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The total length of the edges of a polyhedron

by

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Fejes Tóth, in a paper [1]) to which I have not had access, has conjectured that L, the sum of the lengths of the edges of a convex polyhedron containing a sphere of unit diameter, satisfies $L \geqq 12$; and he has proved that $L > 10$ for all such polyhedra, and $L > 14$ for polyhedra with triangular faces only. In this note I prove that, if no face is a polygon of more than n sides, then

$$L > \frac{10}{3}\sqrt{\left(\pi n \tan\frac{\pi}{n}\right)}. \qquad (1)$$

For triangular faces only, this is weaker ($L > 13.47\ldots$) than Fejes Tóth's result; for triangular and/or quadrilateral faces it gives $L > 11.82\ldots$; and for faces with any number of sides it gives

$$L \geqq 10\pi/3 = 10.47\ldots, \qquad (2)$$

which is slightly stronger than Fejes Tóth's result.

Let S be the surface and the area of the sphere, centre 0, radius $\frac{1}{2}$. Let P be the plane containing any face of the polyhedron. Let p denote the perimeter of this face and its length. The area A′ of this face cannot exceed that of a regular polygon of n sides with perimeter p; so

$$A' \leqq \frac{p^2}{4n}\cot\frac{\pi}{n}. \qquad (3)$$

Let A denote the projection (and its area) from 0 of the interior of p upon S. Define θ by

$$A = \tfrac{1}{2}\pi(1 - \cos\theta), \qquad (0 \leqq \theta \leqq \tfrac{1}{2}\pi). \qquad (4)$$

Let C be the cone of semi-vertical angle θ, with vertex at 0 and axis normal to P. Let B and B' be the areas cut off by C upon S and P respectively. Since $B = A$, we have

$$A' > B' \geqq \tfrac{1}{4}\pi\tan^2\theta. \qquad (5)$$

[1]) *Norske. Vid. Selsk. Forh., Trondhjem* (1948) **21**, 32—4. See *Math. Rev.* (1950) **11**, 386.

Strict inequality holds in (5) because A has not got a circular boundary. From (3) and (5)

$$p > \left(n\pi \tan \frac{\pi}{n}\right)^{1/2} \tan \theta. \tag{6}$$

Use the suffix $i = 1, 2, \ldots$ for the various faces of the polyhedron. Summing we have

$$\sum_i A_i = S = \pi = \tfrac{1}{2}\pi \sum_i (1 - \cos \theta_i), \quad (0 \leqq \theta_i \leqq \tfrac{1}{2}\pi). \tag{7}$$

$$2L = \sum_i p_i > \left(n\pi \tan \frac{\pi}{n}\right)^{1/2} \sum_i \tan \theta_i = T. \tag{8}$$

A minimum of T cannot occur unless either

$$\sec^2 \theta_i + \lambda \sin \theta_i = 0 \tag{9}$$

where λ is a Lagrangian undetermined multiplier, or θ_i is an endpoint of the interval $0 \leqq \theta_i \leqq \frac{1}{2}\pi$. If $\theta_i = \frac{1}{2}\pi$, T is infinite. Suppose that exactly N of the θ_i are not zero. Since these values must then satisfy (9), they are equal; whence, from (7), for these θ_i

$$\cos \theta_i = 1 - \frac{2}{N}, \quad \tan \theta_i = \frac{2(N-1)^{1/2}}{(N-2)}, \tag{10}$$

the first of the relations (10) implying $N \geqq 2$. Then

$$T \geqq \left(n\pi \tan \frac{\pi}{n}\right)^{1/2} \frac{2N(N-1)^{1/2}}{(N-2)} = \left(n\pi \tan \frac{\pi}{n}\right)^{1/2} U(N). \tag{11}$$

When $N \geqq 2$ ranges over the positive integers, $U(N)$ attains its minimum for $N = 5$; whereupon (8) and (11) yield (1).

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