and non-empty. Conversely, if $f$ is surjective, $Y$ is normal and the fibres are connected, one has an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} f_*\mathcal{O}_X$.*

**Proof:** Assume that the fibre of $f$ above $y \in Y$ is not connected. Since $f$ is proper you can find a neighbourhood $V$ of $y$ (in the complex topology) such that $f^{-1}V$ is not connected. But then the canonical map

$$\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$$

cannot be surjective. If the fibre at $y$ would be empty, this map would not be injective.

For the converse, let $V \ni y$ be open and connected in the complex topology and let $g \in \mathcal{O}(f^{-1}V)$ be bounded. The function $g$ has the same value on each smooth fibre $f^{-1}y$, since such a fibre is connected. So there is a bounded continuous function $h'$ on an open dense subset $V'$ of $V$ such that $g = h' \circ f$ on $f^{-1}V'$. You can take $V'$ to be the set of the smooth points of $V$ over which $f$ has maximal rank. This means that every point in $f^{-1}V'$

has a neighbourhood of the form $U \times V''$ with $f$ the projection onto the second factor. But then, in $V''$ the function $h'$ is holomorphic and so $h$ is holomorphic in $V'$ entirely. In view of the normality of $V$, one can extend the bounded function $h'$ to a holomorphic function $h$ on $V$ (see [Gr-Re, Chapter 7 §4.2]) and so the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism at $y \in Y$. ■

**Corollary 5.** *Suppose that $f : X \to Y$ is a surjective morphism between projective varieties, that $Y$ is normal and that the fibres of $f$ are connected. For any line bundle $\mathcal{L}$ on $Y$ there is a natural isomorphism*

$$f^* : \Gamma(Y, \mathcal{L}) \longrightarrow \Gamma(X, f^*\mathcal{L})$$

*given by $f^*(t) = t \circ f$.*

**Proof:** The map $f^*$ as given above is clearly injective. To see that it is surjective, invoke the following special case of the Projection Formula.

$$f_*f^*\mathcal{L} \cong \mathcal{L} \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X.$$

Since $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, there is a canonical isomorphism

$$f_*f^*\mathcal{L} \cong \mathcal{L}.$$

This holds in particular for the global sections so that

$$\dim \Gamma(X, f^*\mathcal{L}) = \dim \Gamma(Y, f_*f^*\mathcal{L}) = \dim \Gamma(Y, \mathcal{L}).$$

It follows that $f^*$ must be surjective. ■

**Corollary 6.** (Zariski's Main Theorem) *Let $Y$ be a normal projective variety and let $f : X \to Y$ be a birational morphism. Then $f$ has connected fibres.*

**Proof:** By the preceding proposition, one only has to verify that $f_*\mathcal{O}_X = \mathcal{O}_Y$. The question is local and so one may assume that $Y$ is affine with coordinate ring $A := R[Y]$ and $B := \Gamma(f_*\mathcal{O}_X)$ is a finitely generated $A$-module (since $f_*\mathcal{O}_X$ is coherent). Both $A$ and $B$ are integral domains with the same field of fractions $\mathbb{C}(Y)$ and $A$ is integrally closed. So $A = B$ and thus $f_*\mathcal{O}_X = \mathcal{O}_Y$. ■

Let me apply this to the situation of general rational map between projective varieties $f : X \dashrightarrow Y$. Let $\Gamma_f \subset X \times Y$ be the closure of the graph and let $p : \Gamma_f \to X$ be the projection onto the second factor.

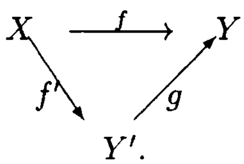
**Corollary 7.** *Let $f : X \dashrightarrow Y$ be a rational map between projective varieties. If $X$ is normal, for every $x \in X$ the set $f(x) := (f \circ p)(p^{-1}x)$ is connected.*

**Proof:** The projection $p$ is a birational morphism to which Zariski's main theorem can be applied. So $p^{-1}(x)$ and hence $f(x)$ is connected. ■

**Corollary 8.** *Let $f : X \to Y$ be a morphism between projective varieties and let $(X', i)$, resp. $(Y', j)$ be the normalisation of $X$, resp. $Y$. There exists a morphism $f' : X' \to Y'$ such that $j \circ f' = f \circ i$.*

