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by

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Summary

Let $\bar{X}_1, \bar{X}_2, \ldots$ be strictly d-dimensional independent random vectors with common distribution function F, with finite second moments and nonzero first moment vector $\bar{\mu}$. Let $U(A) = \Sigma_1^\infty F^m(A)$, where F^m denotes the m-fold convolution of F. The paper studies the asymptotic behaviour as $|\bar{x}| \to \infty$ of $U(A+\bar{x})$ for bounded A. The results of Doney (Proc. London Math. Soc. 26 (1966), 669—684) are derived under more general conditions by a new technique, viz. by first studying the more easily manageable generalized renewal measure $W_F = \Sigma_1^\infty m^\rho F^m$, where $\rho = \frac{1}{2}(d-1)$. This is done by comparing W_F and W_G for F and G having the same first and second moments, using local central limit theorems.

1. Introduction

Throughout this paper F, G and H will denote distribution functions of strictly d-dimensional probability measures — also denoted by F, G, H — with characteristic functions φ, ψ, χ, respectively. A measure on the Borelsets of R_d is called strictly d-dimensional if its support is not contained in a hyperplane of dimension lower than d.

Convolutions will be written as products or powers. Vectors, random or not, will be distinguished from scalars by a bar. The inner product of the vectors $\bar{x}$ and $\bar{y}$ will be written $(\bar{x}, \bar{y})$, and $|\bar{x}| = (\bar{x}, \bar{x})^{\frac{1}{2}}$.

The second moments of F, G, H will be finite and the first moment vector

$$(1.1) \qquad \bar{\mu} = \int \bar{x} F(d\bar{x})$$

of F will be nonzero.

We consider the sequence $\bar{X}_k \equiv (X_{k1}, \cdots, X_{kd})$, $k = 1, 2, \cdots$, of independent random vectors with common distribution function F, and the random walk $\bar{S}_n = \bar{X}_1 + \cdots + \bar{X}_n$, $n = 1, 2, \cdots$. Let $N(A)$ be the number of n with $\bar{S}_n \in A$ and

$$(1.2) \qquad U_F(A) \stackrel{\text{df}}{=} E\{N(A)\} = \sum_{m=1}^{\infty} F^m(A).$$

The corresponding quantities for G and H will be denoted by U_G and U_H. A similar convention will apply to W_F defined below in (1.8).

It was shown by Doney [2] that for $|\bar{c}| = 1$ and bounded A with boundary having volume zero,

$$(1.3) \qquad \lim_{t\to\infty} t^\rho U(A+t\bar{c}) = \begin{cases} \beta \operatorname{Vol}(A), & (\bar{c}, \bar{\mu}) = |\bar{\mu}|, \\ 0, & (\bar{c}, \bar{\mu}) < |\bar{\mu}|, \end{cases}$$

where

$$(1.4) \qquad \rho = \tfrac{1}{2}(d-1),$$

if $\limsup_{|\bar{\mu}|\to\infty} |\varphi(\bar{\mu})| < 1$ and sufficiently many moments of F exist. He also gave a version of (1.3) for integer valued $X_{11}, \cdots, X_{1d}$. The constant β depends on the first and second moments of F.

That the number of moments required increases with d, follows from (1.2) since its first term $F(A+t\bar{c})$ may spoil (1.3) if it does not tend to zero fast enough. Now consider the case $d = 3$, $\mu_1 > 0$, $\mu_2 = \mu_3 = 0$. By symmetry

$$(1.5^a) \qquad \begin{aligned} &\int_E y_1 F^m(d\bar{y}) \\ &= \int \cdots \int I_E(\bar{x}_1 + \cdots + \bar{x}_m)(x_{11} + \cdots + x_{m1}) F(d\bar{x}_1) \cdots F(d\bar{x}_m) \\ &= mQF^{m-1}(E) \end{aligned}$$

with

$$(1.5^b) \qquad Q(E) = \int_E y_1 F(d\bar{y}),$$

so that

$$(1.6) \qquad \int_E y_1 U_F(d\bar{y}) = \sum_{m=0}^{\infty} (m+1) QF^m(E).$$

From (1.3) and (1.6) it follows then that

$$(1.7) \qquad \lim_{t\to\infty} \sum_{m=0}^{\infty} (m+1) QF^m(A+t) = \beta \operatorname{Vol}(A),$$

a relation that cannot be disturbed by misbehaviour of the first term. We are led to consider the asymptotics of

$$(1.8) \qquad W_F(A) \stackrel{\text{df}}{=} \sum_{m=1}^{\infty} m^\rho F^m(A),$$

and from these to derive Doney's results by extensions of (1.6). For $d = 1$ expressions of the form (1.8) were studied by Smith [6], [9] and Kalma [4]. That $W_F(A) < \infty$, will be shown in lemma 2.4.

In order to describe the different possibilities of lattice behaviour of F, we introduce the following terminology

DEFINITION 1.1. If $\{\bar{u} : \varphi(\bar{u}) = 1\} = \{0\}$, we will call F nonarithmetic, if $\{\bar{u} : |\varphi(\bar{u})| = 1\} = \{0\}$, it will be called nonlattice. If

(i) $\{\bar{u} : |\psi(\bar{u})| = 1\} = \{\bar{u} : |\chi(\bar{u})| = 1\}$,

(ii) $\psi(\bar{u}) = \chi(\bar{u})$ if $|\psi(\bar{u})| = 1$,

G and H are called equilattice.

