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Open subsets of Hilbert space

by

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In this paper we prove the following

THEOREM. *If M is an open subset of the separable Hilbert space. l_2, which we shall call H, then M is homeomorphic to an open set $N \subset H$ where ("$\cong$" denotes "is homeomorphic to" and $R \equiv$ Reals)*

(a) $H - N \cong H - \mathrm{cl}(N) \cong H$,

(b) $N \cong \mathrm{cl}(N) \cong \mathrm{cl}(N) - N (\equiv \mathrm{bd}(N)) \cong H \times |K|$, *where K is a countable locally-finte simplicial complex (clfsc), and*

(c) *there is an embedding h: $\mathrm{bd}(N) \times R \to H$ such that* (i) $h(\mathrm{bd}(N) \times R)$ *is open in H,* (ii) $h|\mathrm{bd}(N) \times \{0\}$ *is the inclusion of* $\mathrm{bd}(N)$ *into H, and* (iii) $h(\mathrm{bd}(N) \times (-\infty, 0)) = N$.

Recent results of Eells and Elworthy [7] show that each separable C^∞ Hilbert manifold is C^∞-diffeomorphic to an open subset of H, and, since Hilbert C^∞-structures are unique [7], we may assert, in this case, that M is C^∞-diffeomorphic to N. Also, each separable, infinite-dimensional, Fréchet space, F, is homeomorphic to H (see [1]). In a later paper [9] the author will use the Theorem to show that all separable manifolds modeled on F are homeomorphic to open subsets of H (and thus have unique Hilbert C^∞-structures).

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1. Special cases

It is natural to ask when a manifold M modeled on H has the form $P \times H$, for a finite-dimensional manifold P. In this section we give several answers based on the following result:

[1] The author is an Alfred P. Sloan Fellow.

If X and Y are manifolds modeled on H, then any homotopy equivalence $f : X \to Y$ is homotopic to a homeomorphism (diffeomorphism) of X onto Y. This result is a combined effort of [7], [9], [11], and [12]. (The proof in [9] uses the Theorem of this paper but not the results of this section.) It is the first special case that motivates the proof of the Theorem; however, the proof of the Theorem is otherwise independent of this section.

SPECIAL CASE 1. *M has the homotopy type of a finite-dimensional clfsc L, if and only if, M is homeomorphic (diffeomorphic) to $P \times H$, where P is an open subset of a finite-dimensional Euclidean space.*

PROOF. Embed L in a sufficiently high dimensional Euclidean space E^n and let P be its open regular neighborhood. Then M has the homotopy type of $P \times H$, and thus $M \cong P \times H$. In fact, M is homeomorphic (though not diffeomorphic)to $\text{cl}(P) \times H$.

SPECIAL CASE 2. *Let M be a connected separable manifold modeled on H. Then M is homeomorphic (diffeomorphic) to $P \times H$, where P is an open subset of a Euclidean space, if and only if the following conditions are satisfied:* (a) *If $\tilde{M}$ denotes the universal covering space of M, then there is an integer n such that $H_i(\tilde{M}) = 0$, for $i > n$; and* (b) *$H^{n+1}(M, \mathscr{B}) = 0$, for all local coefficients $\mathscr{B}$ (possibly non-abelian, if $n = 1$).*

PROOF. First of all, M and $\tilde{M}$ have the homotopy type of connected clfsc's. Theorem E of Wall [14] insures that M has the homotopy type of a finite-dimensional clfsc L and thus we may apply the first special case.

SPECIAL CASE 3. In the above cases P is an open manifold. Conditions which insure that M is homeomorphic (diffeomorphic) to $P \times H$, for P a closed (compact) manifold, are much more delicate. As above, we can use Wall's work [14] to reduce the problem to a characterization of those clfsc having the homotopy type of such M. Sufficient conditions are then given by the theorems of W. Browder [6] and Novikov [13]. In another direction, we can use a theorem of Berstein-Ganea [5] which asserts that the conditions (a), (b) below imply that the map $f : M \to P$ is a homotopy equivalence. From it we obtain:

There is a closed manifold P such that M is homeomorphic (diffeomorphic) to $P \times H$, if and only if, there is an integer n such that (a) *$H^n(M) \neq 0$, and* (b) *M is dominated by a manifold* P (i.e., there are maps $f : M \to P$ and $g : P \to M$ such that $g \circ f$ is homotopic to the identity map on M.)

