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SMOOTHNESS AND REGULARITY

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Some applications of the cotangent complex to smoothness and regularity are given; in particular, the proof of a criterion for formal smoothness which was conjectured in [8] (see 2.1), and some generalisations of this criterion for the non-noetherian and noetherian cases (2.2, 2.6, 2.7). Also considered is the descent of formal smoothness.

0. All the rings considered are commutative with unity; the topologies are linear. The definitions and notations used are as in EGA, 0_{IV}, §§ 19–20, and [1]. The following facts about the cotangent complex will be needed:

Let $A \to B$ be a morphism of rings; to any B-module M are associated the B-modules $H_i(A, B, M)$, $H^i(A, B, M)$. (For the definitions, see: [8] for $i = 0, 1$; [9] for $i = 0, 1, 2$; [1] or [14] for $i \geqq 0$. At least for $i = 0$, 1, the various definitions give isomorphic modules. This follows from the properties 0.1–0.4 below ([6], 3.5).) $\{H_i(\text{resp. } H^i),\ i \geqq 0\}$ is a (co) homological functor and has the properties.

0.1. $H_0(A, B, M) = \Omega_{B/A} \otimes_B M$ (where $\Omega_{B/A}$ is the module of A-differentials in B); $H^0(A, B, M) = \mathrm{Der}_A(B, M)$(= the module of A-derivations of B in M; see [9], 2.3).

0.2. If $A \to B$ is surjective with kernel $\mathfrak{b}$, then $H_1(A, B, M) = \mathfrak{b}/\mathfrak{b}^2 \otimes_B M$ and $H^1(A, B, M) = \mathrm{Hom}_B(b/b^2, M)$([9], 3.1.2).

0.3. If B is a polynomialring over A, then $H_i(A, B, .) = 0 = H^i(A, B, .)$ for $i \geqq 1$([9], 3.1.1 or [1], 16.3).

0.4. If $A \to B \to C$ are morphisms of rings and M is a C-module, then the sequence

$$\cdots \to H_i(A, B, M) \to H_i(A, C, M) \to H_i(B, C, M) \to H_{i-1}(A, B, M) \to \cdots \to H_0(B, C, M) \to 0$$

is exact ([9], 2.3.5 or [1], 18.2), and similarly for H^i, with arrows reversed.

0.5. If B is an A-algebra and S a multiplicatively closed system in B, then the canonical morphism $H_i(A, B, M) \otimes_B S^{-1}B \tilde{\to} H_i(A, S^{-1}B, S^{-1}M)$ is an iscmorphism ([9], 2.3.4. or [1], 16).

0.6. If $A \to B \to C$ are morphisms of rings and M a flat C-module, then the canonical morphism $H_i(A, B, C) \otimes_C M \tilde{\to} H_i(A, B, M)$ is an isomorphism.

0.7. Let $A' \overset{v}{\leftarrow} A \overset{u}{\to} B$ be morphisms of rings, where u or v is flat, $B' = A' \otimes_A B$, and M a B'-module. Then the canonical morphism $H_i(A, B, M) \tilde{\to} H_i(A', B', M)$ is an isomorphism ([9], 2.3.2. or [1], 19.2).

0.8. If $A \to B$ are fields, then $H_i(A, B, \cdot) = O = H^i(A, B, \cdot)$ for $i \geqq 2$ ([9], 3.5 or [1], 22.2) and $A \to B$ is separable iff it is formally smooth (EGA, O_{IV}, 19.6.1 or 9, 3.5).

0.9. Let A be a local, noetherian ring, B its residue class field and K a field which is an extension of B. Then the following three statements are equivalent: A is regular: $H_2(A, K, K)$ is zero; $H_i(A, K, K) = 0$ for $i \geqq 2$. This follows from [9], 3.2.1 or [1], [1], 27.1 and 27.2, using 0.6 and 0.8.

