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# Distribution modulo 1 of the powers of real numbers larger than 1 \*

by

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## 1. A class of algebraic integers

Since the fundamental work of H. Weyl in 1916 on the theory of uniform distribution modulo 1, exhaustive studies have been made about the distribution of  $f(n)$  when  $f(n)$  is a polynomial in  $n$ , or when  $f(n)$  increases slower than a polynomial in  $n$  (e.g.  $f(n) = n^\alpha$ , ( $\alpha > 0$ ) or  $f(n) = \log^\alpha n$ ).

In comparison, very little is known on the distribution of the simpler functions increasing faster than any power of  $n$ , in particular of the distribution of  $\theta^n$  when  $\theta$  is a real number larger than one. Whereas Koksma [1] proved in 1935 that  $\theta^n$  is uniformly distributed except for a set of  $\theta$  of measure zero, nothing is known for the simplest individual values of  $\theta$ , such as, e.g.  $\theta = e$ , or  $\theta = \frac{3}{2}$ .

Pisot, in 1938 ([2]) concentrated his work on the study of those values of  $\theta$ , for which, far from being uniformly distributed,  $\theta^n$  is, in a certain sense, as badly distributed as possible. It had been known, for a time, that if we denote by  $S$  the class of algebraic integers whose conjugates lie all (except  $\theta$  itself) *inside* the unit circle then  $\theta^n \pmod{1}$  tends to zero for  $n = \infty$ . This is true also for  $\lambda\theta^n$  when  $\lambda$  is any algebraic integer of the field  $K(\theta)$  generated by  $\theta$  over the rationals.

Let us denote by  $\|a\|$  the distance between  $a$  and the nearest integer. Pisot [2] proved that if  $\theta > 1$  is such that there exists a real  $\lambda$  with the property that

$$(1) \quad \sum \|\lambda\theta^n\|^2 < \infty$$

then  $\theta$  belongs to the class  $S$  and  $\lambda$  is an algebraic number of the field  $K(\theta)$ .

The question whether this important theorem is true if we replace (1) by the weaker condition

\* Nijenrode lecture.

† Professor Raphaël Salem died suddenly on the 20th of June 1963.

$$(2) \quad \|\lambda\theta^n\| \rightarrow 0 \quad (n = \infty)$$

is open, except if we know beforehand that  $\theta$  is algebraic [3], in which case we can again assert that  $\theta \in S$  and  $\lambda \in K(\theta)$ .

In other words, the open problem can be stated as follows: Do there exist transcendental  $\theta$  with the property that, for some  $\lambda$ , (2) holds?

## 2. Another remarkable class of algebraic integers

Instead of considering the class  $S$  of algebraic integers  $\theta$  such that all the conjugates of  $\theta$  (except  $\theta$  itself) have moduli strictly inferior to 1, Salem [4] has introduced the class  $T$  of algebraic integers  $\tau > 1$  such that the conjugates  $\alpha_K$  of  $\tau$  have their moduli  $|\alpha_K| \leq 1$ , the equality being permitted.

One sees immediately, (since if  $|\alpha_K| = 1$ ,  $\overline{\alpha_K} = 1/\alpha_K$ ), that if  $\tau$  does not belong to  $S$ , i.e. if there exist actually conjugates with moduli 1,  $\tau$  is the root of an irreducible reciprocal equation of even degree, whose roots lie all on the circumference of the unit circle, the only conjugate lying inside the unit circle being  $1/\tau$ .

The distribution modulo 1 of  $\tau^m$  has interesting properties:

1°) *The numbers  $\tau^m$  reduced modulo 1 are everywhere dense.*

The proof of this result is based on the following lemma. Let  $2K$  be the degree of  $\tau$  and  $\alpha_j = e^{2\pi i \omega_j}$ ,  $\bar{\alpha}_j = e^{-2\pi i \omega_j}$  ( $j = 1, \dots, K-1$ ) be the conjugates of  $\tau$  having moduli 1. Then  $\omega_1, \dots, \omega_{K-1}$  and 1 are linearly independent. It will be enough to prove that, if the  $A_j$  are rational integers,

$$(3) \quad \alpha_1^{A_1} \dots \alpha_{K-1}^{A_{K-1}} = 1$$

is an impossible equality.