2. Lemmas

We need two lemmas concerning Property Z which was first introduced by R. D. Anderson in [2]. (A closed set $Y \subset W$ is said to have *Property Z in W* if, for each homotopically trivial and non-void open set U in W, $U-Y$ is homotopically trivial and non-void.)

LEMMA 1. *If $X = A \cup B$, where A, B, and $A \cap B$ are homeomorphic to H and $A \cap B$ is closed with Property Z in both A and B, then, for any neighborhood N of $A \cap B$, there is a homeomorphism of A onto X whicq is fixed on $A-N$.*

The proof (which is left to the reader) follows easily from Corollary 10.3 of [2], which asserts that any homeomorphism between two closed sets with Property Z in H can be extended to a homeomorphism of all of H onto itself.

LEMMA 2. *If X is a separable manifold modeled on H and A is a closed subset of X such that $A = \bigcup\mathscr{F}$, where $\mathscr{F}$ is a locally-finite collection of closed sets with Property Z in X. Then A has Property Z in X.*

PROOF. A has Property Z in X if and only if each $x \in A$ has a neighborhood W such that $A \cap U$ has Property Z in W. (Lemma 1 of [4].) Choose W so small that it intersects only finitely many members of $\mathscr{F}$, say $\{F_1, F_2, \cdots, F_n\}$. Let U be a homotopically trivial and non-void open subset of W. Then, since each F_i is closed and has Property Z in X,

$$U-F_1,\ (U-F_1)-F_2,\ \cdots,\ U-\bigcup\{F_1, F_2, \cdots, F_n\}$$

are successively homotopically trivial and non-void open sets. Therefore, $W \cap \bigcup\{F_1, F_2, \cdots, F_n\} = W \cap A$ has Property Z in W.

3. Proof of theorem

The proof uses several basic results from the Theory of Combinatorial Topology. *Most* of these results can be found in any treatise in the subject. *All* of the needed results can be found in [15].

Let L be any countable locally-finite simplicial complex (clfsc) which has the homotopy type of M. (See, for example, [8].) Consider $|L|$ linearly embedded in H so that, if $\{v_1, v_2, \cdots\}$ are the vertices of L, then $|v_i|$ is the point of H whose i-th coordinate is 1 and whose other coordinates are 0. Define E^i to be the set of all points of H whose j-th coordinates are 0 for each $j > i$. Let T be any triangulation of $\bigcup\{E_i \mid i = 1, 2, 3, \cdots\}$ such that $T \cap E^i$

is a combinatorial triangulation of E^i and a subcomplex of T and such that L is a subcomplex of T. Note that T (but not $T \cap E^i$) will, of necessity, fail to be locally finite. If Q is a simplicial complex, then let Q'' denote the second barycentric subdivision of Q; and, if α is a simplex of Q, let $\mathrm{st}(\alpha, Q)$ be the smallest subcomplex of Q which contains all those simplices of Q which have α as a face. For each $\alpha \in L$, let $b(\alpha)$ be the barycenter of α and define

$$D(\alpha) = |\mathrm{st}(b(\alpha), (T \cap E^{n(\alpha)})'')|,$$

where $n(\alpha)$ is the smallest integer such that $|\mathrm{st}(\alpha, L)| \subset E^{n(\alpha)}$. If β is a face of α (written "$\beta < \alpha$"), then $\mathrm{st}(\beta, L) \supset \mathrm{st}(\alpha, L)$; and, thus,

$$D(\beta) \cap E^{n(\alpha)} = |\mathrm{st}(b(\beta), (T \cap E^{n(\alpha)})'')|.$$

There follows a list of those properties of the $D(\alpha)$'s which we shall use. These properties may be verified by standard combinatorial arguments. (See, in particular [15], pages 196 and 197, where C is a simplex and $C^* = D(C)$.)

