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1. Introduction

Let μ be a regular complex valued Borel measure on a locally compact topological (LC) group G whose value on compact sets is finite, and suppose X is a right (left) translation invariant subspace of $C_0(G)$, the space of continuous complex valued functions on G which vanish at infinity, that is, $h \in X$ implies $T_s h(t) = h(ts) \in X (T^s h(t) = h(st) \in X)$, $s \in G$. μ is said to *act right almost invariantly on* X if X is right translation invariant, $\int_G |h| d|\mu| < \infty$, $h \in X$, and

$$\int_G h(ts^{-1}) d\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(ts_i^{-1}) d\mu(t) \qquad (s \in G, h \in X),$$

where $s_1, s_2, \cdots, s_n$ are fixed elements of G. In other words, the linear (not necessarily continuous) functionals $\{F_s | s \in G\}$ on X defined by $F_s(h) = \int_G h(ts^{-1}) d\mu(t)$, $h \in X$, span a space of finite dimension. A similar definition defines the notion of μ acting left almost invariantly. We have shown elsewhere [2, Theorem 1] that if μ acts right almost invariantly on X then there exists a continuous function f such that

$$\int_G h(t) d\mu(t) = \int_G h(t) f(t) dm(t) \qquad (h \in X)$$

where dm denotes right Haar measure on G. The analogous result is valid when μ acts left almost invariantly and right Haar measure is replaced by left Haar measure. In some instances the function f can be so chosen that $\{T_s f | s \in G\}$ spans a finite dimensional space of functions. For example this is the case when μ is a right almost invariant measure, that is, a measure for which $\{T_s \mu | s \in G\}$ spans a finite dimensional space where $T_s \mu(E) = \mu(Es)$ [2, Theorem 4]. However, there also exist situations in which the translates of f span a finite dimensional space but μ is not an almost invariant measure [2, p. 1299]. In this paper we shall examine the question of when, given a measure which acts right (left) almost invariantly, one can find a function f as described above whose right (left) translates span a finite dimensional space. Equivalently, we wish to know when the functional determined by μ can be obtained by means of integrating with respect to some almost invariant measure [2].

In particular, whenever X is invariant under both right and left translations, we shall show that if the translates of f span a finite dimensional space then μ must act both right and left almost invariantly, and establish some sufficient conditions when μ acts both right and left almost invariantly for the existence of f. When G is compact and μ acts right (left) almost invariantly we shall construct a function f with the desired properties. Thus, for compact groups the answer to our question is always in the affirmative.

We shall denote by $V(G)$ the linear space of all regular complex valued Borel measures on the LC group G, and by $M(G)$ the subspace of measures in $V(G)$ with finite total mass. $FDT(G)$ will stand for the space of all continuous complex valued functions on G whose translates span a finite dimensional space of functions. We shall see below that in the definition of $FDT(G)$ no distinction needs be made between right and left translates. m and m' shall denote, respectively, right and left Haar measure on G.

Remark. It should, perhaps, be noted that the first portion of the proof of [2, Theorem 1] is misleading, since the continuity of the α_i appears to be deduced from that of $\int_G h(ts^{-1})d\mu(t)$. However, the latter functions are not *a priori* continuous, and, indeed, for arbitrary μ may fail to be so. Nevertheless, when μ acts almost invariantly the continuity of the α_i, and hence of $\int_G h(ts^{-1})d\mu(t)$, can be established by constructing a certain finite dimensional group representation whose entries are continuous and in terms of which the α_i can be expressed. If one substitutes $C_c(G)$, the space of continuous complex valued functions with compact support, for $C_0(G)$ then the proof as given in [2] is valid.

2. Noncompact groups

Though stated in the context of LC groups the majority of the results of this section are of interest only for noncompact groups.