Since the equation having the root  $\tau$  is irreducible, its Galois group is transitive, i.e. there exists an automorphism  $\sigma$  of the Galois group sending, e.g. the root  $\alpha_1$  into the root  $\tau$ . This automorphism cannot send  $\alpha_j$  ( $j \neq 1$ ) into  $1/\tau$  for since  $\sigma(\alpha_1) = \tau$ ,  $\sigma(1/\alpha_1) = 1/\tau$ , and thus this would imply  $\alpha_j = 1/\alpha_1$  which is not the case. Thus the automorphism applied to (3) would give

$$\tau^{A_1} \alpha_2^{A_2} \dots \alpha_{K-1}^{A_{K-1}} = 1$$

if  $\sigma(\alpha_j) = \alpha'_j$  ( $j \neq 1$ ). This is clearly impossible since  $\tau > 1$  and  $|\alpha'_j| = 1$ . Hence the proof of the linear independence of  $\alpha_1 \dots \alpha_{K-1}$  and 1.

Now, we have modulo 1

$$\tau^m + \frac{1}{\tau^m} + \sum_{j=1}^{K-1} (e^{2\pi i m \omega_j} + e^{-2\pi i m \omega_j}) \equiv 0$$

or

$$(4) \quad \tau^m + 2 \sum_{j=1}^{K-1} \cos 2\pi m \omega_j \rightarrow 0 \pmod{1}$$

as  $m \rightarrow \infty$ . But, by Kronecker's theorem and the linear independence just proved we can determine  $m$  arbitrarily large, such that

$$2 \sum_{j=1}^{K-1} \cos 2\pi m \omega_j$$

be arbitrarily close to any number given in advance (mod. 1). Take, e.g.  $m$  such that

$$\begin{aligned} |m\omega_1 - \delta| &< \varepsilon \pmod{1} \\ |m\omega_j - \frac{1}{4}| &< \varepsilon \pmod{1} \quad (j = 2, 3, \dots, K-1). \end{aligned}$$

Thus  $\{\tau^m\} \pmod{1}$  is everywhere dense.

(The same argument, applied to  $\lambda\tau^m$ ,  $\lambda$  being an integer of  $K(\tau)$  shows that  $\{\lambda\tau^m\}$  is everywhere dense in a certain interval).

2°) *The powers  $\tau^m$  of a number  $\tau$  of the class  $T$ , although everywhere dense modulo 1, are not uniformly distributed modulo 1.*

We prove first the following lemma.

LEMMA. If the  $p$ -dimensional vector  $\{u_n^j\}_{n=1}^\infty$  ( $j = 1, 2, \dots, p$ ) is uniformly distributed (mod. 1) in the  $p$ -dimensional euclidean space  $R^p$ , the sequence

$$v_n = \omega(u_n^1) + \omega(u_n^2) + \dots + \omega(u_n^p),$$

where  $\omega(x)$  is continuous with period 1, is uniformly distributed if and only if the condition

$$\int_0^1 e^{2\pi i h \omega(x)} dx = 0$$

is satisfied for all integers  $h \neq 0$ .

(A different form of this condition is that the distribution function of  $\omega(x) \pmod{1}$  should be linear.)

To prove the lemma, remark that  $\{v_n\}$  is uniformly distributed (mod. 1) if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h v_n} = 0 \quad (h \text{ integer } \neq 0).$$

But

$$\frac{1}{N} \sum_{n=1}^N e^{2\pi i h \{\omega(u_n^1) + \dots + \omega(u_n^p)\}} \rightarrow \left\{ \int_0^1 e^{2\pi i h \omega(x)} dx \right\}^p.$$

Now in our case, in view of (4) we have to prove that the sequence

$$v_m = 2 \cos 2\pi m \omega_1 + \dots + 2 \cos 2\pi m \omega_{K-1}$$

is not uniformly distributed (mod. 1).

In view of the lemma it is enough to prove that

$$\int_0^1 e^{4\pi i h \cos 2\pi x} dx = J_0(4\pi h),$$

$J_0$  being the Bessel function of order zero, does not vanish for all integers  $h \neq 0$ , which is true.

(If we use the second form of the lemma, we would have to show that the distribution function (mod. 1) of  $2 \cos 2\pi x$  is not linear, which can be proved by direct computation.)

### *Unsolved problems.*

Besides the unsolved problem quoted at the end of § 1, we quote the following one:

I) Salem has proved [5] that the set of all numbers  $\theta$  of the class  $S$  is closed.

Little is known about the set of numbers  $\tau$  of the class  $T$ , except [4] that every number  $\theta$  of the class  $S$  is a limit point (on both sides) for numbers of  $T$ . It is not known what are the other limit points of  $T$ , if any.

II) Pisot [6] has proved that if there exists  $\lambda \geq 1$  such that

$$(5) \quad \|\lambda \theta^n\| \leq \varepsilon \quad \text{for all } n = 0, 1, \dots,$$

where

$$\varepsilon = \frac{1}{2e\theta(\theta+1)(1+\log \lambda)}$$

then  $\theta \in S \cup T$ .

It is still an open question whether there exists a theorem including both results (1) and (5).

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