The following criteria (0.10 and 1.1) are essentially the discrete and non-discrete forms of the Jacobian criterion of smoothness (EGA, O_{IV}, 22.6.1 and 22.6.2).

0.10. A morphism of rings $A \to B$ is formally smooth in the discrete topologies if and only if $\Omega_{B/A}$ is a projective B-module and $H_1(A, B, B) = 0$ ([8], 9.5.7 or [9], 3.1.3).

1.1. Let $A \to B$ be a morphism of topological rings: the topology of B is $\mathfrak{c}$-adic for some ideal $\mathfrak{c}$ in B, and that of A is also adic. Let $C = B/\mathfrak{c}$. If $A \to B$ is formally smooth then $\Omega_{B/A} \otimes_B C$ is a projective C-module and $H_1(A, B, C) = 0$: the converse is also true if B is a noetherian ring.

1.2. For A and B noetherian, 1.1 is proved in [2], 5.4. The proof of 1.1 in general is based on the same ideas. Let $A \to B$ be formally smooth. Then $\Omega_{B/A}$ is a formally projective B-module (EGA, O_{IV}, 20.4.9), hence $\Omega_{B/A} \otimes_B C$ is a projective C-module (II, IX, 1.25). Let R be a polynomialring over A, and $R \to B$ a surjection of A-algebras with kernel $\mathfrak{b}$. Then the following sequence is exact (0.4 and 0.1–0.3).

$$(1.2.1) \quad 0 \to H_1(A, B, C) \to \mathfrak{b}/\mathfrak{b}^2 \underset{B}{\otimes} C \xrightarrow[\delta_{B/R/A} \otimes C]{} \Omega_{R/A} \otimes C \to \Omega_{B/A} \underset{B}{\otimes} C \to 0.$$

But $\delta_{B/R/A}$ is formally left invertible (EGA, O_{IV}, 20.7.8 and 19. 4.4), so $\delta_{B/R/A} \otimes_B C$ is injective, and $H_1(A, B, C) = 0$.

Now, let B be noetherian, $\Omega_{B/A} \otimes_B C$ be a projective C-module, and $H_1(A, B, C) = 0$. Let R and $\mathfrak{b}$ be as above. Then ([9], 3.1.2) and (1.2.1) imply that $H^1(A, B, M) = 0$ for any C-module M, and hence for any discrete B-module M with open annihilator. Let A_d, B_d denote the rings A, B with the discrete topology, and A_t, B_t the rings A, B with the given topologies. Then $H^1(A, B, M) = 0$ means $H^1(A_d, B_d, M) = 0$. $A_d \to B_d \to B_t$ and M give the exact sequence ([2], 2 or EGA, $\mathrm{O_{IV}}$, 20.3.7)

$$0 \to H_t^0(A_d, B_t, M) \xrightarrow{u} H_t^0(A_d, B_d, M) \to H_t^1(B_d, B_t, M) \xrightarrow{v} H_t^1(A_d, B_t, M) \to H^1(A_d, B_d, M) = 0$$

But u is an isomorphism (EGA, $\mathrm{O_{IV}}$, 20.3.3), so v also is. B is noetherian, hence $H_t^1(B_d, B_t, M) = 0$([2], 5.1): it follows that $H_t^1(A_d, B_t, M) = 0$. Then $A_d \to A_t \to B_t$ and M give the exact sequence

$$H_t^0(A_d, A_t, M) \to H_t^1(A_t, B_t, M) \to H_t^1(A_d, B_t, M),$$

where the first and third terms are zero. The formal smoothness of $A_t \to B_t$ now follows (EGA, $\mathrm{O_{IV}}$, 19.4.4).