Lemma. *Let G be a LC group and f a continuous function on G. Then the following are equivalent*:

i) $\{T_s f | s \in G\}$ *spans a finite dimensional space.*
ii) $\{T^s f | s \in G\}$ *spans a finite dimensional space.*

Proof. Suppose i) holds. Then we may write

$$T_s f = \sum_{i=1}^{n} \alpha_i(s) T_{s_i} f, \qquad (s \in G),$$

where, without loss of generality, $T_{s_1}f, T_{s_2}f, \cdots, T_{s_n}f$ form a basis for the linear space W spanned by $\{T_s f | s \in G\}$. It follows easily from the continuity of f and the finite dimensionality of W that the α_i, $i = 1, 2, \cdots, n$, are continuous. Moreover it is evident that translation by an element of G defines a linear mapping from W onto W and the matrix associated with translation by t, with respect to the given basis, is $A(t) = (\alpha_i(ts_j))_{i,j=1}^{n}$. From the fact that $A(t)A(s) = A(ts)$ and the independence of $T_{s_i}f$, $i = 1, 2, \cdots, n$, one readily deduces the identity

$$\alpha_i(ts) = \sum_{j=1}^{n} \alpha_i(ts_j)\alpha_j(s), \qquad i = 1, 2, \cdots, n \ (s, t \in G).$$

Consequently, $\{T^t \alpha_i | t \in G\}$ spans a finite dimensional space, and since

$$f(s) = T_s f(e) = \sum_{i=1}^{n} \alpha_i(s) T_{s_i} f(e) = \sum_{i=1}^{n} f(s_i)\alpha_i(s),$$

we conclude that $\{T^t f | t \in G\}$ spans a finite dimensional space.

The proof that ii) implies i) is similar.

This lemma justifies our use of $FDT(G)$ to denote the space of continuous functions whose right or left translates span finite dimensional spaces.

We shall say that $X \subset C_0(G)$ is *translation invariant* if it is invariant under both right and left translations.

Theorem 1. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, and suppose $\mu \in V(G)$ acts right (left) almost invariantly on X. If there exists an $f \in FDT(G)$ such that*

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \left(\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm'(t)\right),$$

$h \in X$, then μ acts left (right) almost invariantly on X.

Proof. Assume μ acts right almost invariantly. Then for each $h \in X$ we have

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t)$$
$$= \int_G h(t)f(t)\Delta_l(t)dm'(t),$$

where Δ_l is the left modular function of G. It follows from the Lemma that μ acts left almost invariantly.

An immediate corollary of the theorem and [2, Theorem 3] is the

COROLLARY. *Let G be a LC group and $\mu \in V(G)$. Then μ is right almost invariant if and only if μ is left almost invariant.*

When X is translation invariant Theorem 1 shows that a necessary condition for the existence of an $f \in FDT(G)$ with the desired properties with respect to μ is that μ act both right and left almost invariantly. Consequently, we shall now restrict our attention to such μ and establish several sufficient conditions for the existence of f.

If μ acts both right and left almost invariantly on X we shall say that μ *acts almost invariantly on* X, and write for each $s \in G$, $h \in X$,

$$\int_G h(ts^{-1})d\mu(t) = \sum_{i=1}^{n} \alpha_i(s)\int_G h(ts_i^{-1})d\mu(t),$$
$$\int_G h(s^{-1}t)d\mu(t) = \sum_{j=1}^{m} \beta_j(s)\int_G h(r_j^{-1}t)d\mu(t),$$

where the α_i and β_j are continuous functions [2]. Moreover, it is not difficult to show, much in the same manner as was done in the proof of the Lemma, that α_i and β_j belong to $FDT(G)$.

We shall state the sufficient conditions in terms which utilize the fact that μ acts right almost invariantly. It will be apparent what the analogous statements should be if one chooses to use the property that μ acts left almost invariantly. The two types of conditions are, of course, equally valid to insure the existence of $f \in FDT(G)$.

THEOREM 2. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, and suppose $\mu \in V(G)$ acts almost invariantly on X. If there exists a function $g \in X$ such that:*

i) $\int_G |g(t)|dm(t) < \infty$,

ii) $\int_G |g(t)\alpha_i(t)|dm(t) < \infty, \quad i = 1, 2, \cdots, n$,

iii) $\int_G g(t)\alpha_i(t)dm(t) = \alpha_i(e), \quad i = 1, 2, \cdots, n$, *then there exists an $f \in FDT(G)$ for which*

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X).$$

PROOF. If $h \in X$ then

$$\begin{aligned}\int_G g(s)\int_G h(ts^{-1})d\mu(t)dm(s) &= \int_G g(s)\left[\sum_{i=1}^n \alpha_i(s)\int_G h(ts_i^{-1})d\mu(t)\right]dm(s)\\ &= \sum_{i=1}^n \alpha_i(e)\int_G h(ts_i^{-1})d\mu(t)\\ &= \int_G h(t)d\mu(t).\end{aligned}$$