**Proof:** The morphism $j$ is birational, let $j^{-1}$ be its inverse and define $f' = j^{-1} \circ f \circ i$. Clearly, $j^{-1}(f \circ i)(x)$ is a finite set of points and so $f'(x) \subset j^{-1}(f \circ i)(x)$ is finite as well. But it is connected by the previous Corollary and so it consists of one point, i.e. $f'$ is a morphism. ■

**Theorem 9.** (Stein factorisation) *Let $f : X \to Y$ be a surjective morphism between projective varieties. There exists a variety $Y'$, a finite surjective morphism $g : Y' \to Y$ and a morphism $f' : X \to Y'$ with connected fibres such that the following diagram is commutative.*




*If $X$ is normal, then so is $Y'$.*

**Proof:**

Let $Y$ be projective and let $L := \mathbb{C}(X)$, a finite extension of $\mathbb{C}(Y)$. As in the proof of the existence of a normalisation, one finds a projective variety $Y'$ with $\mathbb{C}(Y')$ equal to the algebraic closure of $\mathbb{C}(Y)$ in $L$ and a finite morphism $Y' \to Y$. The variety $Y'$ is in fact constructed from open affine pieces $V'$ with coordinate ring equal to the integral closure of $\mathcal{O}(f^{-1}V)$ in $\mathbb{C}(Y)$ where $V$ is an open affine piece of $Y$. In particular, $\mathcal{O}(Y')$ is canonically isomorphic to $f_*\mathcal{O}_X$. This is a coherent sheaf of $\mathcal{O}_Y$-algebras and so the natural morphism $g : Y' \to Y$ is finite. By construction $f = g \circ f'$ and $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$. So the fibres of $g$ are connected by the previous proposition. Finally, if $X$ is normal, then so is $Y'$. ■

**Corollary 10.** *Let $X$ and $Y$ be projective varieties and $f : X \to Y$ a morphism. If $Y$ is normal and the generic fibre of $f$ is connected, then so is every fibre.*

**Proof:** The map $g$ appearing in the Stein normalisation must have degree one in this case and hence must be birational. By Zariski's Main Theorem all fibres must consist of one point and so $g$ is an isomorphism and the map $f$ must have connected fibres. ■

## 9. Kodaira-dimensions

Any coarse classification proceeds according to the Kodaira-dimension. Its characterisation by a certain growth-behaviour is essential as well as the fact that the Kodaira dimension does not change under finite unramified coverings. Full proofs of these facts are given,

In this section $X$ is a normal projective variety and $\mathcal{L} = \mathcal{O}_X(D)$ with $D$ an effective divisor. Associated to $D$ there is the ring

$$R(X,D) := \oplus_{k\geq 0} H^0(X, \mathcal{L}^{\otimes k})$$

and its homogeneous field of fractions

$$Q(X,D) := \{ \frac{s}{t} \mid s,t \in H^0(X,\mathcal{L}^{\otimes k}) \quad k \geq 0 \}.$$

If $D$ is the hyperplane divisor, one has $Q(X,D) = \mathbb{C}(X)$. Its transcendence degree is precisely the dimension of $X$ (see [Reid, §9]). In general one has:

**Proposition 1.** *$Q(X,D)$ is algebraically closed in $\mathbb{C}(X)$. In particular, its transcendence degree is finite and at most equal to* dim $X$.

**Proof:** Assume that $f \in \mathbb{C}(X)$ satisfies an equation

$$f^r + a_1 f^{r-1} + \ldots + a_r = 0, \quad a_i = \frac{s_i}{t}, \quad s_i, t \in H^0(X, \mathcal{L}^{\otimes k}).$$

Then $h := f \cdot t$ is a meromorphic section of $\mathcal{L}^{\otimes k}$ which satisfies

$$h^r + s_1 h^{r-1} + \ldots + s_r t^{r-1} = 0$$

with holomorphic coefficients. Now $X$ being normal implies that locally at every point $x \in X$ the section $h$ is in $\mathcal{O}_{X,x}$, i.e. $h$ is a regular section of $\mathcal{L}^{\otimes k}$ and so $f = \dfrac{h}{t}$ belongs to $Q(X,D)$. ■

**Definition 2.** *The $D$-dimension $\kappa(D)$ is the transcendence degree of the field $Q(X,D)$. In the special case when $D$ is a canonical divisor, $\kappa(D)$ is called the Kodaira dimension of $X$ and denoted $\kappa(X)$.*

**Remark 3.** This definition can be extended to cases where $D$ is not effective. If $H^0(X,\mathcal{L}^{\otimes k}) = 0$ for all $k \geq 0$ you simply set

$$\kappa(X,D) := -\infty.$$

Otherwise you introduce the set $\mathbb{N}(D) \subset \mathbb{N}$ of natural numbers $k$ for which $\mathcal{L}^{\otimes k}$ does have sections and restrict the preceding discussion to the rational maps $\varphi_{\mathcal{L}^{\otimes k}}$ for $k \in \mathbb{N}(D)$. The definition is then easily modified.