REMARK. Let $\bar{X}$ have distribution function F. Then there is a nonsingular homogeneous linear coordinate transformation $\bar{X} \to \bar{Y}$, such that the characteristic function ζ of $\bar{Y}$ has the following properties: $|\zeta(\bar{u})| < 1$ except if $u_1, \cdots, u_s$ are integer multiples of 2π and $u_{s+1} = \cdots = u_d = 0$. Then $\bar{Y} = \bar{a} + \bar{Z}$ with $\bar{a}$ deterministic and the first s components of $\bar{Z}$ a.s. integer valued. If $a_1, \cdots, a_s$ are irrational, F is nonarithmetic but not nonlattice. We refer to Spitzer [7], Ch. II. 7.

If two distribution functions are equilattice, the same s and $\bar{a}$ apply.

Our main results are the following: If G and H have the same (nonzero) first and second moments and are nonarithmetic or equilattice, W_G and W_H have the same asymptotic behaviour (section 3). This result avoids the tedious classification of lattice behaviour and opens a way to more refined estimates, to be derived in a subsequent paper. From these theorems we then obtain $\lim_{t\to\infty} W_F(A+t\bar{c})$ for two special cases: F nonarithmetic (theorem 4.2) and F "totally arithmetic" (theorem 4.3) and then $\lim_{t\to\infty} t^\rho U_F(A+t\bar{c})$ under the extra assumption $E|(\bar{c}, \bar{X}_1)|^\rho < \infty$, which in a certain sense is best possible (section 6). It is noted that, in the same way as for $d = 1$, the relation (1.3) is connected with F being nonarithmetic, not nonlattice. The second part of (1.3) is independent of lattice properties.

Techniques of proof were inspired by the proof of theorem P1, § 26, in Spitzer [7], using local central limit theorems.

2. Preliminary lemmas

LEMMA 2.1. *Let G and H have the same first and second moments and let $|\psi(\bar{u})| < 1$, $|\chi(\bar{u})| < 1$ on $D - \{0\}$, where $D = \{\bar{u} : |\bar{u}| \leqq a\}$.*

If $\eta(\bar{u})$ has continuous second derivatives on D and

$$J_m(\bar{x}) \overset{\text{df}}{=} \int_D \eta(\bar{u})\{\psi^m(\bar{u})-\chi^m(\bar{u})\} \exp\{-i(\bar{u}, \bar{x})\}d\bar{u},$$

then

(a) $$\lim_{m\to\infty} m^{d/2} J_m(\bar{x}) = 0,$$

uniformly in $\bar{x}$ and

(b) $$\lim_{m\to\infty} |\bar{x}-m\bar{\mu}|^2 m^{\frac{1}{2}d-1} J_m(\bar{x}) = 0,$$

uniformly in $\bar{x}$. Here $\bar{\mu} = \int \bar{x}dG$.

REMARK. The lemma also holds if D is replaced by any sufficiently regular bounded domain.

PROOF. We refer to the proofs of theorem P 9 (remark) and P 10 in Spitzer [7], Ch. II. 7. In (b) we have to write

$$\psi(\bar{u}) = \psi_0(\bar{u}) \exp\{i(\bar{u}, \bar{\mu})\}, \qquad \chi(\bar{u}) = \chi_0(\bar{u}) \exp\{i(\bar{u}, \bar{\mu})\}.$$

It is noted that the boundary terms arising by the application of Green's theorem tend to zero exponentially as $m \to \infty$, uniformly in $\bar{x}$.

LEMMA 2.2. *If F is gaussian, the density $w_F(\bar{x})$ of W_F is bounded and*

(a) $$\lim_{|\bar{x}|\to\infty} w_F(\bar{x}) = 0,$$

uniformly in every closed sector not containing $\bar{\mu}$. Furthermore

(b) $$\lim_{t\to\infty} w_F(\bar{\xi}+t\bar{\mu}) = (2\pi)^{-\rho}(\text{Det } B)^{-\frac{1}{2}}|\bar{\mu}|^{-1},$$

uniformly with respect to $\bar{\xi}$ in bounded sets. Here B is the covariance matrix of $Y_2, \cdots, Y_d$ determined as follows: Let the random vector $\bar{X}$ have distribution F. Then $Y_1, \cdots, Y_d$ are the components of $\bar{X}$ in a Cartesian coordinate system with y_1-axis in the direction of $\bar{\mu}$.

PROOF OF (a) and boundedness. It is no restriction to assume that a nonsingular homogeneous linear coordinate transformation has been carried out so that $X_1, \cdots, X_d$ are independent with unit variances and $\mu_1 = \mu > 0$, $\mu_2 = \cdots = \mu_d = 0$. Then

$$w_F(\bar{x}) = a_0 \sum_{m=1}^{\infty} m^{-\frac{1}{2}} \exp\left[-\frac{1}{2m}\{(x_1-m\mu)^2+\eta^2\}\right],$$

where $\eta^2 = x_2^2 + \cdots + x_d^2$, so that $w_F(\bar{x})$ is majorized by the one-dimensional renewal density of $N(\mu, 1)$. In the sector $\{|\eta| \geqq 2\theta x_1\}$ with $0 < \theta \leqq 1$ we have

$$(x_1-m\mu)^2+\eta^2 \geqq \theta^2(x_1-m\mu)^2+\eta^2 \geqq \tfrac{1}{2}(\theta x_1-\theta\mu m-|\eta|)^2,$$

so that $w_F(\bar{x})$ is majorized by the one-dimensional renewal density at $\theta x_1-|\eta|$ of $N(\theta\mu, 2)$, where $\theta x_1-|\eta| \to -\infty$ as $|\bar{x}| \to \infty$, uniformly in the sector.