On the other hand, because of i) and ii), we may apply Fubini's theorem and obtain,

$$\begin{aligned}&\int_G g(s)\int_G h(ts^{-1})d\mu(t)dm(s)\\ &\quad= \int_G\int_G g(s)h(ts^{-1})dm(s)d\mu(t)\\ &\quad= \int_G\int_G g(st)h(s^{-1})dm(s)d\mu(t)\\ &\quad= \int_G h(s^{-1})\int_G g(st)d\mu(t)dm(s)\\ &\quad= \int_G h(s^{-1})\left[\sum_{j=1}^m \beta_j(s^{-1})\int_G g(r_j^{-1}t)d\mu(t)\right]dm(s)\\ &\quad= \int_G h(s)\left[\Delta_r(s)\sum_{j=1}^m\left(\int_G g(r_j^{-1}t)d\mu(t)\right)\beta_j(s)\right]dm(s),\end{aligned}$$

where Δ_r is the right modular function of G.

Thus, $f(t) = \Delta_r(t)\sum_{j=1}^m (\int_G g(r_j^{-1}t)d\mu(t))\,\beta_j(t)$ is a continuous function such that $\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t)$, and $f \in FDT(G)$ since $\beta_j \in FDT(G)$, $j = 1, 2, \cdots, m$.

Let X be translation invariant and consider the linear functionals $\{F_s | s \in G\}$ introduced in section one. If μ acts almost invariantly on X then we may write $F_s = \sum_{i=1}^n \alpha_i(s)F_{s_i}$, $s \in G$, where the functionals $F_{s_1}, F_{s_2}, \cdots, F_{s_n}$ may be assumed to be linearly independent. Furthermore, the development in [2, p. 1297—98] guarantees the existence of a function $k \in C_c(G)$ such that $F_e = \int_G k(s)F_s dm(s)$.

Having made these remarks we can now state and prove our next result.

Theorem 3. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, and suppose $\mu \in V(G)$ acts almost invariantly on X. If $k \in X$ then there exists an $f \in FDT(G)$ such that*

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t).$$

Proof. From the remarks preceding the theorem we conclude that

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i(e)F_{s_i} = F_e &= \int_G k(s)F_s dm(s)\\ &= \int_G k(s)\left[\sum_{i=1}^{n} \alpha_i(s)F_{s_i}\right] dm(s)\\ &= \sum_{i=1}^{n}\left[\int_G k(s)\alpha_i(s)dm(s)\right] F_{s_i}.\end{aligned}$$

But since $F_{s_1}, F_{s_2}, \cdots, F_{s_n}$ are independent we see that

$$\int_G k(s)\alpha_i(s)dm(s) = \alpha_i(e), \qquad i = 1, 2, \cdots, n,$$

and an application of Theorem 2 completes the proof.

Remarks. a) When G is abelian and the measure μ acts invariantly on X, that is, $\int_G h(ts^{-1})d\mu(t) = \int_G h(t)d\mu(t)$, $h \in X$, then Theorem 2 reduces to Theorem 2 in [1].

b) The condition in Theorem 2 though sufficient is not necessary. For instance let $G = R$, the additive group of the real line, X the space spanned by all $h \in C_c(R)$ for which $\hat{h}(0) = 0$, and set $d\mu(t) = (1+t)dm(t)$. Then μ clearly acts almost invariantly on X since μ is an almost invariant measure, indeed,

$$T_s\mu = (1-s)T_0\mu+sT_1\mu = \alpha_1(s)T_0\mu+\alpha_2(s)T_1\mu, \qquad s \in R.$$

Thus $f(t) = 1+t$ satisfies the conclusion of Theorem 2 but there is no $g \in X$ such that $\int_G g(t)\alpha_i(t)dm(t) = \alpha_i(0)$, $i = 1, 2$. As if there were, then

$$\begin{aligned}1 = \alpha_1(0) &= \int_G g(t)\alpha_1(t)dm(t)\\ &= -\int_G g(t)tdm(t)\\ &= -\int_G g(t)\alpha_2(t)dm(t) = -\alpha_2(0) = 0,\end{aligned}$$

which is absurd.

