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On finite primary rings and their groups of units

by

Christine W. Ayoub

In a recent paper [1] Gilmer determined those rings R which have a cyclic group of units. He showed that it is sufficient to consider (finite) primary rings. In this note after proving a preliminary result (Theorem 1) we restrict attention to finite primary rings and show some connections between the additive group of N , the radical of the ring R , and the multiplicative group $1+N$. In Theorem 2 we prove that if either N or $1+N$ is cyclic, R is homogeneous (provided $N \neq 0$ — i.e. R is not a field) in the sense that there is a positive integer k such that

$$R/N, N/N^2, \dots, N^k/N^{k+1}$$

are isomorphic elementary abelian groups under addition and $N^{k+1} = 0$. Furthermore, if $p \geq 3$, N is cyclic if, and only if $1+N$ is cyclic. As a consequence of this theorem we are able to determine the rings for which N is cyclic and those for which $1+N$ is cyclic (Corollary to Theorem 2). Thus we obtain a quite different proof of Gilman's results as well as a proof of the well-known fact that there is a primitive root, mod p^k when $p \geq 3$. In a subsequent paper we hope to discuss finite homogeneous rings in general and to determine conditions under which the radical N is isomorphic (as an additive group) to the multiplicative group $1+N$.

1. Terminology and notation

We recall that a primary ring is a commutative ring with 1 which contains a unique prime ideal N (see [2] p. 204). The facts we need about primary rings are:

- (1) A finite primary ring is a p -ring — i.e. every element has additive order a power of a prime p .
- (2) R/N is a field
- (3) N is nilpotent

The notation used is standard. We mention only the following:

\otimes is used for direct product (of multiplicative groups), \oplus is used for direct sum (of additive groups); and for a finite set S , $|S|$ denotes the cardinality of S .

2. A preliminary result

THEOREM 1. *Let R be a ring with 1 and N a nil ideal. If G is the group of units of R then $H = 1+N$ is a normal subgroup of G and G/H is isomorphic to the group of units of R/N . Furthermore, the additive group N^i/N^{i+1} is isomorphic to the multiplicative group $1+N^i/1+N^{i+1}$ (for each integer $i \geq 1$).*

PROOF. We show first that $1+N$ is contained in G . Let $a \in 1+N$ so that $a = 1+x$ with $x \in N$. Since x is nilpotent, x is regular in the sense of Jacobson. Hence a has an inverse. Thus $1+N \subseteq G$.

If ν is the natural map from R to $\bar{R} = R/N$, ν maps G homomorphically onto a multiplicative subgroup \bar{G} of \bar{R} . Let H be the kernel of the mapping from G to \bar{G} . It is clear that $H = 1+N$ so that $H = 1+N$ is a normal subgroup of G and $G/H \simeq \bar{G}$.

We verify next that \bar{G} is the group of (all) units of \bar{R} . In fact, let $r+N$ be a unit of \bar{R} ; then there is an

$$\begin{aligned} s \in R \ni (r+N)(s+N) &= (s+N)(r+N) = 1+N \Rightarrow rs+N = sr+N \\ &= 1+N \Rightarrow rs, sr \in 1+N \Rightarrow rs, sr \in G \Rightarrow r \in G. \end{aligned}$$

Hence \bar{G} is the group of units of \bar{R} .

Since N^i and N^{i+1} ($i \geq 1$) are nil ideals, $1+N^i$ and $1+N^{i+1}$ are normal subgroups of G and $1+N^{i+1} \triangleleft 1+N^i$; hence we can form the quotient group $1+N^i/1+N^{i+1}$.

Now consider the mapping η from N^i onto $1+N^i/1+N^{i+1}$ defined by: $x\eta = (1+x)(1+N^{i+1})$ for $x \in N^i$. Let $x, y \in N^i$ and let $z \in N^i$ be such that $(1+x+y)(1+z) = 1$ (z exists since $1+N^i$ is a multiplicative group). Then

$$(1+x)(1+y) = (1+x+y)(1+(1+z)xy)$$

so that:

$$[(1+x)(1+N^{i+1})][(1+y)(1+N^{i+1})] = (1+x+y)(1+N^{i+1})$$

since $1+(1+z)xy \in 1+N^{i+1}$. But this last equation shows that:

$$(x\eta)(y\eta) = (x+y)\eta \text{ for } x, y \in N^i \text{ — i.e. } \eta \text{ is a homomorphism.}$$

Now $K(\eta)$, the kernel of η , $= \{x \in N^i | 1+x \in 1+N^{i+1}\} = N^{i+1}$
Hence $N^i/N^{i+1} \simeq 1+N^i/1+N^{i+1}$ as we claimed.

REMARK. The same method establishes the isomorphism $N^i/N^{2i} \simeq 1+N^i/1+N^{2i}$.

3. Finite primary rings

PROPOSITION 1. Let R be a finite primary p -ring with prime ideal N . Let G be the group of units of R and $H = 1+N$. Then

(a) $H \leq G$ and $G/H \simeq (R/N)^*$ = the group of non-zero elements of R/N . Furthermore, $G = H \otimes U$, where $U \simeq (R/N)^*$.