Proof of (b). It is no restriction to assume that $\mu_1 = \mu > 0$, $\mu_2 = \cdots = \mu_d = 0$. Let C be the covariance matrix of $X_1, \cdots, X_d$ and $A = C^{-1}$. Then

$$w_F(\bar{x}) = \sum_{m=1}^{\infty} (2\pi)^{-\frac{1}{2}d} m^{-\frac{1}{2}} (\text{Det } C)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2m} Z(\bar{x})\right\},$$

with

$$Z(\bar{x}) = a_{11}\{x_1-m\mu+\alpha(\bar{x})\}^2+\beta(\bar{x}),$$

where $\alpha(\bar{x})$ and $\beta(\bar{x})$ are bounded if $|\bar{x}| \to \infty$ in the way stated in the conditions. It is noted that $a_{11} > 0$ since F is strictly d-dimensional. If $\beta(\bar{x})$ were zero, the one-dimensional renewal theorem would give

$$\lim_{t\to\infty} w_F(\xi+t\mu) = (2\pi)^{-\rho}(a_{11} \text{ Det } C)^{-\frac{1}{2}}\mu^{-1},$$

from which the theorem follows with the relation

$$a_{11} \text{ Det } C = \begin{vmatrix} C_{22} & \cdots & C_{2d} \\ \vdots & & \\ C_{d2} & \cdots & C_{dd} \end{vmatrix}.$$

But $\exp\{-(1/2m)\beta(\bar{x})\} = 1+m^{-1}\zeta(\underline{x})$ with $\zeta(\bar{x})$ bounded and the contribution of the terms with $m^{-1}\zeta(\bar{x})$ tends to zero for $t \to \infty$, again by the one-dimensional renewal theorem, as is seen by writing

$$\sum_{1}^{\infty} = \sum_{1}^{M} + \sum_{M+1}^{\infty} \text{ with } M \text{ large.}$$

Definition 2.3. A continuous function $g \in L_1$ on R_d belongs to class K_d if its Fourier transform vanishes outside a bounded set $B(g)$ and has continuous second derivatives.

Remark. We make use of K_d since weak convergence of measures on R_d is implied by convergence of their integrals of elements of K_d. See Breiman [1], Ch. 10.2.

An example of a nonnegative element of K_d is a product of sufficiently high even powers of $x_k^{-1} \sin a_k x_k$, $k = 1, \cdots, d$.

LEMMA 2.4. Under the assumptions of section 1 we have $W_F(A) < \infty$ for bounded A, in fact $W_F(A+\bar{x})$ is bounded with respect to $\bar{x}$.

PROOF. Let G have the same first moments and covariance matrix as F. The proof compares W_F and W_G. It is lengthy but large parts of it will serve again in proving deeper results below.

Let $g \geqq 0$, $g \in K_d$, such that $|\varphi(u)| < 1$, $|\psi(u)| < 1$ on $B(g)-\{0\}$. We consider

$$(2.1)\qquad \begin{aligned} T_m(\bar{x}) &= \int g(\bar{y}-\bar{x})F^m(d\bar{y})-\int g(\bar{y}-\bar{x})G^m(d\bar{y}) \\ &= (2\pi)^{-d}\int \gamma(-\bar{u})\{\varphi^m(\bar{u})-\psi^m(\bar{u})\}\exp\{-i(\bar{u},\bar{x})\}d\bar{u}, \end{aligned}$$

where γ is the Fourier transform of g. By lemma 2.1:

$$(2.2)\qquad m^\rho|T_m(\bar{x})| \leqq m^{-\frac{1}{2}}\delta(m),$$

$$(2.3)\qquad \begin{aligned} m^\rho|T_m(\bar{x})| &\leqq m^{\frac{1}{2}}\varepsilon(m)|\bar{x}-m\mu|^{-2} \\ &\leqq \mu^{-2}m^{\frac{1}{2}}\varepsilon(m)(m-\mu^{-1}|x|)^{-2}, \end{aligned}$$

with

$$(2.4)\qquad \mu = |\bar{\mu}|$$

$$(2.5)\qquad \lim_{m\to\infty}\delta(m) = \lim_{m\to\infty}\varepsilon(m) = 0.$$

Putting

$$(2.6)\qquad a(\bar{x}) = \mu^{-1}|\bar{x}|-|\bar{x}|^{\frac{1}{2}},\ b(\bar{x}) = \mu^{-1}|\bar{x}|+|\bar{x}|^{\frac{1}{2}},$$

we have by (2.1)—(2.5)

$$(2.7)\qquad V_1(|\bar{x}|) \overset{\text{df}}{=} \sum_{m=1}^{[a(\bar{x})]} m^\rho|T_m(\bar{x})| \leqq \int_1^{a(\bar{x})} y^{\frac{1}{2}}\varepsilon_1(y)(y-\mu^{-1}|\bar{x}|)^{-2}dy,$$

$$(2.8)\qquad V_2(|\bar{x}|) \overset{\text{df}}{=} \sum_{m=1+[a(\bar{x})]}^{[b(\bar{x})]} m^\rho|T_m(\bar{x})| \leqq \int_{a(\bar{x})}^{b(\bar{x})} y^{-\frac{1}{2}}\delta_1(y)dy,$$

$$(2.9)\qquad V_3(|\bar{x}|) \overset{\text{df}}{=} \sum_{m=1+[b(\bar{x})]}^{\infty} m^\rho|T_m(\bar{x})| \leqq \int_{b(\bar{x})}^{\infty} y^{\frac{1}{2}}\varepsilon_1(y)(y-\mu^{-1}|\bar{x}|)^{-2}dy,$$

with

$$(2.10)\qquad \lim_{y\to\infty}\delta_1(y) = \lim_{y\to\infty}\varepsilon_1(y) = 0.$$

It is easily seen that the $V_i(|\bar{x}|)$ are finite. By taking for G a gaussian distribution, it follows then from lemma 2.2 that

$$\int g(\bar{y}-\bar{x})W_F(d\bar{y}) < \infty$$

and therefore $W_F(A) < \infty$ for bounded A.