c) Suppose μ acts almost invariantly on X. It is possible, even though $f \in FDT(G)$ has the desired properties, for the dimension of the span of $\{T_s f | s \in G\}$ to be strictly greater than the dimension of the span of $\{F_s | s \in G\}$. An example of this is given in [1, p. 420].

d) It was necessary in this section to restrict our attention to measures which acted both right and left almost invariantly. A class of measures in which acting right or left almost invariantly is equivalent to acting almost invariantly is indicated here. If $\mu \in V(G)$ define $\tilde{\mu} \in V(G)$ by $\tilde{\mu}(E) = \mu(E^{-1})$. It is elementary to prove that if X is invariant under translation and reflection, that is, $h \in X$ implies $\tilde{h}(t) = h(t^{-1}) \in X$, and μ acts right (left) almost invariantly on X then $\tilde{\mu}$ acts left (right) almost invariantly on X. Thus whenever $\mu = \tilde{\mu}$ the measure μ acts almost invariantly.

3. Compact groups

In this section we wish to show that if G is a compact group and μ acts right (left) almost invariantly on a subspace X of $C(G)$, the space of continuous complex valued functions on G, then there always exists a function $f \in FDT(G)$ with the desired properties. It should be noted that now $\mu \in M(G)$ and so the functionals determined by μ are continuous. Before establishing the result we wish to set some notation.

We shall denote by $\{g^\gamma\}_{\gamma \in \Gamma}$ a complete family of finite dimensional continuous irreducible inequivalent unitary representations of the compact group G. For each $t \in G$, $g^\gamma(t) = (g^\gamma_{ij}(t))$ is an $r(\gamma) \times r(\gamma)$ unitary matrix, and the functions g^γ_{ij} belong to $FDT(G)$. We set $\Delta = \{g^\gamma_{ij} | i, j = 1, 2, \cdots, r(\gamma), \gamma \in \Gamma\}$. Results pertaining to the representations g^γ which we shall use, in particular the orthogonality relations, are all available in [3, Chapter V].

In the interest of simplicity we shall again state the next theorem only for measures which act right almost invariantly.

THEOREM 4. *Let G be a compact group, X a right translation invariant subspace of $C(G)$, and suppose $\mu \in M(G)$ acts right almost invariantly on X. Then there exists an $f \in FDT(G)$ such that*

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X).$$

PROOF. Since the functional defined by μ is continuous we may assume, without loss of generality, that X is a closed subspace of $C(G)$.

Set $\Delta' = \Delta \cap X$. Then $\Delta' \neq \emptyset$ and the linear span of Δ' is uniformly dense in X. Define $\Delta'' = \{g^\gamma_{ij} | \int_G g^\gamma_{ij}(t)d\mu(t) \neq 0\}$.

If $\Delta' \cap \Delta'' = \emptyset$ the theorem is trivially true, because $\int_G g^\gamma_{ij}(t)d\mu(t) = 0$, $g^\gamma_{ij} \in \Delta'$, implies $\int_G h(t)d\mu(t) = 0$, $h \in X$, and hence $f = 0$ satisfies the conclusion of the theorem.

On the other hand, if $\Delta' \cap \Delta'' \neq \emptyset$ then we claim it is finite. Note first that if $g^{\gamma}_{ij} \in \Delta'$ then, since μ acts right almost invariantly on X, we have

$$\int_G g^{\gamma}_{ij}(ts^{-1})d\mu(t) = \sum_{k=1}^{n} \alpha_k(s) \int_G g^{\gamma}_{ij}(ts_k^{-1})d\mu(t).$$

Moreover, since g^{γ} is a homomorphism,

$$\int_G g^{\gamma}_{ij}(ts^{-1})d\mu(t) = \int_G \sum_{k=1}^{r(\gamma)} g^{\gamma}_{ik}(t)g^{\gamma}_{kj}(s^{-1})d\mu(t).$$
$$= \sum_{k=1}^{r(\gamma)} \left[\int_G g^{\gamma}_{ik}(t)d\mu(t)\right] \overline{g^{\gamma}_{jk}(s)}.$$

