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Chapter 5. Divisors on Surfaces

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11. The Picard variety and the Néron-Severi group

I prove the Lefschetz theorem on (1,1)-classes, give a Hodge theoretic proof of the Index Theorem for surfaces, prove Nakai's ampleness criterion and rephrase it in terms of properties of the nef-cone.

Let me recall that by means of the exponential sequence on any projective manifold you get an isomorphism

$$\mathrm{Pic}^0 M \cong \frac{H^1(M, \mathcal{O}_M)}{\mathrm{im}\, H^1(M, \mathbb{Z})},$$

where the map $H^1(M,\mathbb{Z}) \to H^1(M, \mathcal{O}_M$ is induced from the inclusion $\mathbb{Z}_M \to \mathcal{O}_M$. I want to show how Hodge theory can be used to show that the Picard variety $\mathrm{Pic}^0(M)$ is a torus.

Indeed, the Hodge decomposition (Appendix A3) theorem says that De Rham group $H^k(M,\mathbb{C})$ decomposes into a direct sum $H^{k,0}(M) \oplus \cdots \oplus H^{0,k}(M)$, where $H^{p,q}(M)$ denotes the group spanned by classes represented by closed forms of type (p,q). Furthermore, the groups $H^{p,q}(M)$ can be computed as the cohomology of the complex of global sections of the Dolbeault resolution of Ω^p

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots.$$

The complex De Rham cohomology can be computed using the De Rham resolution $\mathcal{E}^\bullet_M$ of $\mathbb{C}_M$. Sending a complex valued form to its $(0,q)$-component defines a homomorphism $\mathcal{E}^\bullet \to \mathcal{E}^{0,\bullet}$ extending the inclusion $i : \mathbb{C}_M \to \mathcal{O}_M$. This just means that the $(0,q+1)$-component of $d\alpha$ is equal to $\overline{\partial}$ of the $(0,q)$-component of α. Passing to global sections and taking cohomology, one gets

Lemma 1. *The inclusion $j : \mathbb{C}_M \to \mathcal{O}_M$ induces the projection $H^k(M,\mathbb{C}) \to H^{0,k}(M)$ onto the Hodge $(0,k)$-component.*

Corollary 2. *The Picard variety $\mathrm{Pic}^0(M)$ is a torus.*

Proof: Let $[\alpha] \in H^1(M,\mathbb{C})$ be a class of a $(1,0)$-form α and assume that $[\alpha]$ is the image of the class of a real form. Then $\overline{[\alpha]} = [\alpha]$ and so if such classes are independent over the complex numbers they are also independent over the reals. This holds in particular for the images of a basis for $H^1(M,\mathbb{Z})$ in $H^{1,0}$ and so these form a lattice of maximal rank (since $\mathrm{rank}\, H^1(M,\mathbb{Z}) = \dim_{\mathbb{R}} H^{1,0}$). ∎

Remark 3. The torus $\mathrm{Pic}^0(M)$ is an algebraic torus. This is a deeper fact which follows from Lefschetz' theory of primitive cohomology. See [G-H, Chapter 2 §6].

The Néron-Severi group $\mathrm{NS}\, M$ of a smooth projective manifold of M is the group of isomorphism classes of divisors modulo homological equivalence, where two divisors are said to be homologically equivalent if they have the same first Chern class. The exponential sequence yields an exact sequence

$$0 \to \mathrm{NS}\, M \to H^2(M, \mathbb{Z}) \xrightarrow{k^*} H^2(\mathcal{O}_M).$$

Now look at the chain of inclusions $\mathbb{Z}_M \xrightarrow{i} \mathbb{C}_M \xrightarrow{j} \mathcal{O}_M$ inducing the triangle

$$\begin{tikzcd} H^2(M,\mathbb{Z}) \arrow[rr, "k^*"] \arrow[dr, "i^*"'] & & H^2(\mathcal{O}_M) \\ & H^2(M,\mathbb{C}) \arrow[ur, "j^*"'] & \end{tikzcd}$$

Every De Rham class $[\alpha]$ in the image of i^* is a class of a real form and hence, if the $(0,2)$-component vanishes, the $(2,0)$-component is zero as well. This observation in conjunction with the previous lemma shows that the kernel of k^* consists precisely of the integral classes having type $(1,1)$ in the de Rham group $H^2_{DR}(M, \mathbb{C})$. This is the content of the following theorem which says that the Hodge Conjecture is true for divisors.