It will be shown that the V_i are bounded and

$$\lim_{|\bar{x}|\to\infty} V_i(|\bar{x}|) = 0, \qquad i = 1, 2, 3. \tag{2.11}$$

The boundedness of $W_F(A+\bar{x})$ with respect to $\bar{x}$ then follows from lemma 2.2.

That $V_1(|\bar{x}|) \to 0$, follows from a theorem of Toeplitz (Loève [5], § 16, Hardy [3], Ch. III), since (2.10) holds,

$$\lim_{|\bar{x}|\to\infty} y^{\frac{1}{2}}(\mu^{-1}|\bar{x}|-y)^{-2} = 0$$

for fixed y and

$$\int_1^{a(\bar{x})} y^{\frac{1}{2}}(\mu^{-1}|\bar{x}|-y)^{-2}dy$$

is bounded in $|\bar{x}|$.

That $V_i(|\bar{x}|) \to 0$, $i = 2, 3$, may be derived from (2.6), (2.10) and the boundedness in $|\bar{x}|$ of

$$\int_{a(\bar{x})}^{b(\bar{x})} y^{-\frac{1}{2}}dy \text{ and } \int_{b(\bar{x})}^{\infty} y^{\frac{1}{2}}(y-\mu^{-1}|\bar{x}|)^{-2}dy.$$

3. Comparison theorems

Theorem 3.1. *If G and H have the same nonzero first moment vector and the same covariance matrix, there is a nonnegative $g \in K_d$, positive on a neighbourhood of 0, such that*

$$\lim_{|\bar{x}|\to\infty} \left\{\int g(\bar{y}-\bar{x})W_G(d\bar{y}) - \int g(\bar{y}-\bar{x})W_H(d\bar{y})\right\} = 0, \tag{3.1}$$

uniformly in the direction of $\bar{x}$.

Proof. This is implicit in the proof of lemma 2.4 (See (2.11)). We may take g as in the example following definition 2.3 with the a_k sufficiently small.

Theorem 3.2. *If G and H are nonarithmetic and have the same nonzero first moment vector and the same covariance matrix, then (3.1) holds, uniformly in the direction of $\bar{x}$, for any $g \in K_d$.*

Proof. First assume that d is odd, so that ρ is an integer. Let γ be the Fourier transform of g. Then

$$\int g(\bar{y}-\bar{x})W_G(d\bar{y}) - \int g(\bar{y}-\bar{x})W_H(d\bar{y}) = A(\bar{x})+B(\bar{x}), \tag{3.2}$$

(3.3)
$$A(\bar{x}) = (2\pi)^{-d}\sum_{m=1}^{\infty} m^{\rho}\int_{D}\gamma(-\bar{u})\{\psi^m(\bar{u})-\chi^m(\bar{u})\}\exp\{-i(\bar{u},\bar{x})\}d\bar{u},$$

(3.4)
$$B(\bar{x}) = (2\pi)^{-d}\sum_{m=1}^{\infty} m^{\rho}\int_{D^c}\gamma(-\bar{u})\{\psi^m(\bar{u})-\chi^m(\bar{u})\}\exp\{-i(\bar{u},\bar{x})\}d\bar{u},$$

where $D = \{\bar{u} : |\bar{u}| \leqq \xi\}$ is such that $|\psi(\bar{u})| < 1$ and $|\chi(\bar{u})| < 1$ on $D-\{0\}$. In the same way as in the proof of lemma 2.4 we majorize $A(\bar{x})$ by $V_1(|\bar{x}|)+V_2(|\bar{x}|)+V_3(|\bar{x}|)$, so that by (2.11)

$$\lim_{|\bar{x}|\to\infty} A(\bar{x}) = 0, \tag{3.5}$$

uniformly in the direction of $\bar{x}$. By lemma 2.4 both terms on the left in (3.2) are finite. As seen above, the series (3.3) converges. So the same is true for the series (3.4) and with Abel's theorem

$$B(\bar{x}) = \lim_{r\uparrow 1}\,(2\pi)^{-d}\sum_{m=1}^{\infty} r^m m^{\rho}\int_{D^c}\cdots,$$

$$\begin{aligned} B(\bar{x}) = \lim_{r\uparrow 1}\,(2\pi)^{-d}\int_{D^c}\gamma(-u)\{&\Lambda(r\psi(\bar{u})) \\ &-\Lambda(r\chi(\bar{u}))\}\exp\{-i(\bar{u},\bar{x})\}d\bar{u},\end{aligned} \tag{3.6}$$

with

$$\Lambda(z) = \sum_{m=1}^{\infty} m^{\rho} z^m.$$

Now $\Lambda(z)$ is a finite sum of powers of $(1-z)^{-1}$. Since G and H are nonarithmetic, $\psi(\bar{u})$ and $\chi(\bar{u})$ are bounded away from 1 on $D^c B(g)$ and the limit in (3.6) may be taken under the integral sign. By the same argument the Riemann-Lebesgue lemma then applies to the limiting integral, so that

$$\lim_{|\bar{x}|\to\infty} B(\bar{x}) = 0, \tag{3.7}$$

uniformly in the direction of $\bar{x}$, and (3.1) follows from (3.2), (3.5) and (3.7).