(b) $N^i/N^{i+1} \simeq 1+N^i/1+N^{i+1}$ for each integer $i \geq 1$ (the left hand side as an additive group and the right hand side as a multiplicative group).

(c) N^i/N^{i+1} is an elementary p -group (under $+$) and

$$|R/N| \leq |N^i/N^{i+1}|$$

for each $i \geq 1$ such that $N^i \neq 0$.

PROOF. (a) The first statement follows from Theorem 1 since $(R/N)^*$ is the group of units of the field R/N . Now R/N is a Galois field with p^l elements and hence $|(R/N)^*| = p^l - 1$; on the other hand, $|H| = |N| =$ a power of p . Hence $|G| = |H|(p^l - 1)$ and thus $G = H \otimes U$, where $U \simeq G/H \simeq (R/N)^*$.

(b) This follows directly from Theorem 1.

(c) N^i/N^{i+1} is an R -module but since $N(N^i) = N^{i+1}$, it can also be considered as an R/N -module — i.e. as a vector space over the field R/N . But R/N has characteristic p so that $p(N^i/N^{i+1}) = 0$ which shows that N^i/N^{i+1} is an elementary p -group — provided $N^i \neq 0$.

Since $N^i \neq 0$ implies N^i/N^{i+1} is a vector space over R/N of dimension ≥ 1 , it has a basis of t elements, say ($t \geq 1$). Then $|N^i/N^{i+1}| = tp^l$, where $|R/N| = p^l$. Hence $|R/N| \leq |N^i/N^{i+1}|$ provided $N^i \neq 0$.

DEFINITION. The finite primary ring R with radical N is homogeneous of type p if \exists an integer k such that

$$R/N, N/N^2, \dots, N^k/N^{k+1}$$

all have order p and $N^{k+1} = 0$.

THEOREM 2. Let R be a finite primary p -ring with prime ideal $N \neq 0$ and let $H = 1+N$. Then

(a) if either the additive group N or the multiplicative group H is cyclic, R is homogeneous of type p .

(b) For $p \geq 3$, N is cyclic if, and only if H is cyclic.

(c) For $p = 2$:

(i) If N is cyclic, H is cyclic if, and only if $N^2 = 0$. In case $N^2 \neq 0$, $H = (-1) \otimes H^{(2)}$, where $H^{(2)} = 1 + N^2$ is cyclic.

(ii) If H is cyclic and N is not cyclic, $N \simeq$ Klein 4-group.

PROOF. Let $0 = N^{k+1} < N^k$

(a) Since $N^i/N^{i+1} \simeq 1 + N^i/1 + N^{i+1}$ by Proposition 1 (b), either of our hypotheses guarantees that N^i/N^{i+1} is cyclic. But by Proposition 1 (c) N^i/N^{i+1} is an elementary p -group for $N^i \neq 0$, and $|R/N| \leq |N^i/N^{i+1}|$. Hence each of the groups

$$R/N, N/N^2, \dots, N^k/N^{k+1}$$

has order p . Note that $|N| = p^k$.

We prove next the following assertion: (*) Assume that H is cyclic and that N^{i+1} is cyclic. If $p \geq 3$ and $i \geq 1$ or if $p = 2$ and $i \geq 2$, N^i is cyclic.

PROOF OF (*). We can assume $i < k$ since we already know that N^i is cyclic for $i \geq k$. We show that every element of order p in N^i is in N^{i+1} ; this will establish that N^i has a unique subgroup of order p — since by assumption N^{i+1} is cyclic. Indeed, let $x \in N^i$ and assume that $px = 0$. Then $(1+x)^p = 1+x^p$ and $x^p \in N^{i+2}$. Since $(1+x)^p \in 1+N^{i+2}$ and since $1+N^i/1+N^{i+2}$ is cyclic and hence has $1+N^{i+1}/1+N^{i+2}$ as its only subgroup of order p , $1+x \in 1+N^{i+1}$. Thus $x \in N^{i+1}$. This proves the validity of (*). In particular, applying induction we have that if $p \geq 3$ and H is cyclic, N is cyclic (i.e. the “if” part of (b)), and if $p = 2$ and H is cyclic, N^2 is cyclic.

Now assume that H is cyclic and that N is not cyclic. Then $p = 2$, $k \geq 2$ (since N^k is cyclic); we show that $N^3 = 0$. Assume to the contrary that $N^3 \neq 0$ and let $x \in N$ with $2x = 0$. Then $(1+x)^4 = 1+x^4 \in 1+N^4$. $|1+N^4| = 2^{k-3}$ so that

$$1 = (1+x^4)^{2^{k-3}} = (1+x)^{2^{k-1}}$$

and this implies that $x \in N^2$. Thus N is cyclic. Hence if N is not cyclic, $p = k = 2$ and N is isomorphic to the Klein 4-group. This establishes (c) (ii).