Theorem 4. (Lefschetz' Theorem on (1,1)-classes) *The Néron-Severi group of a projective manifold consists precisely of the integral classes of Hodge type* $(1,1)$.

The next topic is the intersection form on the Néron-Severi group of a surface S. I first prove a simple instance of the Hodge-Riemann bilinear relations. Assume that $S \subset \mathbb{P}^n$ and let ω the metric form belonging to the Fubini-Study metric. See Appendix A3. It is a $(1,1)$)-form which is pointwise positive definite .

Define
$$H^2_{\mathrm{prim}}(S, \mathbb{Q}) := \{ \; [\alpha] \mid [\alpha \wedge \omega] = 0 \; \} = [\omega]^{\perp}$$
leading to the orthogonal direct sum decomposition

$$H^2(S, \mathbb{Q}) = \mathbb{Q} \cdot [\omega] \oplus H^2_{\mathrm{prim}}(S, \mathbb{Q}).$$

Theorem 5. *The intersection product is negative definite on* $H^2_{\mathrm{prim}}(S, \mathbb{R}) \cap H^{1,1}$.

Proof: As explained in appendix A.3.2, the Kähler identities imply that wedging with ω preserves the primitive forms and so, in the following computation, the use of forms instead of cohomology classes is allowed.

I CLAIM that

> for any real $(1,1)$ form α with $\alpha \wedge \omega = 0$ one has $\alpha \wedge \alpha \leq 0$ with equality if and only if $\alpha = 0$.

The theorem then follows from the compatibility of the intersection product and the wedge-product:
$$\int_S \alpha \wedge \beta = [\alpha] \cdot [\beta] \quad \text{for all closed 2-forms } \alpha, \, \beta.$$

To prove the claim let me choose a local C^∞-trivialisation of the holomorphic cotangent bundle by two 1-forms β^1, β^2 which are everywhere orthonormal with respect to the Kähler form ω. Thus

$$\omega = \frac{i}{2}(\beta^1 \wedge \overline{\beta^1} + \beta^2 \wedge \overline{\beta^2}).$$

Set

$$\alpha = \alpha_{1\bar{1}}\beta^1 \wedge \overline{\beta^1} + \alpha_{1\bar{2}}\beta^1 \wedge \overline{\beta^2} + \alpha_{2\bar{1}}\beta^2 \wedge \overline{\beta^1} + \alpha_{2\bar{2}}\beta^2 \wedge \overline{\beta^2}.$$

The condition that α is real implies that $\alpha_{1\bar{1}}$ and $\alpha_{2\bar{2}}$ are purely imaginary and that $\alpha_{1\bar{2}} + \overline{\alpha_{2\bar{1}}} = 0$. The condition that $\alpha \wedge \omega = 0$ yields $\alpha_{1\bar{1}} + \alpha_{2\bar{2}} = 0$. So

$$\frac{1}{2}\alpha \wedge \alpha = (|\alpha_{1\bar{1}}|^2 + |\alpha_{1\bar{2}}|^2)\beta^1 \wedge \overline{\beta^1} \wedge \beta^2 \wedge \overline{\beta^2}$$

which is a non-positive multiple of the volume form, and zero precisely when $\alpha = 0$. This proves the claim. ∎

Corollary 6. (Algebraic Index Theorem) *The intersection pairing restricts non-degenerately on* $\mathrm{NS}\, S$ mod torsion *and has signature* $(1, \rho - 1)$, *where* $\rho = \operatorname{rank} \mathrm{NS}\, S$ *is the Picard number.*

Proof: Note that $[\omega] \cdot [\omega] > 0$. Since by the theorem the intersection product is negative on $[\omega]^\perp$, the primitive part of the cohomology, the signature is $(1, h^{1,1} - 1)$ on $H^{1,1}$. So it either restricts non-degenerately with the stated signature or it is semi-negative (with rank one annihilator) on the Néron-Severi group. Since the latter always contains the class of an ample divisor this last possibility is excluded. ∎

Remark 7. It follows that two divisors D and D' are *torsion equivalent*, i.e. homologically equivalent up to torsion if and only if they are *numerically equivalent*, i.e. $c_1(D) = c_1(D')$ mod torsion if and only if $(D, E) = (D', E)$ for all divisors E.