Let me now relate the field $Q(X, D)$ to the geometry of the rational maps

$$\varphi_{\mathcal{L}^{\otimes k}} : X \dashrightarrow \mathbb{P}^{N_k}, \quad N_k + 1 = \dim H^0(\mathcal{L}^{\otimes k}).$$

Let $W_k$ be the image of $\varphi_{\mathcal{L}^{\otimes k}}$. Recall that this is the closure in $\mathbb{P}^{N_k}$ of the image of the maximal subset of $X$ on which $\varphi_{\mathcal{L}^{\otimes k}}$ is a morphism. In terms of a basis $\{ s_0, \ldots, s_{N_k} \}$ for the sections of $H^0(X, \mathcal{L}^{\otimes k})$ the function field of $W_k$ is given by

$$\mathbb{C}(W_k) = \mathbb{C}(s_1/s_0, s_2/s_0, \ldots, s_{N_k}/s_0).$$

So the union of these fields is $Q(X, D)$.

**Lemma 4.** *There is a natural number $k_0$ with $\mathbb{C}(W_k) = Q(X, L)$ for all $k \geq k_0$.*

**Proof:** There are natural embeddings $H^0(X, \mathcal{L}^k) \subset H^0(X, \mathcal{L}^{k+1})$ which induce embeddings $\mathbb{C}(W_k) \subset \mathbb{C}(W_{k+1})$. Their union is $Q(X, D)$ as observed before. Now the overfield $\mathbb{C}(X)$ is finitely generated over $\mathbb{C}$ (this follows from the interpretation of $\dim_{\mathbb{C}} X$ as the transcendence degree of the field extension $\mathbb{C}(X)/\mathbb{C}$)) and hence $Q(X, D)$ is finitely generated over $\mathbb{C}$. So from some $k_0$ on the sequence of inclusions $\mathbb{C}(W_k) \subset \mathbb{C}(W_{k+1})$ stabilise and then $\mathbb{C}(W_k) = Q(X, D)$. ■

**Corollary 5.** $\kappa(X, D) = \max \dim W_k$.

**Proof:** $\kappa(X, D)$ is the transcendence degree of the field extension $Q(X, D)/\mathbb{C}$. By the previous Lemma $Q(X, D) = \mathbb{C}(W_k)$ for all $k \geq k_0$ and so $\kappa(X, D) = \dim W_k$ for all $k \geq k_0$. ■

The fact that $Q(X, D)$ is algebraically closed in $\mathbb{C}(X)$ can be translated in terms of the rational maps $\varphi_{\mathcal{L}^{\otimes k}}$ as follows. Let me first assume that the latter map is actually a morphism.

**Proposition 6.** *If for some $k \geq k_0$ the rational map $\varphi_{\mathcal{L}^{\otimes k}}$ is a morphism, its generic fibre is connected.*

**Proof:** Consider the Stein factorisation of $f := \varphi_{\mathcal{L}^{\otimes k}} : X \to W_k$, say $f = g \circ f'$ with $g : Y \to W_k$ finite and $f' : X \to Y$ connected. There is the inclusion of fields

$$\mathbb{C}(X) \supset \mathbb{C}(Y) \supset \mathbb{C}(W_k) = \mathbb{Q}(X, D)$$

and since the last field extension is finite algebraic and $Q(X, D)$ is algebraically closed in $\mathbb{C}(X)$, one must have $\mathbb{C}(Y) = \mathbb{C}(W_k)$ and so $g$ must in fact be birational. So the fibres of $f$ and $f'$ are the same generically and in particular, the generic fibre of $f$ is connected. ■

**Remark 7.** Since $Y$ is normal and since $g$ is also finite, it follows that $g : Y \to W_k$ is the normalisation of $W_k$.