1.3. Remarks. (i) I do not know if the converse part of 1.1 remains valid for B a non-noetherian ring.

(ii) The criterion 1.1 can be reformulated as follows: it B is noetherian, then $A \to B$ is formally smooth iff $\Omega_{B/A}$ is a formally projective B-module and $H_1(A, B, C) = 0$. The results of N. Radu ([11], [12], [13]) shows that the condition $H_1(A, B, C) = 0$ is superfluous in this form of 1.1 if B is a laskerian local ring with the $\mathfrak{m}$-adic topology ($\mathfrak{m}$ the maximal ideal of B), and A is a field of characteristic zero (or arbitrary characteristic if B is noetherian).

2.0. It is known that, if $A \to B$ is a local morphism of local noetherian rings, formally smooth in the topologies given by the maximal ideals, *then A is regular if and only if B is regular.*

Proof. Let K be the residue class field of B, then the maps $A \to B \to K$ give the exact sequence (0.4):

$$H_2(A, B, K) \to H_2(A, K, K) \to H_2(B, K, K) \to H_1(A, B, K).$$

But $H_1(A, B, K) = 0(1.1)$ and $H_2(A, B, K) = 0$ ([4], corollary). Now apply 0.9.

2.1. In ([8], 9.6), the following criterion for formal smoothness is stated and partially proved:

Theorem. *Let $A \to B \to C$ be local homomorphisms of local noetherian rings, A and C regular, $B \to C$ surjective, so that $C = B/\mathfrak{c}$,$\mathfrak{c}$ an ideal of B. Finally, let B be a localisation of a finitely generated A-algebra. Then B is* *formally smooth over A if and only if the following three conditions are satisfied*:

a) *B is regular, i.e.* $\mathfrak{c}$ *is a regular ideal*;

b) $\Omega_{B/A} \otimes_B C$ *is a projective C-module*;

c) *The echaracteristic homomorphism*

$$N_{C/A} \to \mathfrak{c}/\mathfrak{c}^2$$

is injective.

(The morphism from *c*) is $H_1(A, C, C) \to H_1(B, C, C)$ (0.2)

In [8], it was proved that the conditions are necessary. The sufficiency, however, was only proved for A a field and the sufficiency in general conjectured. I gave generalisations of this criterion for the non-noetherian and noetherian case ([5]). For the noetherian case *I* use essentially (1.1), but this does not work in the non-noetherian case. A direct proof for 2.1 is as follows:

Let $A \to B$ be formally smooth (here the topology is arbitrary, cf. EGA, O_{IV}, 22.6.4). Then B is regular (by 2.0), $\Omega_{B/A} \otimes_B C$ is a projective C-module and $H_1(A, B, C) = 0$ (by 0.10). From the exact sequence (0.4),

$$H_2(B, C, C) \to H_1(A, B, C) \to H_1(A, C, C) \xrightarrow{f} H_1(B, C, C) = \mathfrak{c}/\mathfrak{c}^2,$$

it results that f is injective.

Let a), b), c) be satisfied. Then $H_2(B, C, C) = 0$([9], 3.2.1) since $\mathfrak{c}$ is generated by a B-regular sequence. Hence, by the above sequence and c), $H_1(A, B, C) = 0$; this and b) imply that B is a formally smooth A-algebra for the $\mathfrak{c}$-adic topology (1.1).

We now turn to the non-noetherian case. Let $A \to A' \xrightarrow{u} B \xrightarrow{v} C$ be morphisms of rings, u and v surjective, $\mathfrak{b} = \operatorname{Ker} u$, $\mathfrak{c} = \operatorname{Ker} v$.

2.2. Theorem. *Suppose that A' is a formally smooth A-algebra* (in the $\mathfrak{b}$-adic topology) *and that* $\mathfrak{b}/\mathfrak{b}^2$ *is $\mathfrak{c}$-separated* (*or is a B-module of finite type with B local*). *Then $A \to B$ is formally smooth* (in the discrete topologies) *if and only if $\Omega_{B/A} \otimes_B C$ is a projective C-module and* $H_1(A, B, C) = 0$.

Proof. The necessity results from (0.10).

For the converse two facts are necessary.

