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Some thoughts on the history of mathematics

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Abraham Robinson

1.

The achievements of Mathematics over the centuries cannot fail to arouse the deepest admiration. There are but few mathematicians who feel impelled to reject any of the major results of Algebra, or of Analysis, or of Geometry and it seems likely that this will remain true also in future. Yet, paradoxically, this iron-clad edifice is built on shifting sands. And if it is hard, and perhaps even impossible, to present a satisfactory viewpoint on the foundations of Mathematics today, it is equally hard to give an accurate description of the conceptual bases on which the mathematicians of the past constructed their theories. Some of the suggestions that we shall offer here on this topic are frankly speculative. Some may have been arrived at by comparing similar situations at different times in history, a procedure which is open to challenge and certainly should be used with great caution. Another preliminary remark which is appropriate here concerns the use of the word "real" with reference to mathematical objects. This term is ambiguous and has been stigmatised by some as meaningless in the present context. But the fundamental controversies on the significance of this word should not inhibit its use in a historical study, whose purpose it is to describe and analyze attitudes and not to justify them.

2.

It is commonly accepted that the beginnings of Mathematics as a deductive science go back to the Greek world in the fifth and fourth centuries B.C. It is even more certain that in the course of many hundreds of years before that time people in Egypt and Mesopotamia had accumulated an impressive body of mathematical knowledge, both in Geometry and in Arithmetic. Since this knowledge was recorded in the form of numerical problems and answers it is frequently asserted that pre-Greek Mathematics was purely "empirical". However, unless this expression is meant to indicate merely that pre-Greek Mathematics was not deductive and if it is to be taken literally, we are asked to believe, e.g., that the Mesopotamian mathematicians arrived at Pythagoras' theorem by measuring a large number of right triangles and by inspecting the numbers obtained as the squares of their side lengths. Is it not much more likely that these mathematicians, like their Greek successors, were already familiar with one of the arguments leading to a proof of Pythagoras' theorem by a decomposition of areas, but that no such proof was recorded by them since they regarded the reasoning as intuitively clear? To put it facetiously and anachronistically, if a Sumerian mathematician had been asked for his opinion of Euclid he might have replied that he was interested in *real* Mathematics and not in useless generalizations and abstractions. However, some major advances in Mathematics consisted not in the discovery of new results or in the invention of ingenious new methods but in *the codification of elements of accepted mathematical thought*, i.e. *in making explicit arguments, notions, assumptions, rules, which had been used intuitively for a long time previously.* It is in this light that we should look upon the contributions of the Greek mathematicians and philosophers to the foundations of Mathematics.

3.

For our present discussion, the question whether the major contribution to the system of Geometry recorded in Euclid's Elements was due to Hippocrates or to Eudoxus or to Euclid himself is of no importance (except insofar as it may affect the following problem, for chronological reasons). However, it would be important to know to what extent the emergence of deductive Mathematics was due to the lead given by one of the Greek philosophers or philosophical schools of the fifth and fourth centuries. Is it true, as has been asserted by some, that the creation of the axiomatic method was due to the direct influence of Plato or of Aristotle or, as has been suggested recently by Á. Szabó, that it was a response to the teachings of the Eleatic school? In our time, the immediate influence of philosophers on the foundations of Mathematics is confined to those who are willing to handle technical-mathematical details. But even now, a general philosophical doctrine may, almost imperceptibly, affect the direction taken by foundational research in Mathematics in the long run. In classical Greece, the differentiation between Philosophy and Mathematics was less pronounced, but nevertheless, with the possible exception of Democritus, we do not know of any leading philosopher of that period who originated an important contribution to Mathematics as such. When Plato singled out Theaetetus in order to emphasize the generality of mathematical arguments he was, after all, referring to a real person who had died only a few years earlier, and he wished to take no credit for the achievement described by him. Nevertheless, by laying bare some important characteristics of mathematical thought, both he and Aristotle exerted considerable influence on later generations. Thus Aristotle, having studied the mathematics of the day, established standards of rigor and completeness for mathematical reasoning which went far beyond the level actually reached at that time. And although we may assume that Euclid and his successors were aware of the teachings of Plato and Aristotle, their own aims in the development of Geometry as a deductive science were less ambitious than Aristotle's program from a purely logical point of view. It is in fact well known that even in the domain of purely mathematical postulates Euclid left a number of glaring gaps. And as far as the laws of logic are concerned, Euclid confined himself to axioms of equality (and inequality) and did not include the rules of deduction which had already been made available by Aristotle. Thus Euclid, like Archimedes after him, was content to single out those axioms which could not be taken for granted or which deserved special mention for other reasons and then derived his theorems from those axioms *in conjunction with other assumptions whose truth seemed obvious, by means of rules of deduction whose legitimacy seemed equally obvious.* It would be out of place to ask whether Euclid would have been able to include in his list of postulates this or that assumption if he had wanted just as even today it would, in most cases, be futile to ask a working mathematician to specify the rules of deduction that he uses in his arguments. The chances are that the typical working mathematician would reply that he is willing to leave this task to the logicians and that, by contrast, his own intuition is sound enough to get along spontaneously. For example, when proving that any composite number has a prime divisor (Elements, Book VII Proposition 31), Euclid appealed explicitly to the principle of infinite descent (which is a variant of the "axiom of induction") yet he did not include that principle among his axioms. By contrast, the axiom of parallels was included by Euclid (Elements, Book I, Postulate 5) because though apparently true, it was not intuitively obvious. Similarly "Archimedes axiom" was included by Archimedes (On the sphere and cylinder, Book I, Postulate 5) because although required for developing the method of exhaustion, it was not intuitively obvious either. In fact, Euclid did not accept this axiom at all explicitly but instead introduced a definition (Elements, Book V, Definition 5) which implies that he did not wish to exclude the possibility that magnitudes which are non-archimedean relative to one another actually exist, but that he deliberately confined himself to archimedean systems of magnitudes in order to be able to develop the theory of proportions and, to some extent, the method of exhaustion.