The general case needs a little elaboration on the elimination of points of indeterminacy of rational maps given by linear systems. This is a generalisation of the surface case. I won't give all the details, but refer to [Ha, Example 7.17.3] for them. Briefly, if $s_0, \ldots, s_n$ forms a basis for the sections of a line bundle $\mathcal{M}$ on any projective variety $X$, you consider the subsheaf of $\mathcal{M}$ generated by these. Only at points where all sections vanish you'll end up in the maximal ideal at that point (after choosing a local trivialisation of $\mathcal{M}$ at that point). Now you look at the algebraic set defined by the simultaneous vanishing of all of these sections (more precisely, you look at the 'scheme' defined by them, but you just can think of the equations). This is the base-locus of the linear system defined by $\mathcal{M}$. Now you blow up $X$ in this base locus. As in the surface case one shows that one obtains a projective variety $\tilde{X}$ plus morphisms $\sigma : \tilde{X} \to X$ and $\tilde{f} : \tilde{X} \to Y$ fitting into a commutative diagram

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\sigma}\swarrow & & \searrow^{\tilde{f}} \\ X & \overset{}{\underset{f}{\dashrightarrow}} & Y. \end{array}$$

Next, suppose that $X$ is normal. Then by Theorem 8.6 the fibres of $\sigma$ are all connected and hence, by Corollary 8.5 for any line bundle $\mathcal{M}$ on $X$ one has natural isomorphisms

$$f^* : H^0(X, \mathcal{M}) \xrightarrow{\sim} H^0(\tilde{X}, \sigma^*\mathcal{M}).$$

Unfortunately, it is not automatically true that $\tilde{X}$ is normal again, but you can replace $\tilde{X}$ by its normalisation. Let me assume this and consider now the case at hand with $\mathcal{M} = \mathcal{L}^{\otimes k}$. It follows that I can replace $X$ by another normal variety $X'$ such that $\varphi_{\mathcal{L}^{\otimes k}}$ lifts to a morphism $f'_k$ just by lifting the sections. So the preceding proposition just applies to $X', f'_k$ and so I may assume that the rational map $\varphi_{\mathcal{L}^{\otimes k}}$ simply is a morphism.

I admit the following simple theorem about the existence of the Hilbert polynomial, see [Ha, Proposition 7.5].

**Theorem** *Let $X$ be a projective variety of dimension $n$. There exists a polynomial $P_X(t)$ of degree $n$ such that for all sufficienty large $k$ one has*

$$P_X(k) = \dim H^0(X, \mathcal{O}(k))$$

.

Next I can state and prove the main result of this section.

**Theorem 8.** (Characterisation of the $D$- dimension) *Let $X$ be a normal projective variety and let $D$ be an effective divisor on $X$ with $D$-dimension equal to $\kappa$. There exist positive numbers $\alpha$ and $\beta$ such that for all sufficiently large $k$ one has*

$$\alpha k^\kappa \leq \dim H^0(X, \mathcal{O}_X(kD)) \leq \beta k^\kappa.$$

**Proof:** As explained before, I may, on replacing $D$ by a suitable multiple of $D$, assume that the linear system $|D|$ defines a morphism $f$ of $X$ onto a variety $W$ of dimension $\kappa(D)$. So, if $F$ is the fixed part of this system, one has $D = f^*H + F$ with $H$ a hyperplane section of $W$. Hence

$$\dim H^0(kD) = \dim H^0(f^*(kH)) \geq \dim H^0(W, \mathcal{O}(k)) \geq \alpha m^\kappa,$$

by the result on Hilbert polynomials quoted before.

Now I need to prove the other inequality. This is a bit more subtle. The subtlety lies in the fact that the fixed part $F$ of $|D|$ might contain contain components mapping surjectively onto $W$ by $f$. Let me first assume that tere are no such components. then $F$ is entirely contained in the pull back of some divisor $G$ on $W$ and so

$$\dim H^0(\mathcal{O}(kD)) \leq \dim H^0(f^*\mathcal{O}(kH + kG).$$

Now one may, if neccessary, add a very ample divisor to $G$ so that $H + G$ is very ample and then the result on Hilbert polynomials gives a bound

$$\dim H^0(f^*\mathcal{O}(kH + kG) \leq \beta k^\kappa.$$

The only case left is the case where $F = F' + F''$ with $F'$ the non-empty maximal divisor in $F$ with $f(F') = W$. Obviously $|D| = |D - F'| + F'$ and I claim that also $|kD| = |kD - kF'| + kF'$. If this can be shown, you simply replace $D$ by $D - F'$ in the preceding argument and you are done.