(2.2.1) *If $\Omega_{B/A} \otimes_B C$ is a projective C-module and* $H_1(A, B, C) = 0$ *then*

$$\delta_{B/A/A} \underset{B}{\otimes} C : b/b^2 \underset{B}{\otimes} C \to \Omega_{A'/A} \underset{A'}{\otimes} C$$

is left invertible. For $A \to A' \to B$ and C give the exact sequence

$$0 = H_1(A, B, C) \to H_1(A', B, C) \xrightarrow[\delta_{B/A'/A} \otimes_B C]{} H_0(A, A', C) \to$$
$$\to H_0(A, B, C) \to 0,$$

by 0.4 and 0.1, 0.2. Then 2.2.1 results from the projectivity of $\Omega_{B/A} \otimes_B C = H_0(A, B, C)$.

(2.2.2) *Let $h : M \to N$ be a morphism of B-modules, $\mathfrak{c}$ an ideal of B, M a $\mathfrak{c}$ - separated B-module, and N a projective B-module. If $h_1 = h \otimes_B B/\mathfrak{c} : M/\mathfrak{c}M \to N/\mathfrak{c}N$ is left invertible, then h is also left invertible.* To see this, let $g' : N/\mathfrak{c}N \to M/\mathfrak{c}M$ be a left inverse for h_1, Since N is projective, there is a morphism $g : N \to M$, such that the composition $N \xrightarrow{g} M \to M/\mathfrak{c}M$ equals $N \to N/\mathfrak{c}N \xrightarrow{g'} M/\mathfrak{c}M$. It is obvious that $g_1 = g'$, where the subscript 1 means $\otimes_B B/\mathfrak{c}$, i.e. that $(gh)_1 = 1$. It follows that $(gh)_r = gh \otimes_B B/\mathfrak{c}^r$ is equal to 1 (see the proof of IL, XII, 2, 2, 1). Let $x \in M$; then $x-(gh)(x) \in \mathfrak{c}^r M$ for any $r \geqq 1$, and so $gh = 1$.

Now let $\Omega_{B/A} \otimes_B C$ be a projective B-module and $H_1(A, B, C) = 0$. Then $\delta_{B/A'/A} \otimes_B C$ is left invertible (2.2.1); hence $\delta_{B/A'/A}$ is also left invertible (for $\Omega_{A'/A} \otimes_{A'} B$ is a projective B-module by 0.10. Now, if $\mathfrak{b}/\mathfrak{b}^2$ is $\mathfrak{c}$-separated, apply 2.2.2; otherwise apply EGA, 0_{IV}, 19.1.12). Hence B is a formally smooth A-algebra (EGA, 0_{IV}, 20.5.12).

2.2.3. Remarks. (i) The hypotheses of (2.2) are fulfilled if B is an A-algebra essentially of finite presentation and C any quotient ring of B.

(ii) Under the hypotheses of 2.2, assume $H_2(B, C, C) = 0$ (conditions for this are given in [3], 5.1, or [9], 3.2.1); then $A \to B$ is formally smooth if and only if $\Omega_{B/A} \otimes_B C$ is a projective C-module and $H_1(A, B, C) \to \mathfrak{c}/\mathfrak{c}^2$ is injective. (This shows that for the 'only if' part of 2.1, sufficient hypotheses on $A \to B$ are that B be noetherian and an A-algebra essentially of finite presentation.)

2.3 Corollary. *Let $Z \xrightarrow{i} Y \xrightarrow{h} X$ be morphisms of schemes, i being an immersion and h being locally of finite presentation. Then h is smooth in a neighbourhood of Z in Y if and only if $H_0(X, Y, O_Z)$ is a flat O_Z-Module and $H_1(X, Y, O_Z) = 0$.*

Proof. Let $z \in Z$, $y = i(z)$ and $x = h(y)$. Then

$$H_i(X, Y, O_Z)_z = H_i(O_{X,x}, O_{Y,y}, O_{Z,z}), \text{ by } 0.5$$

Since $H_0(O_{X,x}, O_{Y,y}, O_{Z,z})$ is an $O_{Z,z}$-module of finite presentation, flat means projective. Then $h_y : O_{X,x} \to O_{Y,y}$ is smooth for any $z \in Z$ (cf. 2.2.3 and 2.2). But this is an open property (EGA, IV, 17.5). Hence h is locally smooth, i.e. it is smooth ([8] 9.5.6).

