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Certain theorems on n -dimensional operational calculus

by

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In the present paper we obtain a few theorems in n -dimensional operational calculus starting from given operational relations in one variable. These theorems are next applied to the evaluation of a few operational results. In the theorems, we shall take $D(\sigma_{10}, \dots, \sigma_{n0})$ to represent the set of points for which $R(P_i) > R(P_{i0}) = \sigma_{i0} > 0$ and unless otherwise stated, D will represent the set of points for which $R(P_i) > 0$, $i = 1, 2, \dots, n$.

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THEOREM 1. Let

- (i) $\phi(p) \subset f(x)$
- (ii) $p\sigma(p) \subset f(x^2)$
- (iii) $H(p) \subset x^{-n}\phi(1/x^2)$

where $f(x)$, $f(x^2)$ and $x^{-n}\phi(1/x^2)$ are each integrable L in $(0, \infty)$ or the definition integral in each of (i), (ii), (iii) is absolutely convergent for $R(P) > 0$.

If x_1, \dots, x_n , be a set of real numbers > 0 , then

$$(2.1) \quad \frac{\sigma(\sqrt{1/x_1 + \dots + 1/x_n})}{(x_1 \dots x_n)^{\frac{n}{2}}} \supset_n \frac{\pi^{(n-1)/2} 2^n (p_1 \dots p_n)}{\sqrt{p_1 + \dots + p_n}} \cdot H(\sqrt{p_1 + \dots + p_n})$$

provided that

(a) the integral in (2.1) exists as an absolutely convergent integral for $(p_1, \dots, p_n) \in D(\sigma_{10}, \dots, \sigma_{n0})$; or

(b) the original in (2.1) is integrable L in $(0, \infty)$ with respect to the variables x_1, \dots, x_n ; (the operational variables p_1, \dots, p_n corresponding to x_1, \dots, x_n).

From (i), we have

$$(2.2) \quad f(x^2) \supset \frac{p}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{2}p^2 u^2} \phi\left(\frac{1}{u^2}\right) du$$

where the integrals on the right is absolutely convergent. So that, by (ii), we can write (2.2) as

$$(2.3) \quad \sigma(p) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{2}p^2 u^2} \phi\left(\frac{1}{u^2}\right) du$$

writing (2.3) in the form

$$(2.4) \quad \frac{\pi^{(1-n)/2} \sigma(\sqrt{1/x_1 + \dots + 1/x_n})}{(x_1 \dots x_n)^{\frac{1}{2}}} \\ = \pi^{-n/2} \int_0^\infty (x_1 \dots x_n)^{-\frac{1}{2}} \exp\left(-\frac{1}{4} \sum \frac{u^2}{x_i}\right) \phi\left(\frac{1}{u^2}\right) du.$$

We multiply both sides by $\exp(-\sum p_i x_i)$, integrate with respect to x_i between the limits $(0, \infty)$ and then change the order of integration in the resulting integral on the right, permissible by Fubini's theorem, on account of the absolute convergence.

This gives

$$(2.5) \quad \pi^{(1-n)/2} \int_0^\infty \dots \int_0^\infty \exp(-\sum p_i x_i) \frac{\sigma(\sqrt{1/x_1 + \dots + 1/x_n})}{(x_1 \dots x_n)^{\frac{1}{2}}} dx_1 \dots dx_n \\ = \pi^{-n/2} \int_0^\infty \phi\left(\frac{1}{u^2}\right) du \int_0^\infty x_1^{-\frac{1}{2}} \exp\left(-\frac{1}{4} \frac{u^2}{x_1} - p_1 x_1\right) dx_1 \dots \\ \int_0^\infty x_n^{-\frac{1}{2}} \exp\left(-p_n x_n - \frac{1}{4} \frac{u^2}{x_n}\right) dx_n.$$

Evaluating the inner integrals on the right by

$$(A) \quad \int_0^\infty x^{-\frac{1}{2}} e^{-px - \frac{1}{4}(a^2)/x} dx = \frac{2\sqrt{\pi}}{a} e^{-a\sqrt{p}}.$$

We obtain, on account of the operational relation (iii), the required theorem.

