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On Riemann Integrability and Almost Periodic Functions

by

Raouf Doss

Let $f(x)$ be a Bohr almost periodic (Bohr a.p.) function. To every $\epsilon > 0$ we can associate a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

$$
\sup_t |f(t + \tau_i) - f(t)| < \epsilon,
$$

provided

$$
|\tau_i| < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m).
$$

Conversely, if to every $\epsilon > 0$ there corresponds a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that relations (2) imply (1), then $f(x)$ is a Bohr a.p. function.

This suggests the following definition:

**Definition 1.** A bounded function $f(x)$ is called almost periodic in the sense of Riemann-Stepanoff\(^1\) (R.S.a.p.) if to every $\epsilon > 0$ there corresponds a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

$$
\sup_x \int_{x}^{x+1} |f(t + \tau_i) - f(t)| \, dt < \epsilon,
$$

provided

$$
|\tau_i| < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m).
$$

Here $\int^b_a$ means an upper Lebesgue integral.

To define the R.W.a.p. or the R.B.a.p. classes we just replace (3) by

$$
\limsup_{l \to \infty} \frac{1}{l} \int_{x}^{x+l} |f(t + \tau_i) - f(t)| \, dt < \epsilon
$$

or

$$
\lim_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} |f(t + \tau_i) - f(t)| \, dt < \epsilon.
$$

respectively.

It will be seen below (theorem 2) that the R.W.a.p. and the R.B.a.p. classes are identical.

\(^1\) The approximation theorem below (theorem 2) justifies the name of Riemann.
The Stepanoff, Weyl, and Besicovitch distances between two summable functions $f(x), g(x)$ are defined in the usual manner and will be denoted by $D_S(f, g)$, $D_W(f, g)$ and $D_B(f, g)$ respectively.

We denote by $R$ the additive group of reals. Let $E$ be a measurable set in $R$ and let $c_E(t)$ be its characteristic function. We write

$$S(E) = \sup_x \int_x^{x+1} c_E(t) dt$$

$$W(E) = \lim_{l \to \infty} \sup_x \frac{1}{l} \int_x^{x+1} c_E(t) dt$$

$$B(E) = \lim_{l \to \infty} \frac{1}{2l} \int_{-l}^{+l} c_E(t) dt.$$ 

The complementary of $E$ with respect to $R$ will be denoted by $\tilde{E}$.

We have the following theorem:

**Theorem 1.** In order that the bounded function $f(x)$ be $R.S.a.p.$ it is necessary and sufficient that to every $\epsilon > 0$ there corresponds a measurable set $E$ and numbers $\delta > 0, \pi_1, \ldots, \pi_m$ such that

\begin{equation}
S(\tilde{E}) < \epsilon,
\end{equation}

and such that

$$|f(x) - f(x')| < \epsilon,$$

provided $x \in E$ and

$$|x - x'| < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m)$$

To have the corresponding theorem for the $R.W.a.p.$ or the $R.B.a.p.$ classes we just replace (i) by

$$W(\tilde{E}) < \epsilon$$

or

$$B(\tilde{E}) < \epsilon$$

respectively.

We introduce the following definition:

**Definition 2.** A function $f(x)$ is called $K,S.a.p.$ if to every $\epsilon > 0$ there corresponds a measurable set $E$ and numbers $\delta > 0, \pi_1, \ldots, \pi_m$ such that

\begin{equation}
S(\tilde{E}) < \epsilon
\end{equation}

and such that 
\[ |f(x) - f'(x)| < \epsilon \]
provided \( x \in E, \ x' \in E \) and
\[ |x - x'| < \delta \ (\text{mod } \pi_k) \quad (k = 1, \ldots, m). \]

To have the corresponding definition for the K.W.a.p. or the K.B.a.p. classes we just replace (i) by
\[ W(\tilde{E}) < \epsilon \]
or
\[ B(\tilde{E}) < \epsilon \]
respectively.