Remark 8. A very useful alternative formulation of the Algebraic Index Theorem runs as follows

If D is a divisor with $(D, D) > 0$ and $(C, D) = 0$ then $(C, C) \leq 0$ with equality if and only if C is numerically equivalent to zero.

Remark 9. The preceding theorem is just a special case of the Lefschetz-decomposition theorem valid for the cohomology of any Kähler manifold. See [We, Chapt. V, sect. 6].

From the Algebraic Index Theorem it follows that the intersection pairing on the real vector space $N_{\mathbb{R}}(S) := \mathrm{NS}\, S \otimes \mathbb{R}$ has signature $(1, \rho - 1)$. Such quadratic forms have special properties. There is the *light cone* $x \cdot x = 0$ with disconnected interior $C^+(S) \amalg -C^+(S) = \{ x \in N_{\mathbb{R}}(S) \mid x \cdot x > 0 \}$. Each connected part is convex.

Recall that the dual of a cone C in a real vector space V with non-degenerate product is the cone

$$C^\vee := \{ y \in V \mid y \cdot x \geq 0 \quad \text{for all } x \in C \}.$$

If $x \neq 0$ is on the light cone and in the closure of $C^+(S)$, the dual of the half-ray $\mathbb{R}_{\geq 0} \cdot x$ is the half-space bounded by the hyperplane through this ray, tangent to the light cone and containing $C^+(S)$. The intersection of all such half spaces is the closure of $C^+(S)$. Using convexity it follows that the closed cone $\overline{C^+(S)}$ is self dual.

To study divisors inside the light cone, one uses Riemann-Roch.

Proposition 10. *If for a divisor D on a surface one has $(D, D) > 0$, then $(D, H) \neq 0$ for any ample divisor H. If $(D, H) > 0$ some positive multiple of D is effective and if $(D, H) < 0$, some negative multiple of D is effective.*

Proof: The first assertion follows from the Algebraic Index Theorem.

The Riemann-Roch inequality shows that $h^0(mD) + h^0(-mD + K_S) \geq \frac{1}{2} m^2 (D, D) +$ linear term in m. If $(D, H) > 0$, there can be no divisor in $|-mD + K_S|$ for m large and so $|mD|$ must contain effective divisors for m large enough. The proof of the second assertion is similar. ∎

Since the effective divisors are all on the same side of the hyperplane defined by an ample divisor it follows from the preceding proposition that only one component of the interior of the light cone can contain effective divisors. Let me once and for all choose it to be $C^+(S)$. Let me also speak of *$\mathbb{Q}$-divisors* as a formal linear combination of irreducible curves with rational coefficients. Similarly one can speak of $\mathbb{Q}$-divisor classes, the rational points in $\mathrm{NS}(S) \otimes \mathbb{R}$. Such a class is called *effective* if a positive multiple can be represented by an effective divisor. Explicitly, a $\mathbb{Q}$-divisor class $[D]$ is effective if and only if there is an integer $n > 0$ such that there is an effective divisor numerically equivalent to nD. From the preceding Proposition it follows that for divisors with positive self-intersection in this definition one can replace "numerically equivalent" by "linearly equivalent", i.e. effectivity is a numerical property for divisors with positive self-intersection.

The preceding theorem now can be conveniently reformulated as follows.

Corollary 11. *The rational points in $C^+(S)$ are effective $\mathbb{Q}$-divisors.*

In general, there are more effective divisors in $\mathrm{NS}\, S$ spanning a convex cone $\mathrm{Ef}\, S$ in the real vector space spanned by divisors.

Let me consider the dual cone

$$\mathrm{Nef}\, S := \mathrm{Ef}\, S_\vee = \{ x \in \mathrm{N}_{\mathbb{R}}(S) \mid x \cdot e \geq 0 \quad \forall e \in \mathrm{Ef}\, S \}.$$

Its rational points are the classes of what are called nef-divisors ("numerically effective divisors").