Now assume that there is some $E \in |kD| \setminus (|kD - kF'| + kF')$. Let $G$ be the maximal divisor such that $kF'$ is not contained in $G$ with the property that $E \in |kD - G| + G$. So $E = E' + G$ and $E'$ passes through some but not all points of $F'$. Now, since $F'$ is mapped onto $W$ by $f$, the fibres of $f$ all meet $F'$. Any point $x$ on $E'$ not on $F'$ such that the fibre of $f$ through $x$ meets $F'$ in $y$ will then have the property that $f_k(x) \neq f_k(y)$. Here $f_k$ is the morphism defined by the system $|kD|$

The generic fibre of $f$ as well as of $f_k$ (which is obviously contained in the latter) is connected. So, since $f_k$ is not constant on the generic fibre of $f_k$, the latter must be a strictly lower dimensional subvariety of the generic fibre of $f$. Now, recall the dimension formula

$$\dim W_k + \dim \text{(generic fibre of } f_k) = \dim X.$$

This formula then implies that $\dim W_k > \dim W$ which is impossible, since $W$ already had maximal dimension $\kappa$. ■

The final result treated in this section will be the behaviour of the Kodaira dimension under finite unramified coverings.

**Proposition 9.** *Let $f : X \to Y$ be an unramified covering between smooth projective varieties. Then $\kappa(X) = \kappa(Y)$.*

**Proof:** Since $f$ is unramified, one has $K_X = f^*K_Y$. The result now follows from a more general result, namely that $\kappa(f^*D) = \kappa(D)$ for any divisor $D$ on $Y$. Since $f$ is surjective, the induced map

$$f^* : H^0(Y, \mathcal{O}_Y(kD)) \to H^0(X, \mathcal{O}_X(f^*(kD)))$$

is injective and one has the inequality

$$\kappa(f^*D) \geq \kappa(D).$$

I need to show the reverse inequality. First I reduce to the case that $f$ is a Galois covering. From the theory of covering spaces one knows that $f_* : \pi_1(Y) \to \pi_1(X)$ is injective and there is a normal subgroup $N$ of $\pi_1(X)$ contained in $f_*(\pi_1(Y))$ such that the quotient $\pi_1(X)/N$ is a finite group occurring as a group of deck transformations of a Galois covering $f' : X' \to Y$ which factors over $f : X \to Y$. By the preceding inequality one easily reduces to the case of a Galois cover $f : X \to Y$, say with group $G$.

To handle this case, you first observe that the case $\kappa(f^*D) = -\infty$ is treated by the already known inequality and so you may assume that this $D$-dimension is nonnegative and you choose a basis $\{ s_0, \ldots, s_n \}$ for $H^0(X, f^*(kD))$ with $k$ large enough so that

$$\kappa(f^*D) = \text{transc. deg } \mathbb{C}(s_1/s_0, \ldots, s_n/s_0).$$

Now you let the group $G$ (of order $m$) act on the latter field $L$ and consider the $G$-invariant subfield $K$. I claim that $K$ can be considered as a subfield of the function field of $W_{km}$. From this the desired inequality easily follows:

$$\kappa(f^*D) = \text{transc. deg } L \leq \text{transc. deg } K \leq \text{transc. deg } \mathbb{C}(W_{km}) = \kappa(D).$$

To prove the claim, consider

$$\prod_{g \in G} (X - g^*(s_i/s_0)) = X^m + a_1(s_i/s_0)X^{m-1} + \ldots + a_m(s_i/s_0), \quad i = 1, \ldots, n.$$

Then $K$ is generated over $\mathbb{C}$ by $a_l(s_i/s_0)$ $l = 1, \ldots, m$, $i = 1, \ldots, n$. Furthermore, $t_0 = \prod_{g \in G} g^*(s_0)$ and $t_0 a_l(s_i/s_0)$ define $G$-invariant holomorphic sections of $\mathcal{O}_X(f^*(kmD))$ and hence define sections of $\mathcal{O}_W(km)$. So $a_l(s_i/s_0)$ is a quotient of two sections of $\mathcal{O}_X(f^*(kmD))$ and hence gives a rational function on $W_{km}$. In this way you get a natural embedding of $K$ into the function field of $W_{km}$ as asserted. ■

## 10. The Albanese torus

The Albanese torus and the Albanese map are universal for maps of a projective manifold to a torus, hence the importance of the Albanese. If the image of the Albanese map is a curve in the Albanese much more can be said and this will be used in the sequel.