Now assume that d is even. We write down (3.1) for $d+1$ with $x_{d+1} = 0$ and G, H replaced by the product measures of G, H on R_d and the gaussian probability measure $N(0, 1)$ on R_1. For $g \in K_{d+1}$ we take $g_d(x_1, \cdots, x_d)g_1(x_{d+1})$ with $g_d \in K_d$, $g_1 \in K_1$. This gives

$$\lim_{|\bar{x}|\to\infty}\sum_{m=1}^{\infty} m^{\rho}\zeta(m)\int g_d(\bar{y}-\bar{x})[G^m(d\bar{y})-H^m(d\bar{y})] = 0,$$

with

$$\zeta(m) = \int g_1(t) \exp(-t^2/2m)dt.$$

Taking $t^2 g_1(t) \in L_1$ we have $\zeta(m) = \zeta(\sigma)+0(m^{-1})$, and the proof is finished by noting that

$$\lim_{|\bar{x}|\to\infty} \sum_{m=1}^{\infty} m^{\rho-1} \int g_d(\bar{y}-\bar{x})G^m(d\bar{y}) = 0, \tag{3.8}$$

uniformly in the direction of $\bar{x}$, and similarly for H. The relation (3.8) may be derived from lemma 2.4 by writing

$$\sum_{m=1}^{\infty} = \sum_{m=1}^{M} + \sum_{m=M+1}^{\infty}$$

with M large, and noting that for fixed m

$$\lim_{|\bar{x}|\to\infty} \int g_d(\bar{y}-\bar{x})G^m(d\bar{y}) = 0,$$

uniformly in the direction of $\bar{x}$.

THEOREM 3.3. *If G and H are equilattice and have the same nonzero first moment vector and the same covariance matrix,* (3.1) *holds for any $g \in K_d$, uniformly in the direction of $\bar{x}$.*

PROOF. We assume that the coordinate transformation described in the remark to definition 1.1 has been carried out, so that $|\psi(\bar{u})| < 1$, $|\chi(\bar{u})| < 1$, except if $u_1, \cdots, u_s$ are integer multiples of 2π and $u_{s+1} = \cdots u_d = 0$. Furthermore

$$\psi(\bar{u}) = \psi_1(\bar{u}) \exp\{i(\bar{u}, \bar{a})\},\ \chi(\bar{u}) = \chi_1(\bar{u}) \exp\{i(\bar{u}, \bar{a})\}, \tag{3.9}$$

where ψ_1 and χ_1 are periodic in $u_1, \cdots, u_s$ with period 2π. Let γ be the Fourier transform of g. Then

$$\int g(\bar{y}-\bar{x})G^m(d\bar{y}) = (2\pi)^{-d} \int \gamma(-\bar{u})\psi^m(\bar{u}) \exp\{-i(\bar{u}, \bar{x})\}d\bar{u}.$$

The bounded domain of integration (since $g \in K_d$) is divided into subdomains $\{-\pi+k_j 2\pi \leqq u_j < \pi+k_j 2\pi,\ j = 1, \cdots, s\}$ and by a change of variable each of them is translated to the corresponding domain centered at 0. By (3.9) and the periodicity of ψ_1 we then find

$$\int g(\bar{y}-\bar{x})G^m(d\bar{y}) = \sum_{\bar{k}} \exp\{2\pi i(\bar{k}, m\bar{a}-\bar{x})\}I(m, G, \bar{k}),$$

$$I(m, G, \bar{k}) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int \cdots \int \gamma(-\bar{v}-2\pi\bar{k})\psi^m(\bar{v}) \exp\{-i(\bar{v}, \bar{x})\}d\bar{v},$$

where the finite number of terms in the sum depends only on g. For H a similar relation holds and in the same way as in lemma 2.4 it is shown that

$$\lim_{|\bar{x}|\to\infty} \sum_{m=1}^{\infty} |I(m, G, \bar{k})-I(m, H, \bar{k})| = 0.$$

THEOREM 3.4. *Let G be arithmetic with $\psi(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of 2π and $|\psi(\bar{u})| < 1$ elsewhere. Let H with characteristic function $\chi \in L_1$ have the same nonzero first moment vector and the same covariance matrix as G. Then, as $\bar{k} \to \infty$ through d-dimensional integers,*

$$W_G(\{\bar{k}\})-w_H(\bar{k}) \to 0,$$

uniformly in the direction of $\bar{k}$. Here w_H is the density of W_H.

PROOF. Put $D = \{\bar{u} : -\pi \leqq u_j \leqq \pi, j = 1, \cdots, d\}$. Then

$$W_G(\{\bar{k}\})-w_H(\bar{k}) = -(2\pi)^{-d} \sum_{m=1}^{\infty} m^{\rho} \int_{D^c} \chi^m(\bar{u}) \exp\{-i(\bar{u}, \bar{k})\}d\bar{u}$$
$$+(2\pi)^{-d} \sum_{m=1}^{\infty} m^{\rho} \int_{D} \{\psi^m(u)-\chi^m(\bar{u})\} \exp\{-i(\bar{u}, \bar{k})\}d\bar{u}.$$

The first term tends to zero by the Riemann-Lebesgue lemma since $\chi \in L_1$ and $|\chi(\bar{u})| \leqq \theta < 1$ on D^c. The second term is treated in the same way as in the proof of lemma 2.4.

4. Limits of $W_F(A+\bar{x})$

THEOREM 4.1. *For bounded A, uniformly in every closed sector not containing $\bar{\mu}$,*

$$\lim_{|\bar{x}|\to\infty} W_F(A+\bar{x}) = 0,$$

PROOF. From lemma 2.2[a] and theorem 3.1.