Definition *A divisor D is nef if $(D, C) \geq 0$ for all irreducible curves C.*

The cone $\text{Nef}\, S$ therefore is called the *nef-cone.*

Observation 12. *If for a divisor D one has $(D, C) \geq 0$ for all irreducible curves C then $(D, D) \geq 0$.*

Proof: One has $\text{Nef}\, S = \text{Ef}\, S_\vee \subset C^+(S)_\vee = \overline{C^+(S)}$. So $(D, D) \geq 0$ as desired. ■

Next, let me study the ample divisors. The following technical lemma plays an important role in the proof of the Nakai-moishezon Criterion, which will be treated shortly.

Lemma 13. *Suppose that $C = A + B$ is the sum of two effective divisors on a manifold M. There is an exact sequence of coherent sheaves on M:*

$$0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \xrightarrow{\text{restriction}} \mathcal{O}_B \to 0.$$

For a proof see Problem 1.

Theorem 14. (Nakai-Moishezon) *A divisor D on a surface S is ample if and only if $(D, D) > 0$ and $(D, C) > 0$ for all irreducible curves C.*

Proof: Let H be a very ample line bundle. Since $(D, H) > 0$ and $(D, D) > 0$ by Proposition 11.10 a multiple of D is effective. By replacing D by this multiple one may assume that D is effective. Let me now show that by induction $H^1(D', \mathcal{O}(nD)) = 0$ for all divisors D' supported on D and all n sufficiently large. If D' is irreducible and $\nu : D'' \to D'$ its normalisation, one has $H^1(D', \mathcal{O}(nD)) = H^1(D'', \nu^*(\mathcal{O}(nD))$. Since $\deg \nu^*(\mathcal{O}(nD) = n(D, D')$ and $(D', D) > 0$ by assumption, for all n large enough you have indeed $H^1(D', \mathcal{O}(nD)) = 0$. If D is irreducible one is ready. Otherwise, you write $D' = D'' + R$ for some effective divisor R and irreducible D''. Consider the cohomology sequence associated to the sequence of the previous lemma:

$$\ldots \to H^1(\mathcal{O}_{D''}(nD - R)) \to H^1(\mathcal{O}_{D'} D(nD)) \to H^1(\mathcal{O}_R(nD)) \to \ldots.$$

The first term is zero for $n \geq n_0$ by a similar argument as the argument for irreducible D' while the third term is zero for $n > n_1$ by induction on the number of components (counted with multiplicity) in D'. So the middle term is zero for $n > \max\{ n_0, n_1 \}$. Now one considers the exact cohomology sequence associated to

$$0 \to \mathcal{O}_S((n-1)D) \to \mathcal{O}_S(nD) \to \mathcal{O}_D(nD) \to 0.$$

Let me look at the portion

$$\begin{aligned} H^0(S, \mathcal{O}(nD)) \to & H^0(D, \mathcal{O}(nD)) \to H^1(S, \mathcal{O}((n-1)D)) \to \\ & \to H^1(S, \mathcal{O}(nD)) \to H^1(D, \mathcal{O}_D(nD)) \to \ldots \end{aligned}$$

By the previous vanishing result, one finds for all large enough n:

$$\dim H^1(S, \mathcal{O}(nD)) \leq \dim H^1(S, \mathcal{O}((n-1)D)).$$

But since all these spaces are finite dimensional, their dimensions must eventually stabilise and then the map

$$H^0(S, \mathcal{O}(nD)) \to H^0(D, \mathcal{O}(nD))$$

becomes surjective. Now one can show, again by induction on the number of components of D', that $\mathcal{O}_{D'}(nD)$ is globally generated by its sections if n is large enough. Surjectivity of the preceding map then implies that $\mathcal{O}(nD)$ is generated by its sections along points of D. Now, since D is effective, $\mathcal{O}_S(D)$ has a section vanishing exactly along D and so $\mathcal{O}_S(nD)$ is also generated by sections away from D.

It follows that $\mathcal{O}(nD)$ defines a morphism

$$f : S \to \mathbb{P}^n$$

and I claim that f is a finite morphism. Indeed, if C is a curve which is mapped to a point, you take a hyperplane L in $\mathbb{P}^n$ missing this point and so $(C, L) = (C, nD) = 0$ contradicting our assumptions. Now let me recall Lemma 4.24 which implies that $f^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_S(nD)$ is ample.