Before introducing the Albanese, let me recall briefly a few facts about $g$-tori $T := V/\Gamma$, where $V$ is any $g$-dimensional complex vector space and $\Gamma \subset V$ is a lattice which is of maximal rank (over the reals). The following lemma should be obvious and its proof is left to the reader.

**Lemma 1.** *The homomorphism*

$$t_T : \Gamma \to H_1(T, \mathbb{Z})$$

*defined by assigning to $\gamma \in \Gamma$ the homology class of the 1-cycle on $T$ defined by the straight line segment from 0 to $\gamma$ is an isomorphism. Also the map*

$$\tau_T : V^* \to H^0(T, \Omega_T^1)$$

*which assigns to the functional $f$ on $V$ the one-form $df$ on $T$ is an isomorphism. Moreover*

$$\int_{t_T(\gamma)} \tau_T(f) = f(\gamma). \tag{2}$$

•

Let $X$ be any projective manifold (or, more generally any compact Kähler manifold). The Albanese torus Alb($X$) can be defined very concretely as follows. Let $b = b_1(X)$, the first Betti-number of $X$. The Hodge decomposition for $H^1$ reads as follows: $H^1(X, \mathbb{C}) = H^0(\Omega_X^1) \oplus \overline{H^0(\Omega_X^1)}$, so that $b = 2g$ with $g = \dim H^0(\Omega_X^1)$. Choose a basis for the space of holomorphic 1-forms $\{ \omega_1, \ldots, \omega_g \}$ and choose a basis $\{ \gamma_1, \ldots, \gamma_{2g} \}$ for $H_1(X, \mathbb{Z})$ modulo torsion. The $2g \times g$-matrix

$$\begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}$$

is called the *period matrix* with respect to the one-forms. The $2g$ columns are independent over the reals. Indeed, any linear relation between the columms with real coefficients $a_i$, $i = 1, \ldots, 2g$ implies that $\sum_i a_i \int_{\gamma_i} \omega = 0$ for all $\omega \in H^0(\Omega_X^1)$ and hence, $\sum_i a_i \int_{\gamma_i} \overline{\omega} = 0$ and the Hodge-decomposition then implies that the $\mathbb{C}$-linear functional $\sum_i a_i \int_{\gamma_i}$ is zero on $H^1(X, \mathbb{C})$. The Kronecker pairing (see Appendix A2.3) between $H^1$ and $H_1$ with complex coefficients being perfect, this implies that $\sum_i a_i \gamma_i = 0$ and hence $a_i = 0$ for all $i = 1, \ldots, 2g$.

It follows that the columns of the period matrix span a lattice in $\mathbb{C}^g$ and you can form the quotient $g$-torus, which by definition is the *Albanese torus*. More invariantly

$$\mathrm{Alb}(X) = H^0(\Omega_X^1)^* / \mathrm{im}\ H_1(X, \mathbb{Z}),$$

where $\gamma \in H_1(X, \mathbb{Z})$ is mapped to the functional on $H^0(\Omega_X^1)$ given by integration over $\gamma$. Fixing a point $x_0 \in X$ and choosing any path from $x_0$ to $x$, integration along this path gives a well defined element $\alpha(x) \in \mathrm{Alb}(X)$. This gives then a map, the *Albanese map*

$$\alpha : X \to \mathrm{Alb}(X).$$

This map is holomorphic, as can be seen as follows. Since this is a local matter, one may fix $x \in X$, a path $\gamma$ from $x_0$ to $x$ and compute $\alpha$ in a coordinate ball $U$ about $x$ by integrating over $\gamma$ followed by a straight line segment from $x$ to $y \in U$. This gives a well defined map

$$a : U \to H^0(\Omega_X^1)^*,$$

which clearly is holomorphic and hence $\alpha : U \to \operatorname{Alb} X$ is holomorphic, since $\alpha = q \circ a$, with $q : H^0(\Omega^1_X)^* \to \operatorname{Alb}(X)$ the projection. Note moreover that

$$\tau^{-1}_{\operatorname{Alb} X} = \alpha^* : H^0(\Omega^1_{\operatorname{Alb} X}) \xrightarrow{\cong} H^0(\Omega^1_X).$$