2.4. For the non-dicrete topologies, 2.2 takes the following form:

Proposition. *Let $\mathfrak{a}$ be an ideal of A* s.t. $\mathfrak{m} = u(\mathfrak{a}) \supset \mathfrak{c}$. *Suppose that $A \to A'$ is formally smooth in the $\mathfrak{a}$-adic topology and that the topology of $\mathfrak{b}/\mathfrak{b}^2$ induced by $\mathfrak{b}$ is $\mathfrak{m}$-adic. If $\Omega_{B/A} \otimes_B C$ is a projective C-module and*

$H_1(A, B, C) = 0$, then $A \to B$ is formally smooth in the $\mathfrak{m}$-adic topology; hence $\Omega_{B/A}$ is a formally projective B-module.

PROOF. From 2.2.1 it results that $\delta_{B/A'/A} \otimes_B C$ (and hence also $\delta_{B/A'/A} \otimes_B K$, where $K = B/\mathfrak{m}$) is left invertible. $\Omega_{A'/A}$ is a formally projective A-module in the $\mathfrak{a}$-adic topology (0.10), so $\Omega_{A'/A} \otimes_A B$ is a formally projective B-module for the $\mathfrak{m}$-adic topology; hence $\delta_{B/A'/A}$ is formally left invertible (EGA, 0_{IV}, 19.1.9). Now the jacobian criterion of smoothness (EGA, 0_{IV}, 22, 5, 1) says that $A \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.

Generalisations for the noetherian case. Let $A \xrightarrow{u} B \xrightarrow{v} C$ be morphisms of rings, B and C noetherian.

2.5. PROPOSITION. *Let $\mathfrak{c} = \operatorname{Ker} v$, $\mathfrak{d} \supset \mathfrak{c}$ an ideal of B. $D' = B/\mathfrak{d}$, and D a $(C/\mathfrak{d}C)$-algebra. Suppose also that the topology of B is $\mathfrak{d}$-adic, that of A is $(\mathfrak{d} \cap A)$-adic and that of C is $\mathfrak{d}C$-adic.*

i) *If $\mathfrak{d} \subset R(B)$(*= the Jacobson radical of B), *A is regular, and $A \to B$ is formally smooth, then B is regular, $\Omega_{B/A} \otimes_B D$ is a projective D-module and $f: H_1(A, C, D) \to H_1(B, C, D)$ is injective.*

ii) *Let $D' \to D$ be faithfully flat. Let $H_2(B, C, D) = 0$* (e.g. if $B \to C$ is a Koszul morphism ([9], 3.2.2); in particular, if v is surjective and $\mathfrak{c}$ generated by a regular sequence ([9], 3.2.1)). *If $\Omega_{B/A} \otimes_B D$ is a projective D-module and f is injective, then $A \to B$ is formally smooth.*

iii) *Let $D' \to D$ be faithfully flat, $H_2(B, C, D) = 0$ and $A \to C$ formally smooth. If $\mathfrak{d}$ is maximal or $B \to C$ is formally étale, then $A \to B$ is formally smooth.*

