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by

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This paper is devoted to a simplification and extension of results obtained in Resnikoff [2].

PROPOSITION. *Let D be any homogeneous domain in $\mathbf{C}^n$, n-dimensional complex space, such that if $z \in D$ and $\lambda \in \mathbf{C}$ and $|\lambda| \leqq 1$ then $\lambda z \in D$ (i.e., D is a complete circular domain). Then given a compact subset H of D there are constants $a_H > 0$ and $b_H < \infty$ such that for all $z \in D$ and $\zeta \in H$*

$$a_H \leqq |K_D(z, \zeta)| \leqq b_H$$

where K_D denotes the Bergman kernel function of the domain D.

COROLLARY. *If E is a domain holomorphically equivalent to a domain D which satisfies the hypotheses of the Proposition and H is any compact subset of E then there exist constants $c_H > 0$ and $d_H < \infty$ such that*

$$c_H|K_E(z, \zeta)| \leqq |K_E(z, \zeta')| \leqq d_H|K_E(z, \zeta)|$$

for all $z \in E$, $\zeta \in H$, and $\zeta' \in H$.

PROOF OF THE PROPOSITION. As justified in Cartan [1] we can choose a sequence of homogeneous polynomials $\varphi_0, \varphi_1, \varphi_2, \cdots$ which constitute a Hilbert basis of the Hilbert space of square integrable holomorphic functions on D. It is known that there are the same number of φ_j's which have homogeneous degree m as there are n-tuples $(a_1, a_2, \cdots, a_n)$ of non-negative integers such that $a_1+a_2+ \cdots +a_n = m$. Choose notation so that φ_0 is constant. Clearly $\varphi_0 \neq 0$. We may then write

$$K_D(z, \zeta) = \sum_{j=0}^{\infty} \varphi_j(z)\bar{\varphi}_j(\zeta)$$

where $\bar{\varphi}_j(\zeta)$ denotes the complex conjugate of $\varphi_j(\zeta)$.

We first show that $K_D(z, \zeta) \neq 0$ for any $z, \zeta \in D$. For if $K_D(z, \zeta) = 0$ by the homogeneity of D and the transformation formula of K_D we could find a $z' \in D$ such that $K_D(z', 0) = 0$ (where 0 also represents the origin in C^n). But $K_D(z', 0) = |\varphi_0|^2$.

Let Δ denote the closure of λD where $0 < \lambda < 1$. Given $z \in D$ and $\zeta \in \lambda\Delta$, $K_D(z, \zeta) = K_D(\lambda z, w)$ where $\lambda w = \zeta$. So

$$|K_D(z, \zeta)| \geqq \inf \{|K_D(p, q)| : p \in \Delta \text{ and } q \in \Delta\} > 0$$

since K_D is continuous and non zero on the compact set $\Delta \times \Delta$. This establishes the first inequality.

Similarly

$$|K(z, \zeta)| \leqq \sup \{|K_D(p, q)| : p \in \Delta, q \in \Delta\} < \infty.$$

Proof of the Corollary. It is clear that D satisfies the conclusion of the corollary, and as indicated in Resnikoff [2] this property is preserved under holomorphic equivalence.

Remark. It is easily seen that the above Proposition does not hold for all bounded domains holomorphically equivalent to those described in the hypothesis. For example consider a domain V:

$$V = \{x+iy : 0 \leqq x \leqq 2 \text{ and } 0 \leqq y \leqq 1, \text{ or } 0 \leqq x \leqq 1 \text{ and } 1 \leqq y \leqq 2\}.$$

Let T be a conformal mapping of V onto the unit disk D. We now study the behaviour of $T'(z) = dT(z)/dz$ near 0 and $1+i$. First let $V' = \{z^2 : z \in V\}$. Let $\varphi(z)$ denote branch of $\sqrt{w}$ defined on the upper half plane such that $\varphi(e^{i\pi/2}) = e^{i\pi/4}$. Then $T \circ \varphi$ is a conformal mapping of V' onto D which by the Schwartz reflection principle can be extended to a function ψ which is conformal in some neighborhood of 0. Writing $T(z) = \psi(z^2)$ it is obvious that $T'(z) \to 0$ as $z \to 0$, so the left hand inequality of the Proposition cannot hold for any $a_H > 0$ for the domain V.

A similar argument shows that $|T'(z)| \to \infty$ as $z \to 1+i$, so the right hand inequality of the Proposition cannot hold for V for any $b_H < \infty$.

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