We have the following approximation theorems:

**Theorem 2.** In order that the function \( f(x) \) be R.S.a.p. it is necessary and sufficient that to every \( \epsilon > 0 \) there corresponds two trigonometric polynomials \( p(x), q(x) \) such that

(i) \[ p(x) \ll f(x) \ll q(x), \]
(here \( a \ll b \) means \( \text{Re } a \leq \text{Re } b \) and \( Ia \leq Ib \)),

(ii) \[ D_5(p, q) < \epsilon. \]

To have the corresponding theorem for the R.W.a.p. or the R.B.a.p. classes we just replace (ii) by
\[ D_w(p, q) < \epsilon \]
or
\[ D_B(p, q) < \epsilon. \]
respectively.

Since for polynomials (or Bohr a.p. functions) \( p(x), q(x) \) we have
\[ D_w(p, q) = D_B(p, q), \]
we see that the two classes R.W.a.p. and R.B.a.p. are identical.

**Theorem 3.** In order that the function \( f(x) \) be K.S.a.p. it is necessary and sufficient that to every \( \epsilon > 0 \) we can associate a trigonometric polynomial \( q(x) \) and a measurable set \( E \) such that

(i) \[ S(\tilde{E}) < \epsilon \]
and

(ii) \[ |f(x) - q(x)| \leq \epsilon, \quad \text{for } x \in E. \]

To have the corresponding theorem for the K.W.a.p. or the K.B.a.p. classes we just replace (1) by
\[ W(\tilde{E}) < \epsilon \]
or

\[ B(\tilde{E}) < \epsilon \]

respectively \(^3\).  

Let \( f(x) \) be a R.B.a.p. function. By means of theorem 2 we can easily extend to \( f(x) \) a classical property due to H. Weyl \(^4\) of \( R \)-integrable, purely periodic functions: we can find two numbers \( \xi \) and \( M \) with the property:

To every \( \epsilon > 0 \) there corresponds an integer \( n \) such that

\[ \left| \frac{1}{n} \sum_{i=0}^{n-1} f(x + l\xi) - M \right| < \epsilon \]

whatever be \( x \).

Combining this property with almost periodicity we obtain

**THEOREM 4.** Let \( f(x) \) be a R.B.a.p. function; then we can find two numbers \( \xi \) and \( M \) possessing the following property:

To every \( \epsilon > 0 \) there corresponds an integer \( n \) and numbers \( \delta > 0, \pi_1, \ldots, \pi_m \) such that

\[ \left| \frac{1}{n} \sum_{i=0}^{n-1} f(x_i + l\xi) - M \right| < \epsilon \]

provided

\[ |x_i - x_j| < \delta \quad (mod \ \pi_k) \]

\( (i, j = 0, \ldots, n - 1) \quad (k = 1, \ldots, m) \)

Conversely, if there are two numbers \( \xi \) and \( M \) with the above property, then \( f(x) \) is a R.B.a.p. function \(^5\).

There is no corresponding theorem for the R.S.a.p. functions.

**Proof of theorem 1**

**Necessity.** Let \( f(x) \) be a R.S.a.p. function. Let \( \epsilon > 0 \) be given, and let \( \delta > 0, \pi_1, \ldots, \pi_m \) be such that

\[ \sup_x \int_{x}^{x+1} |f(t + \tau_i) - f(t)| \ dt < \frac{\epsilon^2}{4}, \]

provided

\[ |\tau_i| < \delta \quad (mod \ \pi_k) \]

\( (k = 1, \ldots, m) \).