■

Proposition 15. *L is ample if and only if $(c_1(L), c) > 0$ for all $c \in \overline{\operatorname{Ef} S} \setminus \{ 0 \}$*

Proof: If L is ample, $(L, D) > 0$ for all effective divisors D and so $(c_1(L), c) \geq 0$ for all c in the closure of the effective cone. If $(c_1(L), c) = 0$ for some c in this closure and $c \neq 0$ one can find an effective $C' \in \operatorname{Pic}(S)$ with $(c, c_1(C')) < 0$ and then $(c_1(L^{\otimes n} \otimes C'), c) < 0$. On the other hand $L^{\otimes n} \otimes C'$ will be ample for n large enough by Nakai-Moishezon (for at worst finitely many of components D of C' you will have $(D, C') < 0$ and these can be taken care of by making n large enough). This is a contradiction and so $(c_1 L), c) > 0$.

Conversely, by the Nakai-Moishezon criterion, one only has to show that $(L, L) > 0$. Fix some ample line bundle H and consider the function $f(c) = (c_1(L), c)/(H, c)$ which is constant under homotheties and so to study its values one can restrict to the (compact) closure of $\operatorname{Ef} S$ in the unit ball with respect to some metric on the real vector space $N_{\mathbb{R}}(S)$. It has a positive (rational) maximum ϵ and so $(L - \frac{1}{2}\epsilon H, c) > 0$ for all $c \in \operatorname{Ef} S$ and in particular $L - \frac{1}{2}\epsilon H$ is nef and so has non-negative selfintersection. But then $(L, L) = (L - \frac{1}{2}\epsilon H, L - \frac{1}{2}\epsilon H) + \epsilon(H, L - \frac{1}{2}\epsilon H) + \frac{1}{4}\epsilon^2(H, H) > 0$. ■

Corollary 16. *The cone consisting of ample $\mathbb{Q}$-divisors forms an open subset in $\operatorname{NS} S \otimes \mathbb{Q}$ and its closure is the nef-cone.*

Proof: If H is ample and D any divisor $(H + tD, c) > 0$ for c in the closure $\mathcal{C}$ of $\operatorname{Ef} S$ in the unit ball in some metric on $N_{\mathbb{R}}(S)$ and for $|t| < t_0$ with t_0 the smaller of the minima of the two functions $f(c) = (-D, c)/(H, c)$ on $\mathcal{C} \cap \{(D, c) \leq 0\}$ and $g(c) = (H, c)/(D, c)$ on $\mathcal{C} \cap \{(D, c) \geq 0\}$. By the proposition $H + tD$ is ample for these values of t.

Conversely, by the Proposition, one has $(a, c') \geq 0$ for all $c' \in \overline{\operatorname{Ef} S}$. But this is the case precisely when $(a, c) \geq 0$ for all $c \in \operatorname{Ef} S$, i.e. when a is nef. ■

Problems.

11.1. Let M be a projective manifold and $C = A + B$ the sum of two effective divisors. Show that the inclusion $C \subset B$ induces an exact sequence

$$0 \to \mathcal{I}_B/\mathcal{I}_C \to \mathcal{O}_C \to \mathcal{O}_B \to 0.$$

Show that there is a canonical isomorphism

$$\mathcal{I}_B/\mathcal{I}_C \cong \mathcal{O}_A(-B).$$

12. Rationality theorem and applications

I state and prove Mori's rationality theorem following the sketch in [Wi] and give applications which are to be considered as the first steps in classification theory, e.g. Castelnuovo's Rationality Criterion.