Thus we see that if $g^{\gamma}_{ij} \in \Delta'$ then the functions of the form

$$\sum_{k=1}^{r(\gamma)} \left[\int_G g^{\gamma}_{ik}(t)d\mu(t)\right] \overline{g^{\gamma}_{jk}}$$

belong to the finite dimensional space of functions spanned by $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Assume $\Delta' \cap \Delta''$ is infinite. Then there exists a sequence γ_l of γ's such that for each γ_l at least one $g^{\gamma_l}_{ij} \in \Delta' \cap \Delta''$. Choose one of these elements and denote it as $g^{\gamma_l}_{i(l)j(l)}$. Define the function h_l as follows,

$$h_l(s) = \sum_{k=1}^{r(\gamma_l)} \left[\int_G g^{\gamma_l}_{i(l)k}(t)d\mu(t)\right] \overline{g^{\gamma_l}_{j(l)k}(s)}.$$

We assert that the functions h_l, $l = 1, 2, \cdots$, are linearly independent.

Indeed, suppose for a finite subset of positive integers we have $\sum_l c_l h_l(s) \equiv 0$. Then the linear independence of $\overline{g^{\gamma_l}_{i(l)k}}$, $k = 1, 2, \cdots, r(\gamma_l)$, $l = 1, 2, \cdots$, allows us to conclude that for each l,

$$c_l \int_G g^{\gamma_l}_{i(l)k}(t)d\mu(t) = 0, \qquad k = 1, 2, \cdots, r(\gamma_l).$$

In particular then $c_l = 0$ since $\int_G g^{\gamma_l}_{i(l)j(l)}(t)d\mu(t) \neq 0$. Thus the h_l are independent.

However, the h_l all belong to the finite dimensional space spanned by $\alpha_1, \alpha_2, \cdots, \alpha_n$, and we have obtained a contradiction. Therefore $\Delta' \cap \Delta''$ is finite.

Let us denote the distinct elements of $\Delta' \cap \Delta''$ as $g^{\gamma_l}_{ij}$, $i = 1, 2, \cdots, m(l)$, $j = 1, 2, \cdots, n(l)$, $l = 1, 2, \cdots, d$, and define

$$f(s) = \sum_{l=1}^{d} \sum_{i=1}^{m(l)} \sum_{j=1}^{n(l)} r(\gamma_l) \left[\int_G g_{ij}^{\gamma_l}(t) d\mu(t) \right] \overline{g_{ij}^{\gamma_l}(s)}.$$

Evidently $f \in FDT(G)$, and, using the orthogonality relations of the $g_{ij}^{\gamma_l}$, it is easy to show that

$$\int_G g_{pq}^{\gamma}(t) d\mu(t) = \int_G g_{pq}^{\gamma}(t) f(t) dm(t) \qquad (g_{pq}^{\gamma} \in \Delta').$$

It is then an immediate consequence of these equations and the denseness of the span of Δ' in X that

$$\int_G h(t) d\mu(t) = \int_G h(t) f(t) dm(t) \qquad (h \in X).$$

This completes the proof.

REMARKS. a) If G is abelian then f is a linear combination of the continuous characters which are common to the space X and the support of the Fourier-Stieltjes transform of μ.

b) However in the nonabelian case one cannot, in general, obtain f as a linear combination of the characters of the representations g^{γ}. For example let g^{γ} be a representation and let g_{ij}^{γ} be any element such that $i \neq j$. Set X equal to the closed linear span of $\{T_s \overline{g_{ij}^{\gamma}} | s \in G\}$ and $d\mu(t) = g_{ij}^{\gamma}(t) dm(t)$. μ is a right almost invariant measure and hence acts right almost invariantly on X. If

$$f = \sum_{l=1}^{n} c_l \chi_l = \sum_{l=1}^{n} c_l \sum_{k=1}^{r(\gamma_l)} g_{kk}^{\gamma_l}$$

then, since $i \neq j$, the orthogonality relations reveal that

$$\int_G \overline{g_{ij}^{\gamma}(t)} f(t) dm(t) = 0,$$

but

$$\int_G \overline{g_{ij}^{\gamma}}(t) d\mu(t) = \int_G \overline{g_{ij}^{\gamma}(t)} g_{ij}^{\gamma}(t) d\mu(t) = 1/r(\gamma) \neq 0.$$

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