\(^5\) This theorem has been stated without proof in Raouf Doss” Sur une nouvelle classe de fonctions presque-périodiques” C. R. Acad. Sci. Paris, 238, 317–318, (1954).
Put
\[ \varphi(t) = \sup_{\tau} |f(t + \tau) - f(t)|, \]
where \( \tau \) is subject to the condition
\[ |\tau| < \delta \ (\text{mod } \pi_k) \quad (k = 1, \ldots, m). \]
Then, by (1) and the definition of an upper Lebesgue integral
\[ (1, S) \quad \sup_x \int_x^{x+1} \varphi(t)dt < \frac{\epsilon^2}{4}. \]

Let \( n \) be a fixed positive or negative integer and call \( D_n \) the set of points \( t \) of the interval \((n, n+1)\) at which \( \varphi(t) \geq \epsilon \). \( D_n \) is not necessarily measurable, but there exists a partition of \((n, n+1)\) into a finite number of disjoint measurable sets \( E_1, \ldots, E_s \) such that
\[ \sum_{i=1}^{s} M_i \mu(E_i) \leq \int_n^{n+1} \varphi(t)dt + \frac{\epsilon^2}{4}, \]
where \( \mu(E_i) \) is the measure of \( E_i \) and \( M_i \) is the sup. of \( \varphi(t) \) for \( t \) on \( E_i \). The set \( D_n \) above meets a number of \( E_i \), say \( E_1, \ldots, E_r \), \( (r \leq s) \), so that
\[ \epsilon \leq M_i \quad \text{for } i = 1, \ldots, r. \]
By (2)
\[ \sum_{i=1}^{r} \epsilon \mu(E_i) \leq \int_n^{n+1} \varphi(t)dt + \frac{\epsilon^2}{4}; \]
The set
\[ C_n = \bigcup_{i=1}^{r} E_i \]
possesses therefore the property that
\[ (3) \quad \mu(C_n) \leq \frac{1}{\epsilon} \int_n^{n+1} \varphi(t)dt + \frac{\epsilon}{4}, \]
and
\[ \varphi(t) < \epsilon \quad \text{for } t \in (n, n+1), \ t \in C_n. \]
Let
\[ C = \bigcup_{n=-\infty}^{\infty} C_n, \]
and let \( E = \tilde{C} \) be the complementary of \( C \). Then, clearly
\[ (4) \quad \varphi(t) < \epsilon \quad \text{for } t \in E. \]
Also, by (3) and (1, S)

\[(5) \quad S(\tilde{E}) = S(C) \leq \frac{2}{\epsilon} \sup \int_{n}^{n+1} \varphi(t) dt + 2 \frac{\epsilon}{4} < \epsilon.\]

Since \(\epsilon > 0\) is arbitrary, relations (4) and (5) show that \(f(x)\) satisfies the condition of the theorem.

If we start with a R.W.a.p. or a R.B.a.p. function, relation (1, S) should be replaced by

\[(1, W) \quad \lim_{l \to \infty} \sup_{x} \frac{1}{l} \int_{x}^{x+l} \varphi(t) dt < \frac{\epsilon^2}{4},\]

or

\[(1, B) \quad \lim_{l \to \infty} \frac{1}{2l} \int_{-l}^{l} \varphi(t) dt < \frac{\epsilon^2}{4},\]

respectively. Relation (4) is still true, but (3) would then give easily

\[W(C) < \epsilon\]

or

\[B(C) < \epsilon\]

respectively.

The necessity is now proved.

**Sufficiency.** The sufficiency of the condition of the theorem is immediate if we take into account the boundedness of \(f(x)\).

**Lemma.** Let \(f(x)\) be a real function and \(E \subset E'\) be two subsets of \(R\). Let

\[|f(x)| \leq M \quad \text{for} \quad x \in E'.\]

Let \(\epsilon > 0\), \(\delta > 0\), \(\pi_1, \ldots, \pi_m\) be numbers such that

\[|f(x) - f(x')| < \epsilon\]

provided \(x \in E, x' \in E'\) and

\[|x - x'| < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m).\]

Then there exists a Bohr a.p. function \(q(x)\) such that

(i) \(f(x) \leq q(x) \leq M\) for \(x \in E'\)

and

(ii) \(|f(x) - q(x)| \leq \epsilon\) for \(x \in E\).