(i) $D(\alpha)$ is an $n(\alpha)$-cell.

(ii) $D(\alpha) \cap D(\beta) \neq \emptyset$, if an only if, $\alpha < \beta$ or $\beta < \alpha$.

(iii) Let $\bigcup\{D(\beta) | \beta < \alpha \text{ and } \beta \neq \alpha\} = D(\mathrm{bd}\, \alpha)$.
Then $(D(\alpha), D(\alpha) \cap D(\mathrm{bd}\, \alpha))$ is homeomorphic to the pair $[I^{n(\alpha)-\dim(\alpha)} \times I^{\dim(\alpha)}, I^{n(\alpha)-\dim(\alpha)} \times \mathrm{bd}(I^{\dim(\alpha)})]$.

Let K be the subcomplex of T'' such that $|K| = \bigcup\{D(\alpha) | \alpha \in L\}$. We will show that K is the clfsc whose existence is asserted in the Theorem. K also corresponds to the P of the first special case. Let

$$C = ((|K| \times R) \cup (H \times [0, \infty))) \times H \subset (H \times R) \times H.$$

We shall finish the proof of the theorem via several propositions. In the proofs of these propositions we will several times need to use a theorem proved by Klee (Theorem III. 1.3 of [10]); *Hilbert space H is homeomorphic to $H \times (0, 1]$ and to $H \times [0, 1]$.*

PROPOSITION 1. *C is homeomorphic to H.*

PROOF. Define $C_{-1} = (H \times [0, \infty)) \times H \subset C$. C_{-1} is clearly homeomorphic to a half-space of H; and, therefore by Klee's Theorem, $C_{-1} \cong H$. Inductively define

$$C_n = C_{n-1} \cup [(\bigcup\{D(\alpha) | \alpha \in L \text{ and } \dim(\alpha) = n\} \times (-\infty, 0]) \times H].$$

and, for a fixed ordering $(\alpha_1^n, \alpha_2^n, \alpha_3^n, \cdots\}$ of the n-simplices of L, define

$$C_{n,0} = C_{n-1} \text{ and } C_{n,i} = C_{n,i-1} \cup (D(\alpha_i^n) \times (-\infty, 0] \times H).$$

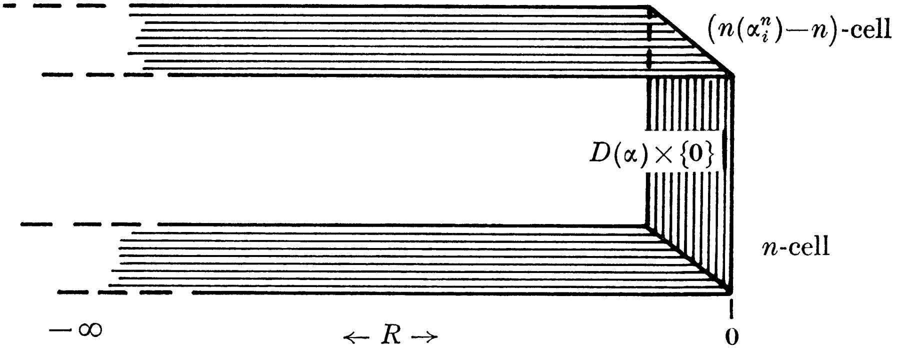
Let $n \geqq 0$ and $i \geqq 1$, and assume inductively that $C_{n,i-1} \cong H$. $(D(\alpha_i^n) \times (-\infty, 0] \times H) \equiv D_i^n$ is homeomorphic to H by Klee's Theorem and (i) above.

$$C_{n,i-1} \cap D_i^n = [(D(\alpha_i^n) \cap D(\mathrm{bd}\, \alpha_i^n) \times (-\infty, 0]) \cup (|D(\alpha_i^n)| \times \{0\})] \times H,$$

which by using (ii) and (iii) above can be seen (see figure) to be homeomorphic to

$$(\text{open } n\text{-cell}) \times (\text{closed } (n(\alpha_i^n) - n)\text{-cell}) \times H,$$

which in turn is homeomorphic to H by Klee's Theorem.