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THEOREM 2. *Let*

- (i) $\phi(p) \subset f(x)$
 (ii) $\psi(p) \subset x^{-\frac{1}{2}}\phi\left(\frac{1}{x}\right)$
 (iii) $\lambda(p) \subset x^{1-n}f(x^4)$

where $f(x)$, $x^{-\frac{1}{2}}\phi(1/x)$ & $x^{1-n}f(x^4)$ are each integrable L in $(0, \infty)$ or the definition integral in each of (i), (ii) and (iii) is absolutely convergent for $R(P) > 0$. If x_1, \dots, x_n be a set of positive real numbers, then

$$(3.1) \quad \frac{\psi\left[\frac{1}{64}(1/x_1 + \dots + 1/x_n)^2\right]}{(x_1 \dots x_n)^{\frac{1}{2}}(1/x_1 + \dots + 1/x_n)^2} \supset_n \frac{2^{n-4}\pi^{(n+1)/2}(p_1 \dots p_n)}{\sqrt{p_1 + \dots + p_n}} \cdot \lambda(\sqrt{p_1 + \dots + p_n})$$

provided that

- (a) the definition integral in (3.1) is absolutely convergent for $(p_1, \dots, p_n) \in D(\sigma_{10}, \dots, \sigma_{n0})$; or
 (b) the original in (3.1) is integrable L in $(0, \infty)$ with respect to the variables x_1, \dots, x_n ; (the operational variables p_1, \dots, p_n corresponding to x_1, \dots, x_n).

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THEOREM 3. *Let*

- (i) $\phi(p) \subset f(x)$
 (ii) $\phi(\sqrt{p}) \subset F\left(\frac{1}{x}\right)$
 (iii) $H(p) \subset \frac{1}{x^n}f(x)$.

Then

$$(4.1) \quad \frac{F(1/x_1 + \dots + 1/x_n)}{(x_1 \dots x_n)^{\frac{1}{2}}(1/x_1 + \dots + 1/x_n)^{\frac{1}{2}}} \supset_n \frac{2^n \pi^{(n-1)/2}(p_1 \dots p_n)}{\sqrt{p_1 + \dots + p_n}} \cdot H(\sqrt{p_1 + \dots + p_n}).$$

The proofs of theorems 2 and 3 are on the same lines.

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As applications of theorems 1, 2 and 3, we shall find some two-dimensional correspondences.

(a) Let

$$f(x) = J_{2n,n}(\mathfrak{B}\sqrt[n]{x}); \text{ so that } \phi(p) = J_{2n}\left(\frac{2}{\sqrt{p}}\right)$$

$$f(x^2) = J_{2n,n}(\mathfrak{B}\sqrt[n]{x^2}); \text{ so that } \sigma(p) = \frac{1}{p} e^{-(2/p^2)} I_n\left(\frac{2}{p^2}\right)$$

and $n = 2$, then

$$x^{-n}\phi\left(\frac{1}{x^2}\right) = x^{-2}J_{2n}(2x);$$

so that

$$H(p) = \frac{\Gamma(2n-1)}{\Gamma(2n+1)p^{2n-2}} {}_2F_1\left[\begin{matrix} n-\frac{1}{2}, n; \\ 2n+1; \end{matrix} -\frac{4}{p^2}\right].$$

Hence we obtain, from (2.1)

$$\begin{aligned} \frac{e^{-(2xy)/(x+y)}}{xy(x+y)^{\frac{1}{2}}} I_n\left(\frac{2xy}{x+y}\right) \supset_2 \frac{2\sqrt{\pi}}{n(2n-1)} \frac{pq}{(\sqrt{p}+\sqrt{q})^{2n-1}} \\ \cdot {}_2F_1\left[\begin{matrix} n-\frac{1}{2}, n; \\ 2n+1; \end{matrix} -\frac{4}{(\sqrt{p}+\sqrt{q})^2}\right]. \end{aligned}$$