**Proof.** Denote by \(T_k\) the additive group of reals modulo \(\pi_k\) and let \(\varphi_k(x)\) be the canonical homomorphism of \(R\) on \(T_k\).
is metrized and the distance between two elements $\xi, \bar{\xi}$ will be denoted by $\varrho_k(\xi, \bar{\xi})$. We introduce in $R$ a new distance $\varrho(x, \bar{x})$ defined as follows

$$\varrho(x, \bar{x}) = \sum_{k=1}^{m} \varrho_k(\varphi_k(x), \varphi_k(\bar{x})).$$

It is clear that to every $\alpha > 0$ there corresponds a $\beta > 0$ such that

$$\varrho(x, \bar{x}) < \beta$$

implies

$$|x - \bar{x}| < \alpha \pmod{\pi_k} \quad (k = 1, \ldots, m).$$

Conversely, to every $\beta > 0$ corresponds an $\alpha > 0$ such that relations (2) imply relation (1).

We now put for a positive integer $n$

$$f_n(x) = \sup_{x' \in E'} [f(x') - n\varrho(x, x')].$$

We shall show that $f_n(x)$ is a Bohr a.p. function. In fact

$$f_n(\bar{x}) = \sup_{x' \in E'} [f(x') - n\varrho(\bar{x}, x')].$$

Hence, for $x' \in E'$

$$f_n(\bar{x}) \geq f(x') - n\varrho(\bar{x}, x').$$

$$f_n(\bar{x}) \leq f(x') + n\varrho(\bar{x}, x') \leq f(x') + n\varrho(\bar{x}, x) + n\varrho(x, x')$$

$$f(x') - n\varrho(x, x') \leq f_n(\bar{x}) + n\varrho(\bar{x}, x).$$

This relation holding for any $x' \in E'$, we conclude

$$f_n(x) \leq f_n(\bar{x}) + n\varrho(x, \bar{x}).$$

In the same way we prove

$$f_n(\bar{x}) \leq f_n(x) + n\varrho(\bar{x}, x),$$

so that

$$|f_n(x) - f_n(\bar{x})| \leq n\varrho(x, \bar{x}).$$

Let $\eta > 0$ be given; take $\beta = \eta/n$ and let $\alpha > 0$ be the number associated to $\beta$ in such a way that relations (2) imply relation (1). Relations (2) imply therefore

$$|f_n(x) - f_n(\bar{x})| \leq n(\eta/n) = \eta,$$

and this proves that $f_n(x)$ is a Bohr a.p. function.
It is clear that (whatever be $n$)

(i$'$) \[ f(x) \leq f_n(x) \leq f \quad \text{for} \quad x \in E'. \]

To complete the proof of the lemma we shall show that for some $n$ we have

(ii$'$) \[ |f(x) - f_n(x)| \leq \epsilon \quad \text{for} \quad x \in E. \]

In fact, by hypothesis

\[ f(x') \leq f(x) + \epsilon \]

provided $x \in E$, $x' \in E'$ and

\[ |x - x'| < \delta \quad \text{(mod} \; \pi_k) \quad (k = 1, \ldots, m). \]

Let $\delta' > 0$ be the number associated to $\delta$ in such a way that

\[ q(x, x') < \delta' \]

implies relations (4). Thus relation (5), combined with $x \in E$, $x' \in E'$ implies (3).

Take $n$ such that $n\delta' > 2M$; then, for a fixed $x \in E \subset E'$ we have

\[ \sup_{x' \in E', q(x, x') \geq \delta'} [f(x') - nq(x, x')] < -M \leq f_n(x). \]

We conclude

\[ f_n(x) = \sup_{x' \in E', q(x, x') < \delta'} [f(x') - nq(x, x')], \]

\[ f_n(x) \leq \sup_{x' \in E', q(x, x') < \delta'} [f(x')], \]

so that, by (3)

\[ f_n(x) \leq f(x) + \epsilon \quad \text{(for} \quad x \in E). \]

This, combined with (i$'$) gives the required relation (ii$'$).

The lemma is now proved.