You see this as follows. Since $\tau_{\operatorname{Alb} X}$ is an isomorphism, it suffices to prove that $\alpha^*(\tau_{\operatorname{Alb} X}(\omega)) = \omega$ for all holomorphic 1-forms $\omega$ on $X$. Pick $x \in U$, where $U$ is as before and evaluate the 1-form $\alpha^* \circ \tau_{\operatorname{Alb} X}\omega$ at $x \in U$ and get $a^* \circ q^*(\tau_{\operatorname{Alb} X}(\omega))(x) = a^* d(\langle \omega, - \rangle)(x) = d(\langle \omega, a(x) \rangle) = d(\int_{x_0}^x \omega) = \omega(x)$ and so $\alpha^*(\tau_{\operatorname{Alb} X}\omega) = \omega$. Here $\langle\, ,\, \rangle$ denotes the pairing between $H^0(\Omega^1_X)$ and its dual.

The pair $(\operatorname{Alb} X, \alpha)$ satisfies a universal property with respect to maps $f : X \to T$ of $X$ to a complex torus: such a map factors uniquely over $\alpha$, i.e there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & T \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\tilde f} & \\ & \operatorname{Alb}(X). & \end{array}$$

and $\tilde{f}$ is uniquely determined by commutativity of the diagram.

To show this, first look at morphisms between any two tori $T = V/\Gamma$ and $T' = V'/\Gamma'$. A linear map $H : V \to V'$ which sends the lattice $\Gamma$ to $\Gamma'$ induces a morphism $h : T \to T'$. Any other morphism is obtained by composing such a map with a translation, as shown by the next lemma.

**Lemma 3.** *Any morphism $h : T = V/\Gamma \to T' = V'/\Gamma'$ is induced by an affine-linear map $V \to V'$ which is the composition of a translation by a linear map $H : V \to V'$ with $H(\Gamma) \subset \Gamma'$. Invariantly, the transpose $H^T$ of $H$ fits into the commutative diagram*

$$\begin{array}{ccc} V'^* & \xrightarrow{H^T} & V^* \\ \downarrow{\scriptstyle\tau_{T'}} & & \downarrow{\scriptstyle\tau_T} \\ H^0(T', \Omega^1_{T'}) & \xrightarrow{h^*} & H^0(T, \Omega^1_T). \end{array}$$

**Proof:** By the lifting properties of the universal coverings $V$ and $V'$ there exists some holomorphic map $\bar{h} : V \to V'$ such that $\bar{h}(v + \gamma) - \bar{h}(v) \in \Gamma'$ for all $v \in V$ and $\gamma \in \Gamma$. By continuity, $\bar{h}(v + \gamma) - \bar{h}(v)$ is independent of $v$ and so all the partial derivatives of $\bar{h}$ are invariant under translation by $\Gamma$ and hence define holomorphic functions on $T$. So they must be constant and $\bar{h}(v) = H(v) +$ a constant vector (defining a translation) and $H$ a linear map with $H(\Gamma) \subset \Gamma'$. The last assertion should be obvious. ∎

Apply this lemma in the situation of the Albanese torus. In this situation, since $\alpha^*$ is an isomorphism with inverse $\tau_{\mathrm{Alb}\,X}$, the linear map $g = \tau_{\mathrm{Alb}\,X} \circ f^* : H^0(T, \Omega^1_T) \to H^0(\mathrm{Alb}\,X, \Omega^1_{\mathrm{Alb}\,X})$ makes the diagram

$$\begin{array}{ccc} H^0(X,\Omega^1_X) & \xleftarrow{f^*} & H^0(T,\Omega^1_T) \\ {\scriptstyle \alpha^*}\nwarrow & & \swarrow{\scriptstyle g} \\ & H^0(\mathrm{Alb}\,X, \Omega^1_{\mathrm{Alb}\,X}). & \end{array}$$

commutative. Now let $\tilde{F} : H^0(\Omega^1_X)^* \to V$ be defined by the requirement $\tau_{\mathrm{Alb}\,X} \circ \tilde{F}^T = g \circ \tau_T$ as suggested by the commutative diagram of the previous lemma. Note that by construction