(b) Let

$$f(x) = x^{\nu/2} J_{\nu}(2a\sqrt{x});$$

so that

$$\phi(p) = a^{\nu} p^{-\nu} e^{-a^2/p},$$

$$f(x^2) = x^{\nu} J_{\nu}(2ax);$$

so that

$$p\sigma(p) = \frac{a^{\nu} 2^{2\nu} \Gamma(\nu+\frac{1}{2})p}{\sqrt{\pi}(p^2+4a^2)^{\nu+\frac{1}{2}}}, \quad R(\nu) > -\frac{1}{2},$$

$$x^{-n}\phi\left(\frac{1}{x^2}\right) = a^{\nu} x^{2\nu-2} e^{-a^2 x^2};$$

$$\text{so that } H(p) = \frac{\Gamma(2\nu-2)p}{(8a^2)^{\nu-\frac{1}{2}}} e^{(p^2/8a^2)} D_{2\nu-1}\left(\frac{p}{\sqrt{2a}}\right), \quad R(a) < 0.$$

Hence from (2.1), we get

$$\begin{aligned} \frac{(xy)^{\nu-1}}{(x+y+4a^2xy)^{\nu+\frac{1}{2}}} \supset_2 \frac{\sqrt{\pi}\Gamma(2\nu-1)}{a^{3\nu-1}2^{5\nu-\frac{1}{2}}} \\ pq e^{(\sqrt{p}+\sqrt{q})^2/8a^2} D_{2\nu-1}\left(\frac{\sqrt{p}+\sqrt{q}}{\sqrt{2a}}\right), \quad \begin{matrix} R(\nu) > -\frac{1}{2} \\ R(a) < 0 \end{matrix}. \end{aligned}$$

(c) Let

$$f(x) = \sqrt{x} L_1(\sqrt{x});$$

so that
$$\phi(p) = \frac{1}{2p} e^{\frac{1}{2}p} \operatorname{Erf} e \left(\frac{1}{2\sqrt{p}} \right)$$

$$f(x^2) = x L_1(x);$$

so that
$$p\sigma(p) = \frac{p}{(p^2-1)^{\frac{3}{2}}} - \frac{2\sqrt{2}p^{-\frac{1}{2}}}{\sqrt{\pi}(1-p^2)^{-\frac{1}{2}}} P^{-\frac{1}{2}} \left(\frac{1}{p} \right)$$

$$x^{-n} \phi \left(\frac{1}{x^2} \right) = \frac{1}{2} e^{x^2/4} \operatorname{Erf} c \left(\frac{x}{2} \right);$$

so that
$$H(p) = \frac{p}{2\sqrt{\pi}} e^{-p^2} \operatorname{Ei}(p^2).$$

Hence from (2.1), we get

$$\begin{aligned} & 2pq e^{-(\sqrt{p}+\sqrt{q})^2} \operatorname{Ei}(\sqrt{p}+\sqrt{q}) C_2 \frac{1}{(x+y-xy)^{\frac{3}{2}}} \\ & - \frac{2\sqrt{2}(xy-x-y)^{\frac{3}{2}}}{\sqrt{\pi}(xy)^{\frac{3}{2}}(x+y)^{\frac{3}{2}}} P^{-\frac{1}{2}} \left(\sqrt{\frac{xy}{x+y}} \right). \end{aligned}$$

(d) Let

$$f(x) = \frac{x^\nu}{\Gamma(\nu+1)};$$

so that
$$\sigma(p) = \frac{1}{p^\nu}; \quad R(\nu) > -1.$$

$$f(x^2) = \frac{x^{2\nu}}{\Gamma(\nu+1)};$$

so that
$$\sigma(p) = \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)} p^{-1-2\nu}, \quad R(\nu) > -\frac{1}{2}$$

and $n = 3$, then

$$x^{-n} \phi \left(\frac{1}{x^2} \right) = x^{2\nu-3}; \quad \text{so that } H(p) = \Gamma(2\nu-2) p^{3-2\nu}.$$

Thus we obtain, from (2.1)

$$\begin{aligned} & \frac{(xyz)^{\nu-1}}{(xy+yz+zx)^{\nu+\frac{1}{2}}} \supset_3 \frac{8\pi\Gamma(2\nu-2)\Gamma(\nu+1)}{\Gamma(2\nu+1)} \\ & \times \frac{pqr}{(\sqrt{p}+\sqrt{q}+\sqrt{r})^{2\nu-2}}, \quad R(\nu) > -\frac{1}{2}. \end{aligned}$$