Proof of theorem 2

Necessity. Let $f(x)$ be a R.S.a.p. function. It will suffice to prove that to every $\epsilon > 0$ we can associate two Bohr a.p. functions $p(x)$ and $q(x)$ satisfying conditions (i) and (ii) of the theorem. Moreover, we can suppose that $f(x)$ is real.

Let

\[ |f(x)| < M \quad \text{for} \quad x \in R. \]

Let $\epsilon > 0$ be given. By theorem 1 we can find a measurable
set $E$ and numbers $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that

\[(1) \quad S(\tilde{E}) < \frac{\epsilon}{6M}\]

and such that

\[|f(x) - f(x + \tau)| < \frac{\epsilon}{3}\]

provided $x \in E$ and

\[|\tau| < \delta \text{ (mod } \pi_k) \quad (k = 1, \ldots, m)\].

By the lemma, taking $E' = R$ we can find a Bohr a.p. function $q(x)$ such that

\[f(x) \leq q(x) \leq M \quad \text{for } x \in R\]

and

\[|q(x) - f(x)| \leq \frac{\epsilon}{3} \quad \text{for } x \in E.\]

In the same way we can find a Bohr a.p. function $p(x)$ such that

\[-M \leq p(x) \leq f(x) \quad \text{for } x \in R\]

and

\[|f(x) - p(x)| \leq \frac{\epsilon}{3} \quad \text{for } x \in E.\]

Then, by (1)

\[\sup_{x} \int_{x}^{x+1} |q(t) - p(t)| \, dt < \frac{2\epsilon}{3} + 2M \frac{\epsilon}{6M} = \epsilon.\]

For R.W.a.p. or R.B.a.p. functions the proof is quite similar.

**Sufficiency.** Let $f(x)$ satisfy the condition of the theorem. Let $\epsilon > 0$ be given and let $p(x), q(x)$ be two Bohr a.p. functions such that

\[p(x) \ll f(x) \ll q(x)\]

and

\[(1) \quad \sup_{x} \int_{x}^{x+1} |q(t) - p(t)| \, dt < \frac{\epsilon}{3}.\]

Choose $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that

\[|p(t + \tau_i) - p(t)| < \frac{\epsilon}{5}\]

\[|q(t + \tau_i) - q(t)| < \frac{\epsilon}{5}\]
provided

(2) \[| \tau_t | < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m).\]

Then
\[
| f(t + \tau_t) - f(t) | \leq | f(t + \tau_t) - p(t + \tau_t) | + | p(t + \tau_t) - p(t) | \\
+ | f(t) - p(t) |
\]
\[
\leq | q(t + \tau_t) - p(t + \tau_t) | + | p(t + \tau_t) - p(t) | + | q(t) - p(t) | \\
+ | p(t + \tau_t) - p(t) | + | q(t) - p(t) |.
\]

Thus, relations (2) imply
\[
| f(t + \tau_t) - f(t) | \leq \frac{3\epsilon}{5} + 2 | q(t) - p(t) |.
\]

The same relations (2), therefore, imply by (1)
\[
\sup_{x} \int_{x}^{x+1} | f(t + \tau_t) - f(t) | dt \leq \frac{3\epsilon}{5} + 2 \frac{\epsilon}{5} = \epsilon.
\]

This proves, since \( \epsilon \) is arbitrary, that \( f(x) \) is a R.S.a.p. function.

For the R.W.a.p. or the R.B.a.p. classes the proof is quite similar.

Proof of theorem 3

Necessity. Let \( f(x) \) be a K.S.a.p. function and let \( \epsilon > 0 \) be given. We can find a measurable set \( E \) and numbers \( \delta > 0, \pi_1, \ldots, \pi_m \) for which \( S(\tilde{E}) < \epsilon \), and
\[
| f(x) - f(x') | < \epsilon
\]
provided \( x \in E, x' \in E \) and
\[
| x - x' | < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m).
\]
If we show that there is a constant \( M \) such that
\[
| f(x) | \leq M \quad \text{for} \quad x \in E,
\]
then, by the lemma, taking \( E' = E \) we can find a Bohr a.p. function \( q(x) \) such that
\[
| f(x) - q(x) | \leq \epsilon \quad \text{for} \quad x \in E,
\]
and the condition of the theorem will be proved.