THEOREM 4.2. *If F is nonarithmetic and A is bounded and the boundary of A has volume zero,*

$$\lim_{t\to\infty} W_F(A+t\bar{\mu}) = (2\pi)^{-\rho}(\operatorname{Det} B)^{-\frac{1}{2}}|\bar{\mu}|^{-1} \operatorname{Vol}(A), \tag{4.1}$$

where B is defined as in lemma 2.2[b].

PROOF. From lemma 2.2b and theorem 3.2. See the remark to definition 2.3. The theorem may be stated as weak convergence of the measure Z_t with $Z_t(A) = W_F(A+t\bar{\mu})$.

THEOREM 4.3. *Let F be arithmetic with $\varphi(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of 2π and $|\varphi(\bar{u})| < 1$ elsewhere. Then, as $\bar{x} \to \infty$ through d-dimensional integers within bounded distance of the halfline $\bar{y} = t\bar{\mu}$, $t > 0$,*

$$W_F(\{\bar{x}\}) \to (2\pi)^{-\rho}(\text{Det } B)^{-\frac{1}{2}}|\bar{\mu}|^{-1},$$

with B defined as in lemma 2.2[b].

PROOF. From lemma 2.2[b] and theorem 3.4.

5. Limit theorems for $U(A+\bar{x})$

THEOREM 5.1. *Let A be a bounded set.*

(a) *If $\bar{c}/|\bar{c}| \neq \bar{\mu}/|\bar{\mu}|$ and $\int |(\bar{c}, \bar{x})|^\rho F(d\bar{x}) < \infty$,*

$$\lim_{t\to\infty} t^\rho U_F(A+t\bar{c}) = 0.$$

(b) *If $\int |\bar{x}|^\rho F(d\bar{x}) < \infty$, then uniformly in every closed sector not containing $\bar{\mu}$,*

$$\lim_{t\to\infty} |\bar{x}|^\rho U_F(A+\bar{x}) = 0,$$

NOTE. The proof below does not apply to $d = 2$, but for $d = 2, 3, 4$ stronger results hold, due to the existence of second moments (theorem 5.2).

PROOF OF (a). By Minkowski's inequality and the symmetry in $\bar{X}_1, \cdots, \bar{X}_m$

$$\begin{aligned}(5.8)\qquad \int_B |(\bar{c}, \bar{x})|^\rho F^m(d\bar{x}) &= E\left\{\{I_B^\rho(\bar{S}_m)\left|\sum_{j=1}^m (\bar{c}, \bar{X}_j)\right|^\rho\right\} \\ &\leqq m^\rho E\{I_B(\bar{S}_m)|(\bar{c}, \bar{X}_1)|^\rho\} = m^\rho F^{m-1}R(B),\end{aligned}$$

with

$$R(E) = \int_E |(\bar{c}, \bar{x})|^\rho F(d\bar{x}). \tag{5.9}$$

So

$$\int_{A+t\bar{c}} |(\bar{c}, \bar{x})|^\rho U_F(d\bar{x}) \leqq R(A+t\bar{c})+2^\rho W_F R(A+t\bar{c}). \tag{5.10}$$

Here the first term tends to zero since R is a finite measure and the second term tends to zero since in

$$W_F R(A+t\bar{c}) = \int W_F(A-\bar{y}+t\bar{c})R(d\bar{y}) \tag{5.11}$$

the integrand converges to zero for $t \to \infty$ by theorem 4.1 and is bounded by lemma 2.4.

The theorem now follows from (5.10) and the boundedness of A.

PROOF OF (b). In the same way as above

$$\int_B |\bar{x}|^\rho F^m(dx) \leqq m^\rho F^{m-1}K(B), \tag{5.12}$$

with

$$K(E) = \int_E |\bar{x}|^\rho F(d\bar{x}),$$

and

$$\int_{A+\bar{x}} |\bar{y}|^\rho U_F(d\bar{y}) \leqq K(A+\bar{x})+2^\rho W_F K(A+\bar{x}). \tag{5.13}$$

The right-hand side of (5.13) has limit zero for $|\bar{x}| \to \infty$, (uniformly) in any closed sector not containing $\bar{\mu}$. This follows from theorem 4.1 by a relation analogous to (5.11). Since K is a finite measure, we may write $K = K_0+K_1$, with K_0 restricted to a bounded set and $K_1(R_d) < \varepsilon$, and make use of the boundedness of $W_F(A+\bar{z})$ stated in lemma 2.4.

THEOREM 5.2. *For bounded A we have, uniformly in any closed sector not containing $\bar{\mu}$,*

$$\lim_{|\bar{x}|\to\infty} \theta(|\bar{x}|)U_F(A+\bar{x}) = 0, \tag{5.14}$$

where $\theta(y) = y$ if $d = 2$, $\theta(y) = y^{\frac{3}{2}}$ if $d = 3$ and $\theta(y) = y^2/\log y$ if $d = 4$.

PROOF. The theorem is a consequence of the assumed existence of second moments.