Let me recall that the Néron-Severi group $\mathrm{NS}\, S$ is the group of divisor classes modulo homological equivalence on S. The cup product on the real vector space $\mathrm{NS}\, S \otimes \mathbb{R}$ makes it into a self dual vector space. So you may view a divisor either as giving a class in $\mathrm{NS}\, S$ or as giving a hyperplane in $\mathrm{NS}\, S \otimes \mathbb{R}$. One has the real cone $\mathrm{Ef}\, S$ of effective divisors (with real coefficients) whose dual is called the cone of nef-divisors and denoted by $\mathrm{Nef}\, S$. So a divisor D is nef if and only if the cone $\mathrm{Ef}\, S$ is on the non-negative side of the hyperplane which D defines. The cone Nef is a closed cone whose integral points in the interior are the classes of the ample divisors. So H is ample if and only if $(H, H) > 0$ and $\mathrm{Ef}\, S \setminus 0$ is on the positive side of the hyperplane defined by H. If some D is not nef the hyperplane it defines will have some part of $\mathrm{Ef}\, S$ on its negative side and in the pencil $H + sD$ there will be a smallest value for wich the resulting hyperplane no longer has Ef on the positive side. The rationality theorem says that for $D = K_S$ this happens for a rational value. This theorem has surprizingly many consequences for the classification of surfaces as you will see.

Theorem 1. (Rationality theorem) *Let S be a surface and let H be very ample on S. Assume that K_S is not nef. Then there is a rational number b such that the hyperplane corresponding to $H + bK_S$ touches the cone* $\mathrm{Ef}\, S$.

Proof: Introduce

$$b := \sup\{\ t \in \mathbb{R} \mid H_t = H + tK_S \text{ is nef }\ \}.$$

Set

$$P(v, u) := \chi(vH + uK_S).$$

By Riemann-Roch this is a quadratic polynomial in v, u. If u and v are positive integers with $(u-1)/v < b$ the divisor $vH + (u-1)K_S$ is ample and so by Kodaira Vanishing (Appendix A3) $H^i(vH + uK_S) = 0$ for $i = 1, 2$. It follows that $P(v, u) \geq 0$.

Assume now that b is irrational. Number theory ([HW, Theorem 167]) implies that b can be approximated by rational numbers of the form p/q, p and q arbitrarily large integers in such a way that

$$p/q - 1/(3q) < b < p/q.$$

The polynomial $P(kq, kp)$ is quadratic in k. If it is identically zero, $P(v, u)$ must be divisible by $(vp - uq)$. Taking p and q sufficiently large one may assume that this is not the case. For $k = 1, 2, 3$ the numbers $v = kq$ and $u = kp$ satisfy $(u-1)/v < b$ and hence $P(kq, kp) \geq 0$ for these three values of k. Since a quadratic polynomial has at most two zeroes, it follows that for at least one pair of positive integers (v, u) with $t_0 := u/v > b$ one has $\dim H^0(vH + uK_S) > 0$. So there is an effective divisor (with coefficients in $\mathbb{Q}$) $L := H_{t_0} = \sum a_j \Gamma_j$, $a_j > 0$. Now H_{t_0} is not nef. Since L is effective, it can only be negative on the Γ_j. But then one can subtract off a rational multiple of K_S from H_{t_0} to get H_b and so b would be rational contradicting our assumption. ∎

Let me give a first application.

Proposition 2. *A minimal algebraic surface with K not nef is either a geometrically ruled surface or $\mathbb{P}^2$.*

Proof: Let me first look at the positive half ray in $\mathrm{NS}\, S \otimes \mathbb{Q}$ spanned by $-K_S$. There are two possibilities. The first possibility is that all ample classes of S are on this line and hence $-K_S$ is ample and $\mathrm{Pic}\, S$ has rank 1. Kodaira-Vanishing implies that $h^0(K_S) = h^1(K_S) = 0$ and so $p_g = q = 0$. It follows that $\mathrm{Pic}\, S \xrightarrow{\cong} H^2(S, \mathbb{Z})$ has rank one. Moreover $b_2 = 1$ and $b_1 = 0$ imply that $e(S) = 3$ and by Noether's Formula one has $(K, K) + 3 = 12(1 - q + p_g) = 12$ and so $(K, K) = 9$. Next, take an ample generator H of $\mathrm{Pic}\, S$ mod torsion and apply Riemann-Roch to H. Note that since $H - K_S$ is ample, Kodaira-Vanishing gives that $h^1(H) = 0 = h^2(H)$ and one finds $h^0(H) = \frac{1}{2}(H, H - K_S) + 1 = 3$. Indeed, since $(K, K) = 9$, K must be numerically equivalent to $-3H$. One gets a dominant (i.e. the closure of the image is the entire target space) rational map $f : S \dashrightarrow \mathbb{P}^2$ which maps H to the class of a line. Now $(H, H) = \frac{1}{9}(K_S, K_S) = 1$ implies that $|H|$ can have no fixed points and that f is birational (why?). Now f cannot contract any curves to points, since $\mathrm{Pic}\, S$ has rank 1. From the discussion about birational geometry it follows that f must be biregular and so S is isomorphic to $\mathbb{P}^2$.