So suppose there is a sequence \( x_n \) of points of \( E \) for which
\[
(1) \quad \lim_{n \to \infty} | f(x_n) | = \infty.
\]
We can extract from \( x_n \) a subsequence \( \tilde{x}_n \) such that whatever be \( p, q \)
Then
\[ | \tilde{x}_p - \tilde{x}_q | < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m). \]

but this is incompatible with (1).

For the K.W.a.p. or the K.B.a.p. functions the proof is quite similar.

**Sufficiency.** Let \( f(x) \) satisfy the condition of the theorem. Let \( \varepsilon > 0 \) be given. Let the polynomial \( q(x) \) and the measurable set \( E \) be such that
\[ S(E) < \frac{\varepsilon}{3} \]
and
\[ (1) \quad | f(x) - q(x) | < \frac{\varepsilon}{3} \quad \text{for} \quad x \in E. \]

We can find a \( \delta > 0 \) and numbers \( \pi_1, \ldots, \pi_m \) such that
\[ | q(x) - q(x') | < \frac{\varepsilon}{3} \]
provided
\[ (3) \quad | x - x' | < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m). \]

If \( x \in E, x' \in E \) and if relations (3) hold, then, by (1) and (2)
\[ | f(x) - f(x') | \leq | f(x) - q(x) | + | q(x) - q(x') | + | q(x') - f(x') | \]
\[ \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \]

\( f(x) \) is thus a K.S.a.p. function.

For the K.W.a.p. or the K.B.a.p. classes the proof is quite similar.

**Proof of theorem 4**

**Necessity.** Suppose that \( f(x) \) is a R.B.a.p. function. Let \( \varepsilon, \ell \) be a sequence of positive numbers tending to 0 and let \( p_l(x), q_l(x) \) be the polynomials associated by theorem 2 to \( \varepsilon, \ell \). Let \( \lambda_\ell \) be a sequence containing all the non-vanishing exponents of each of the polynomials \( p_l(x), q_l(x), l = 1, 2, \ldots \). Choose \( \xi \) such that \( \xi \lambda_\ell / 2\pi \) is never an integer. We may suppose, by considering separately the real and imaginary parts, that \( f(x) \) is real and that \( p_l(x), q_l(x) \) are real Bohr a.p. functions.

Let
\[ M = \overline{\text{bound} \mathcal{M}\{p_l(x)\}} = \text{bound} \mathcal{M}\{q_l(x)\}. \]
\( \varepsilon > 0 \) being given, let \( p(x), q(x) \) be two real Bohr a.p. functions
chosen among the $p_i(x)$, $q_i(x)$ such that

(1) \[ p(x) \leq f(x) \leq q(x) \]

(2) \[ M\{q(x) - p(x)\} < \frac{\epsilon}{3}. \]

We put

\[ p_0 = M\{p(x)\}, \quad q_0 = M\{q(x)\}. \]

Then, by (1) and (2)

(3) \[ q_0 - \frac{\epsilon}{3} < M < p_0 + \frac{\epsilon}{3}. \]

We see easily, in view of our choice of $\xi^6$), that there exists a number $n_0$ such that for $n \geq n_0$ and every $x_0$

(4) \[ p_0 - \frac{\epsilon}{3} \leq \frac{1}{n} \sum_{i=0}^{n-1} p(x_0 + i\xi) \]

(5) \[ \frac{1}{n} \sum_{i=0}^{n-1} q(x_0 + i\xi) \leq q_0 + \frac{\epsilon}{3}. \]