It is no restriction to assume that $\mu_1 = 1$, $\mu_2 = \cdots = \mu_d = 0$. Take $g \in K_d$ as in the proof of lemma 2.4, and let G be gaussian with the same first and second moments as F. By (2.1) and lemma 2.1b.

$$\begin{aligned} Z(\bar{x}) &\overset{\text{df}}{=} \left| \int g(\bar{y}-\bar{x})U_F(d\bar{y}) - \int g(\bar{y}-\bar{x})U_G(d\bar{y}) \right| \\ &\leqq \sum_{m=1}^{\infty} m^{1-\frac{1}{2}d}\varepsilon(m)[(x_1-m)^2+\eta^2]^{-1} \\ &\leqq \int_1^\infty y^{1-\frac{1}{2}d}\varepsilon_1(y)[(y-x_1)^2+\eta^2]^{-1}dy, \end{aligned} \tag{5.15}$$

with $\eta^2 = x_2^2+\cdots+x_d^2$ and

$$\lim_{y\to\infty} \varepsilon(y) = \lim_{y\to\infty} \varepsilon_1(y) = 0. \tag{5.16}$$

By elementary integration we find that

$$\theta(|\bar{x}|)\int_1^\infty y^{1-\frac{1}{2}d}[(y-x_1)^2+\eta^2]^{-1}dy$$

is bounded in any closed sector not containing $\bar{\mu}$. (Put $\eta^2 \geqq \alpha^2 x_1^2$ and $y = x_1 z$). It follows then with (5.15), (5.16) and the Toeplitz theorem (Loève [5], § 16, Hardy [3], Ch. III) that

$$\lim_{|\bar{x}|\to\infty} \theta(|\bar{x}|)Z(\bar{x}) = 0.$$

The theorem follows from (5.15) since it is easily seen that, with the uniformity stated,

$$\lim_{|\bar{x}|\to\infty} \theta(|\bar{x}|)\int g(\bar{y}-\bar{x})U_G(d\bar{x}) = 0.$$

This may be shown by the method used in the proof of lemma 2.2[a]. It is noted that for' $x_1 \to -\infty$ the renewal density corresponding to $N(\mu, 1)$ decreases exponentially if $\mu > 0$. See Stone [8].

THEOREM 5.3. *Let F be nonarithmetic, and A a bounded set whose boundary has volume zero. Then, if $\bar{c} = \bar{\mu}/|\bar{\mu}|$ and $\int|(\bar{c}, \bar{x})|^\rho F(d\bar{x}) < \infty$,*

$$\lim_{t\to\infty} t^\rho U_F(A+t\bar{c}) = (2\pi)^{-\rho}(\text{Det } B)^{-\frac{1}{2}}|\bar{\mu}|^{\rho-1}\text{ Vol }(A),$$

where B is defined as in lemma 2.2[b].

PROOF. For convenience of notation we assume $\mu_1 = \mu > 0$, $\mu_2 = \cdots = \mu_d = 0$. Let $k \leqq m$, $k \leqq \max(2, \rho)$. By symmetry (cf. (1.5)) we have

$$\int_E y_1^k F^m(d\bar{y}) = \sum_{j=1}^k m^j R_j F^{m-k}(E),$$

where the R_j are finite signed measures, in particular

$$R_k = Q^k,\ Q(E) = \int_E y_1 F(d\bar{y}). \tag{5.17}$$

So

$$\sum_{m=1}^\infty m^{\rho-k}\int_{A+t} y_1^k F^m(d\bar{y}) = \sum_{m=1}^k m^{\rho-k}\int_{A+t} y_1^k F^m(d\bar{y}) + \sum_{j=1}^k\sum_{m=k+1}^\infty m^{\rho+j-k}R_j F^{m-k}(A+t). \tag{5.18}$$

Here the first term tends to zero since $E\{|X_{11}|^k\} < \infty$. In the second term the principal contribution is for $j = k$. It may be written

$$(5.19) \qquad R_k W_F(A+t) + \sum_{m=1}^{\infty} \xi(m) R_k F^m(A+t),$$

where $\xi(m) = 0(m^{\rho-1})$ for $m \to \infty$. Since $\lim_{t\to\infty} W_F(B+t)$ exists (theorem 4.2) we have by (5.17), using an argument similar to the one applied to (5.11)

$$(5.20) \qquad \lim_{t\to\infty} R_k W_F(A+t) = \mu^k \lim_{t\to\infty} W_F(A+t).$$

The second term in (5.19) and the contributions with $j \leqq k-1$ in (5.18) tend to zero for $t \to \infty$. We refer to (3.8). So (5.18), (5.19), (5.20) give

$$(5.21) \qquad \lim_{t\to\infty} t^k \sum_{m=1}^{\infty} m^{\rho-k} F^m(A+t) = \mu^k \lim_{t\to\infty} W_F(A+t).$$

We have now to distinguish:

1° d is odd. The theorem follows from theorem 4.2 and (5.21) with $k = \rho$.

2° d even, $d \geqq 6$. From theorem 4.2 and (5.21) with $k = 1, 2$:

$$(5.22) \qquad \lim_{t\to\infty} \sum_{m=1}^{\infty} m^{\rho} F^m(A+t) = \gamma,$$

$$(5.23) \qquad \lim_{t\to\infty} \sum_{m=1}^{\infty} t m^{\rho-1} F^m(A+t) = \mu\gamma,$$

$$(5.24) \qquad \lim_{t\to\infty} \sum_{m=1}^{\infty} t^2 m^{\rho-2} F^m(A+t) = \mu^2\gamma,$$

where γ is the limit occurring in (4.1). Put

$$(5.25) \qquad \lambda(t) = \sum_{m=1}^{\infty} t^2 \mu^{-2} \gamma^{-1} m^{\rho-2} F^m(A+t), \qquad t > 0,$$

and consider a family of integer valued random variables $\{M_t, t > 0\}$, with

$$(5.26) \quad P\{M_t = m\} = t^2 \mu^{-2} \gamma^{-1} m^{\rho-2} F^m(A+t)/\lambda(t), \quad m = 1, 2, \cdots.$$

Expectations with respect to the probability distribution (5.26) will be denoted by E_1.