PROOF. $A \to B \to C$ and D give the exact sequence

$$(2.5.1) \qquad H_2(B, C, D) \to H_1(A, B, D) \to H_1(A, C, D) \xrightarrow{f} H_1(B, C, D) \to \\ \to \Omega_{B/A} \underset{B}{\otimes} D \to \Omega_{C/A} \underset{C}{\otimes} D \to \Omega_{C/B} \underset{C}{\otimes} D \to 0$$

i) From 1.1, it results that $\Omega_{B/A} \otimes_B D'$ is a projective D-module and $H_1(A, B, D') = 0$. Then $\Omega_{B/A} \otimes_B D$ is a projective D-module and $H_1(A, B, D) = 0$. Indeed, let R be a ring of polynomials over A, and $R \to B$ a surjection of A-algebras with kernel $\mathfrak{b}$. Then $A \to R \to B$ and D', D give the exact sequence (0.4, 0.1–0.3).

$$0 \to H_1(A, B, D') \to b/b^2 \underset{B}{\otimes} D' \to \Omega_{R/A} \underset{R}{\otimes} D' \to \Omega_{B/A} \underset{B}{\otimes} D' \to 0$$

$$0 \to H_1(A, B, D) \to b/b^2 \underset{B}{\otimes} D \to \Omega_{R/A} \underset{R}{\otimes} D \to \Omega_{B/A} \underset{B}{\otimes} D \to 0$$

since $H_1(A, B, D') = 0$ and $\Omega_{B/A} \otimes_B D'$ is projective, it follows that $H_1(A, B, D) = 0$. It follows immediately that f is injective (2.5.1).

Let $\mathfrak{d} \subset R\ (B)$. If $\mathfrak{m}$ is a maximal ideal of B and $\mathfrak{n} = A \cap \mathfrak{m}$ then

$A_\mathfrak{n} \to B_\mathfrak{m}$ is formally smooth in the radicial topologies. Since $A_\mathfrak{n}$ is regular, it follows from 2.0 that $B_\mathfrak{m}$ also is.

ii) We have that $H_1(A, B, D) = H_1(A, B, D')\otimes_{D'} D$, and $\Omega_{B/A}\otimes_B D = (\Omega_{B/A}\otimes_B D')\otimes_{D'} D$. (0.6)

From the fact that f is injective and $H_2(B, C, D) = 0$, it results that $H_1(A, B, D) = 0$. Hence $H_1(A, B, D') = 0$. Since $D' \to D$ is faithfully flat and $\Omega_{B/A}\otimes_B D$ is D-projective, $\Omega_{B/A}\otimes_B D'$ is a projective D'-module ([15]). Hence $A \to B$ is formally smooth (1.1).

iii) From the formal smoothness of $A \to C$, it results that $\Omega_{C/A}\otimes_C D$ is a projective D-module and $H_1(A, C, D) = 0$(1.1). Since $H_2(B, C, D) = 0$, we find from 2.5.1 that $H_1(A, B, D) = 0$. But $H_1(A, B, D) = H_1(A, B, D')\otimes_{D'} D$, hence $H_1(A, B, D') = 0$.

Let $\mathfrak{d}$ be maximal, i.e. D' a field; then $A \to B$ is formally smooth, because of 1.1.

Let $B \to C$ be formally étale; then $H_1(B, C, D) = 0$(1.1) and $\hat{\Omega}_{C/B} = 0$ (and also $H_2(B, C, D) = 0$, if B, C are local ([4])). Hence $\Omega_{C/B}\otimes_B D = 0$. Now from 2.5.1 it results that $\Omega_{B/A}\otimes_B D = \Omega_{C/A}\otimes_C D$; hence $\Omega_{B/A}\otimes_B D'$ is a projective D'-module. Consequently, $A \to B$ is formally smooth (1.1). In particular, we obtain:

2.6. Corollary. *Let A, B, C, u and v be local and L be the residue class field of C; the topologies are adic and given by the maximal ideals.*

i) *If A is regular and $A \to B$ formally smooth, then B is regular and $f: H_1(A, C, L) \to H_1(B, C, L)$ is injective.*

ii) *If $H_2(B, C, L) = 0$* (e.g. if $B \to C$ is Koszul, or if B is regular and $H_3(C, L, L) = 0$ (use 0.4 and 0.9). This last occurs, for instance, if C is regular, by 0.9), *and if f is injective, then $A \to B$ is formally smooth.*

iii) *If $H_2(B, C, L) = 0$ and $A \to C$ is formally smooth, then $A \to B$ is formally smooth.*