Also we can find a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

\[ |p(x_0) - p(x_i)| < \frac{\epsilon}{3} \quad \text{and} \quad |q(x_0) - q(x_i)| < \frac{\epsilon}{3}, \]

provided

\[ |x_0 - x_i| < \delta \quad (\text{mod} \ \pi_k) \quad (k = 1, \ldots, m). \]

We conclude, by (3), (4) and (5) that

\[ M - \epsilon < p_0 - \frac{2\epsilon}{3} \leq \frac{1}{n} \sum_{i=0}^{n-1} p(x_i + i\xi) \]

\[ \frac{1}{n} \sum_{i=0}^{n-1} q(x_i + i\xi) \leq q_0 + \frac{2\epsilon}{3} < M + \epsilon \]

provided

\[ |x_i - x_j| < \delta \quad (\text{mod} \ \pi_k) \quad \left( \begin{array}{c}
  (i, j = 0, \ldots, n-1) \\
  (k = 1, \ldots, m)
\end{array} \right) \]

Thus these last relations imply

\[ \left| \frac{1}{n} \sum_{i=0}^{n-1} f(x_i + i\xi) - M \right| < \epsilon. \]

**Sufficiency.** Suppose that $f(x)$ satisfies the condition of the theorem. We may suppose again that $f(x)$ is real.

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Let $\epsilon > 0$ be given, and let $n, \delta > 0, \pi_1, \ldots, \pi_m$ be such that
\[
\left| \frac{1}{n} \sum_{i=0}^{n-1} f(x_i + l\xi) - M \right| < \frac{\epsilon}{2}
\]
provided
\[
| x_i - x_j | < 2\delta \quad (i, j = 0, \ldots, n - 1) \quad (k = 1, \ldots, m).
\]
Let $\tau_x$ be such that
\[
(1) \quad | \tau_x | < \delta \quad (k = 1, \ldots, m).
\]
Then, if $\theta_x, \theta'_x$ are two functions which take only the values 0 and 1, we have, whatever be $x$
\[
\left| \frac{1}{n} \sum_{i=0}^{n-1} f(x_i + l\xi + \theta_x + \theta'_x \tau_x + l\xi) - M \right| < \frac{\epsilon}{2}
\]
Hence
\[
\left| \frac{1}{n} \sum_{i=0}^{n-1} \left[ f(x_i + l\xi + \theta_x + \theta'_x \tau_x + l\xi) - f(x_i + l\xi + \theta'_x \tau_x + l\xi) \right] \right| < \epsilon.
\]
By an appropriate choice of $\theta_x$ and $\theta'_x$ we see that
\[
\left| \frac{1}{n} \sum_{i=0}^{n-1} \left[ f(x_i + l\xi + \tau_x + l\xi) - f(x_i + l\xi) \right] \right| < \epsilon.
\]
Let $a$ be arbitrary and let $L = n\xi$. Then
\[
\frac{1}{L} \int_a^{a+L} \left| f(x + \tau_x) - f(x) \right| \, dx
\]
\[
= \frac{1}{n\xi} \sum_{i=0}^{n-1} \int_{a+i\xi}^{a+(i+1)\xi} \left| f(x + \tau_x) - f(x) \right| \, dx
\]
\[
= \frac{1}{n\xi} \sum_{i=0}^{n-1} \int_{a}^{a+i\xi} \left| f(x + \tau_x + \xi) - f(x + \xi) \right| \, dx
\]
\[
= \frac{1}{\xi} \int_{a}^{a+\xi} \frac{1}{n} \sum_{i=0}^{n-1} \left| f(x + l\xi + \tau_x + l\xi) - f(x + l\xi) \right| \, dx \leq \epsilon.
\]
Relations (1) therefore imply
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| f(x + \tau_x) - f(x) \right| \, dx \leq \epsilon,
\]
so that $f(x)$ is a R.B.a.p. function.

Remark. The proof shows that $f(x)$ is a R.W.a.p. function. Thus we see again that the R.W.a.p. and the R.B.a.p. classes are identical.

(Oblatum 1-10-'54).