From (5.22)—(5.24)

$$E_1\{M_t\} \approx \mu^{-1} t, \;\; E_1\{M_t^2\} \approx \mu^{-2} t^2,$$

for $t \to \infty$, so

(5.27)
$$\lim_{t\to\infty} t^{-1}M_t = \mu^{-1},$$

in quadratic mean and in probability. Then

(5.28)
$$t^{\rho-2}\, M_t^{2-\rho} \xrightarrow{P} \mu^{\rho-2},$$

and if in (5.28) limit and E_1 may be interchanged, we obtain with (5.26)

$$\lim_{t\to\infty} t^{\rho} \sum_{m=1}^{\infty} F^m(A+t) = \mu^{\rho}\gamma$$

and the proof is finished. It is sufficient to show that to every ε there is C_ε with

$$\limsup_{t\to\infty} E_1\{t^{\rho-2}M_t^{2-\rho}I_{\{t/M_t \geqq C_\varepsilon\}}\} < \varepsilon,$$

or, equivalently,

(5.29)
$$\limsup_{t\to\infty} t^{\rho} \sum_{m=1}^{[t/C_\varepsilon]} F^m(A+t) < \varepsilon.$$

By (5.8) and the boundedness of A it is sufficient for (5.29) that

(5.30)
$$\limsup_{t\to\infty} \sum_{m=1}^{[t/C_\varepsilon]} m^{\rho}\, RF^{m-1}(A+t) < \varepsilon,$$

with R given by (5.9). Since R is a finite measure, the first term in (5.30) tends to zero for $t \to \infty$. We have

$$E_1\{t^{-2}M_t^2 I_{\{M_t \leqq t/C\}}\} \leqq C^{-2}P\{M_t \leqq t/C\},$$

so with (5.27) and (5.26)

(5.31)
$$\lim_{t\to\infty} \sum_{m=1}^{[t/C]} m^{\rho}\, F^m(A+t) = 0,\ C > \mu.$$

The sum from $m = 2$ on in (5.30) is majorized by

(5.32)
$$2^{\rho} \int \left\{ \sum_{m=1}^{[t/C]} m^{\rho}\, F^m(A-\bar{y}+t) \right\} R(d\bar{y})$$

where the integrand for fixed $\bar{y}$ tends to zero as $t \to \infty$, since (5.31) also holds with A replaced by $A-\bar{y}$. Moreover the integrand is bounded (lemma 2.4), so that (5.32) has limit zero for $t \to \infty$. So (5.30) holds.

3° $d = 2$, $d = 4$. The proof of 2° applies up to and including (5.28) and now E_1 and limit are interchanged since $0 < 2-\rho < 2$ and $t^{-1}M_t \to \mu^{-1}$ in quadratic mean.

THEOREM 5.4. *Under the conditions of theorem* 4.3, *if* $\int |(\bar{\mu}, \bar{x})|^\rho F(d\bar{x}) < \infty$,

$$\lim_{\bar{x}\to\infty} |\bar{x}|^\rho U_F(\{\bar{x}\}) = (2\pi)^{-\rho}(\text{Det } B)^{-\frac{1}{2}}|\bar{\mu}|^{\rho-1}.$$

PROOF. The theorem is derived from theorem 4.3 by the same methods as theorem 5.3 from theorem 4.2

REMARK. No theorems for densities are given here, but on inspection of the proofs involved it is easily seen that the condition $g \in K_d$ may be weakened if $\varphi \in L_1$. The theorems for densities then follow by taking $\gamma = \varphi$.

6. Are the conditions essential?

The condition $E|(\bar{c}, \bar{X}_1)|^\rho < \infty$ of section 5 in a sense is best possible as is seen by the following example: Let $\bar{X}_1$ be arithmetic, X_{11} and $(X_{12}, \cdots, X_{1d})$ independent, $\mu_1 > 0$, $\mu_2 = \cdots = \mu_d = 0$, and

$$(6.1) \qquad \begin{aligned} p(k) &\overset{\text{df}}{=} P\{X_{11} = k\} = 0,\ k \neq 2^n, \\ p(2^n) &= P\{X_{11} = 2^n\} \approx n^{-2}2^{-(\rho-\varepsilon)n}. \end{aligned}$$

Then $E|X_{11}|^\theta$ is finite for $\theta \leqq \rho-\varepsilon$ and infinite for $\theta > \rho-\varepsilon$, and the first term in (1.2) causes theorem 5.4 to fail. Leaving out a finite number of leading terms in (1.2) does not help, for we may replace the distribution (6.1) by

$$q(k) = \sum_{n=0}^{\infty} a_n q^{(n)}(k).$$

where the a_k are positive and have sum 1. Along the same lines examples may be constructed where $E|X_{11}|^\rho = \infty$ and $E\{|X_{11}|^\rho \zeta(X_{11})\} < \infty$ with $\zeta(x) \to 0$ sufficiently slowly.

The assumption that all second moments are finite may not be best possible for our results on W_F, in fact it is conjectured that $E(Y_k^2) < \infty$, $k = 2, \cdots, d$, with $Y_1, \cdots, Y_d$ as in lemma 2.2, is sufficient for (4.1). We also conjecture that theorem 5.2 can be improved to give $\theta(x) = x^2$.

Note added in proof. Lemma 2.1^b is not correct. The boundary term arising by the application of Green's theorem is bounded by $|\bar{x}-m\bar{\mu}|\theta^m$, where $0 \leqq \theta < 1$. In the proof of lemma 2.4 extra terms should be added to (2.7) and (2.9). These are easy to handle and (2.11) continues to hold. Similar remarks apply to the proofs of theorems 3.2, 3.3 and 3.4.

The boundary term may vanish by periodicity or if $\eta \in R_a$ and $D = B(\eta)$. See definition 2.3.

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