4.

From the beginnings of the axiomatic method until the nineteenth century A.D. axioms were regarded as statements of fact from which other statements of fact could be deduced (by means of legitimate procedures and relying on other obvious facts, see above). However, there is in Euclid an element of "constructivism" which, on one hand, seems to hark back to pre-Greek Mathematics and, on the other hand, should strike a chord in the hearts of those who believe that Mathematics has been pushed too far in a formal-deductive direction and who advocate a more constructive approach to the foundations of Mathematics. And although the first three postulates of the Elements, Book I can be interpreted as purely existential statements, the "constructivist" tenor of their actual style is unmistakable. Moreover, the cautious formulations of the second and fifth postulates seems to show a trace of the distaste for infinity that we find already in Aristotle. In addition, there are, of course, scattered through the Elements many "propositions" which are actually constructions.

5.

Euclid's geometry was supposed to deal with real objects, whether in the physical world or in some ideal world. The definitions which preface several books in the Elements are supposed to communicate what object the author is talking about even though, like the famous definition of the point and the line, they may not be required in the sequel. The fundamental importance of the advent of non-Euclidean geometry is that by contradicting the axiom of parallels it denied the uniqueness of geometrical concepts and hence, their reality. By the end of the nineteenth century, the interpretation of the basic concepts of Geometry had become irrelevant. This was the more important since Geometry had been regarded for a long time as the ultimate foundation of all Mathematics. However, it is likely that the independent development of the foundations of the number system which was sparked by the intricacies of Analysis would have deprived Geometry of its predominant position anyhow.

An ironic fate decreed that only after Geometry had lost its standing as the basis of all Mathematics its axiomatic foundations finally reached the degree of perfection which in the public estimation they had possessed ever since Euclid. Soon after, the codification of the laws of deductive thinking advanced to a point which, for the first time, permitted the satisfactory formalization of axiomatic theories.

6.

In the twentieth century, Set Theory achieved the position, once occupied by Geometry, of being regarded as the basic discipline of Mathematics in which all other branches of Mathematics can be embedded. And, within quite a short time, the foundations of Set Theory went through an evolution which is remarkably similar to the earlier evolution of the foundations of Geometry. First the initial assumptions of Set Theory were held to be intuitively clear being based on natural laws of thought for whose codification Cantor, at least, saw no need. Then Set Theory was put on a postulational basis, beginning with the explicit formulation of the least intuitive among them, the axiom of choice. However, at that point the axioms were still supposed to describe "reality", albeit the reality of an ideal, or Platonic, world. And finally, the realization that it is equally consistent either to affirm or to deny some major assertions of Set Theory such as the continuum hypothesis led, in the mid-sixties, to a situation in which the belief that Set Theory describes an objective reality was dropped by many mathematicians.

The evolution of the foundations of Set Theory is closely linked to the development of Mathematical Logic. And here also we can see how, in our own time, advances have been made through the codification of notions (such as the truth concept) which were used intuitively for a long time previously. And again it may be left open whether the postulates of a system deal with real objects or with idealizations (e.g. the rules of formation and deduction of a formal language). And there is every reason to believe that the codification of intuitive concepts and the reinterpretation of accepted principles will continue also in future and will bring new advances, into territory still uncharted.

Added March 20, 1968:

In an article published since the above lines were written (Non-Cantorian Set Theory, Scientific American, vol. 217, December 1967, pp. 104—116) Paul J. Cohen and Reuben Hersh compare the development of geometry and set theory and anticipate some of the points made here.

(Oblatum 3-1-'68)

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