(e) Let

$$f(x) = (2x)^{\nu-1} e^{-2\sqrt{x}};$$

so that
$$\phi(p) = \Gamma(2\nu) p^{1-\nu} e^{1/2\nu} D_{-2\nu} \left(\sqrt{\frac{2}{p}} \right)$$

$$x^{-\frac{1}{2}} \phi(x) = \Gamma(2\nu) x^{\nu-\frac{3}{2}} e^{x/2} D_{-2\nu}(\sqrt{2x});$$

so that
$$\psi(p) = \frac{\sqrt{2\pi} \Gamma(2\nu-1) p}{2^{\nu-\frac{1}{2}} (p-1)^{\nu-\frac{1}{2}}} P_{-1}^{1-2\nu} \left(\frac{1}{\sqrt{p}} \right), R(\nu) > \frac{1}{2}$$

and $n = 2$, then

$$x^{1-n} f(x^4) = 2^{\nu-1} x^{4\nu-5} e^{-2x^2};$$

so that
$$\lambda(p) = \frac{\Gamma(4\nu-4)}{2^{8(\nu-1)}} p e^{(p^2/16)} D_{4-4\nu} \left(\frac{p}{2} \right).$$

Hence from theorem 2, we obtain

$$\begin{aligned} & \frac{(xy)^{2\nu-\frac{1}{2}}}{[(x^2+y^2)-64x^2y^2]^{\nu-\frac{1}{2}}} P_{-1}^{1-2\nu} \left(\frac{xy}{8(x+y)} \right) \\ & \supset_2 \frac{\pi \Gamma(4\nu-4) pq}{2^{8\nu-4} \Gamma(2\nu-1)} e^{(\sqrt{p}+\sqrt{q})^2/16} D_{4-4\nu} \left(\frac{\sqrt{p}+\sqrt{q}}{2} \right), R(\nu) > +\frac{1}{2}. \end{aligned}$$

(f) Let

$$f(x) = x^{\nu-1} e^{-(x^2/8a)};$$

so that
$$\phi(p) = 2^\nu a^{\nu/2} \Gamma(\nu) p e^{ap^2} D_{-\nu}(2\sqrt{ap})$$

$$\phi(\sqrt{p}) = \Gamma(\nu) 2^\nu a^{\nu/2} \sqrt{p} e^{ap} D_{-\nu}(2\sqrt{ap});$$

so that
$$F \left(\frac{1}{x} \right) = \frac{\Gamma(\nu) (2a)^{\nu/2}}{\Gamma((\nu+1)/2)} x^{(\nu-1)/2} (x+2a)^{-\nu/2}$$

and $n = 2$, then

$$x^{-n} f(x) = x^{\nu-3} e^{-(x^2/8a)};$$

so that
$$H(p) = 2^{\nu-2} a^{(\nu/2)-1} \Gamma(\nu-2) p e^{ap^2} D_{2-\nu}(2\sqrt{ap}).$$

Hence from (4.1), we obtain

$$\begin{aligned} & \frac{(xy)^{(\nu-3)/2}}{(x+y)^{\nu/2} (xy/(x+y) + 2a)^{(\nu+1)/2}} \supset_2 \frac{\sqrt{\pi} 2^{\nu/2} \Gamma((\nu+1)/2)}{a(\nu-1)(\nu-2)} pq \\ & \exp(a(\sqrt{p}+\sqrt{q})^2) D_{2-\nu}[2\sqrt{a}(\sqrt{p}+\sqrt{q})] R(\nu) > -2. \end{aligned}$$