So one may assume that there exists an ample H such that its class in $\mathrm{NS}\, S \otimes \mathbb{Q}$ does not belong to the positive half-ray spanned by $-K_S$. Now apply the rationality theorem to H and K_S.

Clearing denominators one finds a divisor

$$L = vH + uK_S,\ b = u/v = \sup\{\, t \in \mathbb{R} \mid H_t = H + tK_S \text{ is nef } \}.$$

Now L belongs to the closure of the nef-cone, which- as shown before- is itself closed. So L is a nef divisor and so in particular, $(L, L) \geq 0$ (see Observation 11.12). If you subtract any positive rational multiple of K_S from L you come into the interior of the nef-cone, which is the ample cone. So $mL - K_S$ is ample for all $m \geq 1$. Serre duality implies that $\dim H^2(mL) = \dim H^0(-(mL - K_S)) = 0$ and so by Riemann-Roch

$$\dim H^0(mL) \geq \chi(mL) = \chi(S) + \frac{1}{2}(mL, mL - K_S).$$

One can distinguish two cases, namely $(L, L) > 0$ or $(L, L) = 0$.

i) $(L, L) > 0$. Since L is nef, for any effective divisor, one has $(L, D) \geq 0$. The equality sign can be excluded as follows. Any irreducible curve D for which $(L, D) = 0$ must be an exceptional curve of the first kind. Indeed, from the definition of L one sees that $(K_S, D) < 0$, while the Algebraic Index Theorem applied to L and D shows that $(D, D) < 0$. In combination with the adjunction formula this shows that D has to be an exceptional curve of the first kind. By assumption these don't exist and so $(L, D) > 0$ for all curves D and so, by the Nakai-Moishezon criterion, L is ample, which is impossible by construction (L is on the boundary of the nef-cone).

ii) $(L, L) = 0$. Since L is nef one has $(L, H) \geq 0$, and if $(L, H) = 0$ an application of the Algebraic Index Theorem shows that L is numerically trivial. In this last case, the class of H in $\mathrm{NS}\, S \otimes \mathbb{Q}$ would be on the positive half-ray spanned by the class of $-K_S$, which has been excluded. So $(L, H) > 0$. From $0 = 1/v(L, L) = (L, H + bK_S)$ one infers that $(L, K_S) < 0$ and so dim $H^0(mL)$ grows like a linear function of m. You may replace L by mL and assume that dim $|L| \geq 1$. Now write $L = L' + L_{\text{fixed}}$, where L_{fixed} is the fixed part of $|L|$. I claim that L' is still nef and that still $(L', L') = 0$. The first is clear since L' moves in a linear system. So $(L', L) \geq 0$ and $(L', L_{\text{fixed}}) \geq 0$. From

$$0 = (L, L) = (L', L) + (L_{\text{fixed}}, L) \geq 0$$

one infers $(L', L) = (L_{\text{fixed}}, L) = 0$, while

$$0 = (L', L) = (L', L') + (L', L_{\text{fixed}}) \geq 0$$

implies that $(L', L') = 0$. Moreover, for every irreducible component D of $|L'|$ the equality $(L, L') = 0$ implies that $(L, D) = 0$ and since $(L', L_{\text{fixed}}) = 0$ one also has $(D, L_{\text{fixed}}) = 0$. So $(L', D) = 0$ and from this you easily see that $(D, D) \leq 0$. By definition of L from the equality $(L, D) = 0$ one concludes that $(D, K_S) < 0$. The Adjunction Formula then implies that D is a smooth rational curve with $(D, D) = 0$.