We now prove a statement analogous to (*), viz. (**). Assume that N is cyclic and that $1+N^{i+1}$ is cyclic. If $p \geq 3$ and $i \geq 1$ or if $p = 2$ and $i \geq 2$, $1+N^i$ is cyclic.

PROOF OF ().** We can assume that $i < k$. Let $1+x \in 1+N^i$ and assume $(1+x)^p = 1$. Then

$$1 = (1+x)^p = 1+px + \frac{p(p-1)}{2}x^2 + \dots + x^p = 1+(px)u+x^p,$$

where

$$u = 1 + \frac{p-1}{2}x + \dots \in 1+N \quad (u = 1 \text{ if } p = 2).$$

Letting $uv = 1$ (u is a unit) we obtain $px = -x^p v \in N^{ip} \leq N^{i+2}$ since $x \in N^i$. But N^i/N^{i+2} is cyclic of order p^2 and N^{i+1}/N^{i+2} is its only subgroup of order p . Hence $x \in N^{i+1}$. Therefore $1+x \in 1+N^{i+1}$ and (**) is established. Thus the "only if" part of (b) is proved and we have only (c) (i) left to verify.

So assume that N is cyclic and that $p = 2$. If $N^2 = 0$, $H \simeq N$ and H is cyclic. So assume $N^2 \neq 0$. By (**), $H^{(2)} = 1+N^2$ is cyclic. We show that $-1 \in H \setminus H^{(2)}$. Indeed

$$-1 = 1+(-2) \in 1+N = H$$

but if $-1 \in H^{(2)}$, $2 \in N^2$ and this implies that $2 = 2a$ for some $a \in N$ since $N^2 = 2N$. But then $2(1-a) = 0$ so that $2 = 0$ since $1-a$ is a unit. But this implies that $N^2 = 2N = 0$ — a contradiction. Hence $H = (-1) \otimes H^{(2)}$ and (c) (i) is established.

COROLLARY. *Let R be a finite primary p -ring with prime ideal $N \neq 0$, let G be its group of units and let $H = 1+N$. Then G is cyclic if and only if H is cyclic. Furthermore, G is cyclic if and only if R is isomorphic to one of the following:*

- (i) $Z_p k+1$, where $p \geq 3$ and $k \geq 1$.
- (ii) Z_r
- (iii) $Z_p[x]/(x^2)$
- (iv) $Z_2[x]/(x^3)$
- (v) $\frac{Z[x]}{\text{Id}\{4, 2x, x^2-2\}}$.

On the other hand, N is cyclic if and only if either:

$$(1) \quad R \simeq Z_p k+1$$

or

$$(2) \quad R \simeq Z_p[x]/(x^2)$$

Note: We are using the notation: $Z_n = Z/(n)$.

PROOF. Assume that N is cyclic, and suppose that $p = pa$ for some $a \in N$. Then $p(1-a) = 0$ and this implies that $p = 0$ ($1-a \in 1+N$ is a unit). Thus either p is a generator of N or N is of order p .

In the first case, R has characteristic p^{k+1} , where $p^k = |N|$. But $|R| = p^{k+1}$ so that $R \simeq Z_p k + 1$. Theorem 2(b) and (c) (i) tells us that H is cyclic if, and only if either $p \geq 3$ or if $p = 2$ and $k = 1$.

In the second case, R has characteristic p and $N^2 = 0$. Thus $R \simeq Z_p[x]/(x^2)$ and it follows immediately that in this case H is cyclic.

If the characteristic of R is 2, $R = Z_2 + (a) + (a^2)$ and $R \simeq Z_2[x]/(x^3)$. If the characteristic of R is 4 and if $2 \in N \setminus N^2$, we can take $a = 2$ and then $2^2 = 4 = 0$ — a contradiction. Hence $b = 2$. Then $R = Z_4 + (a)$ with $2a = 0$ and $a^2 = 2$ so that $R \simeq Z[x]/Id\{4, 2x, x^2 - 2\}$.

Finally we verify that for these two rings with 8 elements, H is cyclic. $|H| = 4$ and $(1+a)^2 = 1+a^2 = 1+b \neq 1$ (in both cases). Thus H is not the 4-group so must be cyclic.

If R is an infinite primary ring, its group of units cannot be cyclic. For if $0 = N^{k+1} < N^k$, N^k is a vector space over the field R/N and thus N^k cannot be cyclic. But $N^k \simeq 1 + N^k$, a subgroup of the group G of units of R . Hence G cannot be cyclic if $N \neq 0$. If $N = 0$, R is a field and it is easy to see that its non-zero elements do not form an (infinite) cyclic group.

If R is a commutative ring with identity and with descending chain condition, then R is a direct sum of a finite number of primary rings (see [2] Theorem 3 on p. 205). Now if R has a cyclic group of units each of the primary rings has a cyclic group of units — and hence must be finite. Thus we have proved:

PROPOSITION 2. *Let R be a commutative ring with identity which satisfies the descending chain condition. If the group of units of R is cyclic, R is finite.*

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