(g) Let

$$f(x) = x^{\nu+1} J_{\nu}(ax);$$

so that
$$\phi(p) = \frac{2^{\nu+1} \Gamma(\nu + \frac{3}{2}) a^{\nu} p^2}{\sqrt{\pi} (p^2 + a^2)^{\nu + \frac{3}{4}}}, R(\nu) > -\frac{1}{2}.$$

$$\phi(\sqrt{p}) = \frac{2^{\nu+1} \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi}} \frac{a^{\nu} p}{(p + a^2)^{\nu + \frac{3}{4}}};$$

so that
$$F\left(\frac{1}{x}\right) = \frac{2^{\nu+1} a^{\nu}}{\sqrt{\pi}} x^{\nu + \frac{1}{2}} e^{-a^2 x}$$

and $n = 2$, then

$$x^{-n} f(x) = x^{\nu-1} J_{\nu}(ax);$$

so that
$$H(p) = \frac{\Gamma(2\nu)p}{(p^2 + a^2)^{(\nu/2)}} P_{\nu-1}^{-\nu} \left(\frac{p}{\sqrt{p^2 + a^2}} \right).$$

Hence from (4.1), we get

$$\frac{(xy)^{\nu-\frac{1}{2}}}{(x+y)^{\nu+1}} e^{-a^2 xy/(x+y)} \supset_2 \frac{\pi \Gamma(2\nu)}{2^{\nu-1} a^{\nu} [(\sqrt{p} + \sqrt{q})^2 + a^2]^{-(\nu/2)}} P_{\nu-1}^{-\nu} \left[\frac{\sqrt{p} + \sqrt{q}}{\sqrt{(\sqrt{p} + \sqrt{q})^2 + a^2}} \right], R(\nu) > -\frac{1}{2}.$$

(h) Let

$$f(x) = x^3 J_{\frac{3}{4}}^2(x);$$

so that
$$\phi(p) = \frac{5 \cdot 2^5}{\pi p^6} {}_2F_1 \left[2, \frac{7}{2}; \frac{5}{2}; -\frac{4}{p^2} \right]$$

$$\phi(\sqrt{p}) = \frac{5 \cdot 2^5}{\pi p^3} {}_2F_1 \left[2, \frac{7}{2}; \frac{5}{2}; -\frac{4}{p} \right];$$

so that
$$F\left(\frac{1}{x}\right) = \frac{5 \cdot 2^5 x^3}{\pi \Gamma(4)} {}_2F_2 \left[2, \frac{7}{2}; \frac{5}{2}, 4; -4x \right]$$

and $n = 2$, then

$$x^{-n} f(x) = x J_{\frac{3}{4}}^2(x);$$

so that
$$H(p) = \frac{2^4}{\pi \Gamma(4) p^5} {}_2F_1 \left[([2, 3; 4, -\frac{4}{p^2}] \right).$$

Hence from (4.1), we get

$$\frac{(xy)^2}{(x+y)^{\frac{7}{2}}} {}_2F_2 \left[2, \frac{7}{2}; \frac{5}{2}, 4; -\frac{4(x+y)}{xy} \right] \supset_2 \frac{4\sqrt{\pi} pq}{5(\sqrt{p}+\sqrt{q})^5} {}_2F_1 \left[2, 3; 4; -\frac{4}{(\sqrt{p}+\sqrt{q})^2} \right].$$

(i) In the theorem taking $n = 3$ and let

$$f(x) = \frac{x^\nu}{\Gamma(\nu+1)};$$

so that $\phi(p) = \frac{1}{p^\nu}$, $R(\nu) > -1$

$$x^{-n}f(x) = \frac{x^{\nu-3}}{\Gamma(\nu+1)};$$

so that $H(p) = \frac{\Gamma(\nu-2)}{\Gamma(\nu+1)p^{\nu-3}}$

$$\phi(\sqrt{p}) = p^{-(\nu/2)};$$

so that $F\left(\frac{1}{x}\right) = \frac{x^{\nu/2}}{\Gamma((\nu+2)/2)}$.

Hence from (4.1), we get

$$\frac{(xyz)^{(\nu-2)/2}}{(xy+yz+zx)^{(\nu+1)/2}} \supset_3 \frac{8\pi\Gamma((\nu+2)/2)}{\nu(\nu-1)(\nu-2)} \times \frac{pqr}{(\sqrt{p}+\sqrt{q}+\sqrt{r})^{\nu-2}}, R(\nu) > -2.$$

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