The same reasoning applies to *any* linear subsystem of $|L|$ which has no fixed part. You can for instance take a one-dimensional subsystem of $|L|$, take off the fixed part and end up with a pencil $\mathbb{P}$ without fixed components and with $(F, F) = 0$ for every $F \in \mathbb{P}$. By the preceding discussion every irreducible component of a member of $|F|$ is a smooth rational curve.

Since $(F, F) = 0$ there can be no fixed points and so one gets a morphism $f : S \to \mathbb{P}^1$. Now by taking the Stein factorization of f (see §9) one obtains a fibration $f' : S \to C$ of S onto a curve whose fibres are smooth rational curves. So S is a geometrically ruled surface. ∎

Corollary 3. (Uniqueness of Minimal Model) *If S, S' are two minimal surfaces which are not ruled then any birational map $f : S' \dashrightarrow S$ is an isomorphism. In particular, any surface which is not ruled or rational has a unique minimal model.*

Proof: This follows from the previous theorem and Proposition 6.15 ■

Let me give an application of which the full strength will be shown in the next sections.

Proposition 4. *Let K_S be nef. There are the following possibilities for S.*

1. $(K_S, K_S) > 0$. *Then* $P_m \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g$ *for* $m \geq 2$ *and always* $P_2 > 0$.
2. $(K_S, K_S) = 0$, $q = 0$. *Then* $P_2 > 0$.
3. $(K_S, K_S) = 0$, $p_g > 0$ *and* $q > 0$.
4. $(K_S, K_S) = 0$, $p_g = 0$, $q = 1$ *and* $b_2 = 2$.

Proof: Observe that nefness of K_S implies that $(K_S, K_S) \geq 0$. Now you only have to prove the following three assertions:

i. If $(K, K) > 0$ the stated bound for P_m is valid and $P_m > 0$.

ii. If $(K, K) = 0$, $p_g = q = 0$ implies $P_2 > 0$.

iii. If $(K, K) = 0$, $p_g = 0$ and $q > 0$ one has $b_2 = 2$.

If $p_g > 0$, clearly $P_m > 0$ for all $m \geq 1$ so to, prove that $P_2 > 0$ it suffices to look at the case $p_g = 0$.

So let me first consider the case $p_g = 0$. Noether's formula in this case reads

$$12(1-q) = (2 - 4q + b_2) + (K_S, K_S).$$

So $b_2 = 10 - 8q - (K_S, K_S) \geq 1$ implies that $q \leq 1$.

If $q = 1$ and $(K_S, K_S) = 0$ one must have $b_2 = 2$ and this is case 4. This already proves iii.

In the remaining cases one either has $q = 1$, $(K, K) \geq 1$ or $q = 0$ which makes the right hand side of the Riemann-Roch inequality for mK_S positive in all cases:

$$h^0(mK_S) + h^0(-(m-1)K_S) \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g.$$

In particular, $P_m \geq \frac{1}{2}m(m-1)(K_S, K_S) + 1 - q + p_g$ as soon as $H^0(-((m-1)K_S)) = 0$. Therefore, to prove i. and ii. I only need to see that $H^0(-((m-1)K_S)) = 0$ if $m \geq 2$. This is an immediate consequence of the following Lemma.

Lemma 5. *Let L be a nef line bundle on a surface S such that L^{-1} has a section. Then L is trivial.*

Proof: Suppose L (and hence L^{-1}) is not trivial. Then there would exist a section of L^{-1} vanishing along a divisor and any curve C transversal to this divisor would satisfy $-(L, C) > 0$ which contradicts the nefness of L. ■

■

Corollary 6. (Castelnuovo's Rationality Criterion) *A surface is rational if and only if* $P_2 = q = 0$.

Problems.

12.1. Prove that any surface with S minimal and $-K_S$ ample is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

12.2. Let S_r be the projective plane blown up in r points in general position. Suppose that $r \leq 6$.

1. Show that the linear system of cubics passing through these points corresponds to the linear system $|-K_{S_r}|$ which gives an embedding of S_r in $\mathbb{P}^{9-r}$ as a degree $9-r$ surface. Such a surface is called a *Del Pezzo surface.*
2. Prove that a surface $S \subset \mathbb{P}^N$ for which $|-K_S|$ is very ample is a Del Pezzo surface or the quadric embedded in $\mathbb{P}^8$.