2.7. Corollary. *Let A, B, C, u and v be local, $B \to C$ surjective with kernel $\mathfrak{c}$, $\mathfrak{d} \supset \mathfrak{c}$ an ideal of B and $D = B/\mathfrak{d}$. The topology of B is $\mathfrak{d}$-adic, that of A is $(\mathfrak{d} \cap A)$-adic.*

i) *If A is regular and $A \to B$ is formally smooth, then B is regular, $\Omega_{B/A}\otimes_B D$ is a projective D-module and $f: H_1(A, C, D) \to \mathfrak{c}/\mathfrak{d}\mathfrak{c}$ is injective.*

ii) *If $\mathfrak{c}$ is generated by a B-regular sequence* (e.g. for B and C regular), *and if $\Omega_{B/A}\otimes_B D$ is a projective D-module and f is injective, then $A \to B$ is formally smooth.*

3.0. In EGA, 0_{IV}, 19.7.1 the following smoothness criterion is given: Let $A \to B$ be a local morphism of local noetherian rings and let k be the residue class field of A; the topologies are adic and given by the maximal ideals. Then $A \to B$ is formally smooth if and only if $A \to B$ is flat and $k \to k \otimes_A B$ is formally smooth.

The following counter-example given by *N.* Radu shows that B must be noetherian for this criterion to be valid.

(3.0.1) *Let k be a perfect field and B a k-algebra which is a non-discrete valuation ring of dimension* 1; *let* $\mathfrak{m}$ *be the maximal ideal of* B, *and* $K = B/\mathfrak{m}$. *Then* $B \to K$ *is formally étale.*

Indeed, $\mathfrak{m} = \mathfrak{m}^2$; $k \to B \to K$ and K give the exact sequence (0.4):

$$0 = \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \underset{B}{\otimes} K \xrightarrow{v_{K/B/k}} \Omega_{K/k} \to 0.$$

Hence $v_{K/B/k}$ is left invertible; but this means that K is a formally smooth B-algebra with respect to k(EGA, 0_{IV}, 20.5.7). On the other hand, K is a formally smooth k-algebra (EGA, 0_{IV}, 19.6.1); hence $B \to K$ is formally smooth.

(A purely homological proof of the above criterion is given in [4].)

The following results concern the descent of formal smoothness; from (3.1) results the 'only if' part of 3.0.

3.1. THEOREM. *Let* $A' \overset{v}{\leftarrow} A \overset{u}{\rightarrow} B$ *be ring-morphisms with B noetherian. Let* $B' = A' \otimes_A B$. *Let B be local with maximal ideal* $\mathfrak{m}$, *and* $\mathfrak{q}$ *be a prime ideal of* B' s.t. $\mathfrak{q} \cap B = \mathfrak{m}$. *The topologies of B and* $B'_\mathfrak{q}$ *are adic and given by the maximal ideals. Suppose that u or v is flat. Then* $A' \to B_\mathfrak{q}$ *is formally smooth if and only if* $A \to B$ *is formally smooth.*

PROOF. Let $k = B/\mathfrak{m}$ and $K = B'_\mathfrak{q}/\mathfrak{q}B$. Then

$$H_1(A, B, k) \underset{k}{\otimes} K = H_1(A, B, K) = H_1(A', B'_\mathfrak{q}, K) = H_1(A', B'_\mathfrak{q}, K)$$

(by 0.6, 0.7 and 0.5). Now apply (1.1).