$$\tilde{F}^T = f^* \circ \tau_T. \tag{4}$$

If $\tilde{F}$ induces a morphism $\tilde{f} : \mathrm{Alb}\,X \to T$ between the corresponding tori, by the previous lemma it is unique up to translation, but since $\tilde{f}(0) = f(x_0)$ it is then is completely determined. Again by the lemma, to prove existence, one only needs to see that $\tilde{F}$ preserves the lattices. This turns out to be the case, since $\tilde{F}$ restricted to the image of $H_1(X,\mathbb{Z})$ in $H^0(\Omega^1_X)^*$ coincides with the homomorphism induced by $f$ on the first homology group as the following computation shows. Fix $\gamma \in H_1(X,\mathbb{Z})$ (which is identified with its image in $H^0(\Omega^1_X)^*$) and $h \in V^*$. One has $\langle h, \tilde{F}\gamma\rangle = \langle \tilde{F}^T h, \gamma\rangle = \langle f^*\tau_T h, \gamma\rangle$ (by (4)) $= \int_\gamma f^*\tau_T h$ (by (2)) $= \int_{f_*\gamma} \tau_T h = \langle h, t_T^{-1} f_*\gamma\rangle$ (again by (2)) and so $t_T\tilde{F} = f_*$ and hence lattices are preserved.

**Example 5.** The Albanese of a curve is its Jacobian and the Albanese map is the Abel-Jacobi map. Recall from the theory of Riemann surfaces [G, Chapter V] that the period matrix can always be normalised by taking a symplectic basis for $H_1(C,\mathbb{Z})$ (see Appendix A2, Example A2.16 1.) and a suitable basis for $H^0(\Omega^1_C)$ such that the period matrix reads as follows

$$(\,\mathbb{1}_g \quad Z_g\,),$$

where $Z$ is a symmetric matrix with positive definite imaginary part. Recall also that these matrices form the so-called Siegel upper halfspace $\mathfrak{h}_g$.

Let me now formulate a few useful consequences and additions

1. If $q(X) = 0$ the Albanese reduces to 0 and the universal property of the Albanese implies that any map $X \to T$ must be constant. This applies e.g. to $X = \mathbb{P}^n$.

2. $\mathrm{Alb}\,X$ is the smallest subtorus generated by the image of the Albanese map. Indeed, let $i : A \subset \mathrm{Alb}\,X$ be this torus, the universal property shows that there is a morphism $a : \mathrm{Alb}\,X \to A$ such that $i \circ a = \mathrm{Id}$ and from this it follows that $A = \mathrm{Alb}\,X$.

3. Functoriality: any morphism $f : X \to Y$ induces a morphism between the Albanese tori $a(f) : \mathrm{Alb}\,X \to \mathrm{Alb}\,Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathrm{Alb}\,X & \xrightarrow{a(f)} & \mathrm{Alb}\,Y \end{array}$$

is commutative. This should be obvious. Note that in view of the previous remark, $a(f)$ is surjective if $f$ is surjective.

4. Special case: the image of the Albanese map is a curve.

**Lemma 6.** *If the image of the Albanese map $X \dashrightarrow \operatorname{Alb} X$ is a curve $C$, the fibres are connected. Moreover, $C$ is smooth and has genus $q(S)$.*

**Proof:** Consider the Stein factorisation ( see §9) for the Albanese map.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & C \subset \operatorname{Alb} X \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Y. & \end{array}$$

Since $X$ is normal, $Y$ is a normal and hence smooth curve. I want to show that the finite map $g$ in fact gives an immersion into the Albanese. The map $f$ fits into a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & \searrow{\scriptstyle g} & \\ \operatorname{Alb} X & \xrightarrow{a(f)} & \operatorname{Jac} Y & \xrightarrow{a(g)} & \operatorname{Alb} X \end{array}$$

and since $f$ is surjective, $a(f)$ must be surjective, as remarked before. Moreover since $a(g) \circ a(f) \circ \alpha = g \circ f = \alpha$, the universal property of $\alpha$ implies that $a(g) \circ a(f) = \operatorname{Id}$ and so, since I already know that $a(f)$ is surjective, it must be an isomorphism with inverse $a(g)$ and since the Abel-Jacobi map is an embedding, this then follows for $g = a(g) \circ \alpha'$. ∎