Let $\dot{D}(\alpha)$ be the combinatorial boundary of $D(\alpha)$. Then, since the pair $(D(\alpha), \dot{D}(\alpha))$ is homeomorphic to a ball and its boundary, it is easy to see that $\dot{D}(\alpha) \times (-\infty, 0] \times H$ has Property Z in $D(\alpha) \times (-\infty, 0] \times H$. Also $H \times \{0\} \times H$ is a closed set with Property Z in $H \times [0, \infty) \times H$. Repeated applications of Lemma 2, above, and Lemma 2 of [4] lead to the conclusion that $C_{n,i-1} \cap D_i^n$ has Property Z in both $C_{n,i-1}$ and in D_i^n. Lemma 1 now applies and we conclude that $C_{n,i} \cong H$. For each n, i, let $A_{n,i}$ be a neighborhood of D_i^n in $C_{n,i}$ such that $A_{n,i} \cap A_{m,i} \neq \emptyset$, $m \geqq n$, if and only if $\alpha_i^n < \alpha_j^m$. We may assume by Lemma 1 that there are homeomorphisms $h_{n,i} : C_{n,i-1} \to C_{n,i}$ such that $h_{n,i} | C_{n,i-1} - A_{n,i} = \mathrm{id}$. Then the transfinite sequence

$$\cdots \circ h_{n+1,0} \circ \cdots \circ h_{n,i} \circ h_{n,i-1} \circ \cdots \circ h_{n,0} \circ \cdots \circ h_{1,1} \circ h_{1,0} \circ \cdots \circ h_{0,1} \circ h_{0,0}$$

moves some neighborhood of each point at most finitely often because L is locally-finite; and thus the sequence converges to a $1-1$ map $h : C_{-1} \to C$. It is easy to check that h is a homeomorphism and thus $C \cong H$.

PROPOSITION 2. *Let* $N = (|K| \times (-\infty, -1)) \times H \subset C$. *Then* $N \cong \mathrm{cl}\,(N) \cong \mathrm{bd}\,(N) \cong H \times |K|$. (This is the N of the Theorem.)

PROOF. This follows immediately from Klee's Theorem and the observation that $\mathrm{cl}(N) = (|K| \times (-\infty, -1]) \times H$.

PROPOSITION 3. $M \cong N$.

PROOF. M and N are both open subsets of H and they have the same homotopy type. It follows directly from recent results of Kuiper and Burghelea [11] and Moulis [12] that M and N are homeomorphic (in fact, diffeomorphic).

PROPOSITION 4. $C - N \cong C - \mathrm{cl}(N) \cong H$.

PROOF. Clearly $C - \mathrm{cl}(N)$ is homeomorphic to C and thus, by Proposition 1, to H. The set

$$(|K| \times \{-1\}) \times H = (C - N) - (C - \mathrm{cl}(N))$$

is a closed set with Property Z (because it is collared) in $C - N$. Using Klee's Theorem it is easy to see that $C - N$ is a manifold; therefore, Theorem 4 of [4] asserts that $C - N$ is homeomorphic to

$$(C - N) - ((|K| \times \{-1\}) \times H) = C - \mathrm{cl}(N),$$

and thus homeomorphic to H.

PROPOSITION 5. *Conclusion* (*c*) *of the Theorem is satisfied.*

PROOF. This holds because $\mathrm{bd}\,(N) = (|K| \times \{-1\}) \times H$ and $(|K| \times (-\infty, 0)) \times H$ is an open neighborhood of

$$(|K| \times (-\infty, -1)) \times H = N$$

in C.

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