3.2. PROPOSITION. *Let* $A \to B$ *be a morphism of topological rings, B noetherian, and* $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$ *ideals with* $\mathfrak{a} B \subset \mathfrak{b}$, *such that the topology of A is* $\mathfrak{a}$*-adic and that of B,* $\mathfrak{b}$*-adic. Then, if* (1) *or* (2) *holds,* $A \to B$ *is formally smooth if* $A' \to B'$ *is.*

(1) $A' = A/\mathfrak{a}$, $B' = B/\mathfrak{b}$, $A \to B$ *flat.*

(2) A' *a faithfully flat A-algebra, with the* $(\mathfrak{a}A)$ – *adic topology, and* $B' = A' \otimes_A B$ (which has the $(\mathfrak{b}B)$ – adic topology).

Moreover, in (2), $A \to B$ *is formally étale if* $A' \to B'$ *is.*

PROOF. Let $C = B/\mathfrak{b}$ and $C' = B'/\mathfrak{b}B'$; then $C \to C'$ is faithfully flat. Hence, $\Omega_{B/A} \otimes_B B = \Omega_{B'/A'}$(0.7) so that

$$(\Omega_{B/A} \underset{B}{\otimes} C) \underset{C}{\otimes} C' = \Omega_{B'/A'} \underset{B'}{\otimes} C'),$$

and

$$H_1(A, B, C) \underset{C}{\otimes} C' = H_1(A, B, C') = H_1(A', B', C')$$

(0.6 and 0.7).

i) Let be (1). From 1.1 it results that $\Omega_{B'/A'} \otimes_{B'} C (= \Omega_{B/A} \otimes_B C)$ is a projective C-module and $H_1(A, B, C) = H_1(A', B', C) = 0$; now apply 1.1 again.

ii) Let be (2). Let $A' \to B'$ be formally smooth; then $\Omega_{B'/A'} \otimes_{B'} C'$ is a projective C'-module and $H_1(A', B', C') = 0$ (1.1). Hence, by the above equalities, $\Omega_{B/A} \otimes_B C$ is a projective C-module ([15]) and $H_1(A, B, C) = 0$; then $A \to B$ is formally smooth (1.1)

Let $A' \to B'$ be formally étale, so $\hat{\Omega}_{B'/A'} = 0$ $H_1(A', B', C') = 0$(0.10 and 1.1). Hence, as above, $\Omega_{B/A} \otimes_B C = 0$ and $H_1(A, B, C) = 0$. Let $M'_n = \Omega_{B'/A'} \otimes_{B'} B'/\mathfrak{b}^n B'$ and $M_n = \Omega_{B/A} \otimes_B B/\mathfrak{b}^n$; then $M'_n = M_n \otimes_{B/\mathfrak{b}^n} B'/\mathfrak{b}^n B'$ (0.7. Then $\hat{\Omega}_{B'/A'} = 0$ gives $M'_n = 0$, but $B/\mathfrak{b}^n \to B'/\mathfrak{b}^n B'$ is faithfully flat, and so $M_n = 0$. Hence $\hat{\Omega}_{B/A} = 0$. Now use 0.10 and 1.1. (Observe that 3.1, 3.2 are formally very similar and probably both follow from a more general statement.)

3.3. Remark. i) *Let $A' \leftarrow A \to B$ be morphisms of rings, $B' = A' \otimes_A B$ and $A \to A'$ faithfully flat; the topologies are discrete. Then $A' \to B'$ is formally smooth* (resp. *étale*) *iff $A \to B$ is formally smooth* (resp. *étale*).

Indeed $\Omega_{B/A} \otimes_B B' = \Omega_{B'/A'}$ (0.7) and $H_1(A, B, B) \otimes_B B = H_1(A', B, B') = H_1(A', B', B')$, by 0.6 and 0.7. Now apply 0.10.

ii) *Let $X' \to X \leftarrow Y$ be morphisms of schemes, $Y' = X' \times_X Y$, and $X' \to Y$ faithfully flat. If $Y' \to X'$ is formally étale* (resp. *locally formally smooth*), *then $Y \to X$ is formally étale* (resp. *locally formally smooth*).

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