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IRREGULARITIES OF DISTRIBUTION. VI

by

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1. Introduction

We are interested in the distribution of an arbitrary sequence of numbers in an interval. We are thus returning to questions investigated in the first part [10] of the present series. However, the present paper can be read independently.

Let U be the unit interval consisting of numbers ξ with $0 < \xi \leq 1$, and let $\omega = \{\xi_1, \xi_2, \dots\}$ be a sequence of numbers in this interval. Given an α in U and a positive integer n , we write $Z(n, \alpha)$ for the number of integers i with $1 \leq i \leq n$ and $0 \leq \xi_i < \alpha$. We put

$$D(n, \alpha) = |Z(n, \alpha) - n\alpha|.$$

The sequence ω is called *uniformly distributed* if $D(n) = o(n)$, where $D(n)$ is the supremum of $D(n, \alpha)$ over all numbers α in U . Answering a question of Van der Corput [3], Mrs. Van Aardenne-Ehrenfest [1] showed that $D(n)$ cannot remain bounded. Later [2] she proved that there are infinitely many integers n with $D(n) > c_1 \log \log n / \log \log \log n$ where c_1 is a positive absolute constant, and K. F. Roth [9] improved this to $D(n) > c_2(\log n)^{\frac{1}{2}}$.

For $\kappa \geq 0$ let $S(\kappa)$ be the set of all numbers α in U with

$$D(n, \alpha) \leq \kappa \quad (n = 1, 2, \dots).$$

Further let $S(\infty)$ be the union of the sets $S(\kappa)$, i.e. the set of numbers α in U for which $D(n, \alpha)$ remains bounded as a function of n . Erdős [4, 5] asked whether $S(\infty)$ was necessarily a proper subset of U . This question was answered in the affirmative by the author in the first paper [10] of this series, where among other things it was shown that $S(\infty)$ has Lebesgue measure zero. In the present paper we shall show that $S(\infty)$ is at most a countable set.

Recall that a number γ is a *limit point* of a set S if there is a sequence of distinct elements of S which converge to γ . The *derivative* $S^{(1)}$ of S

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consists of all the limit points of S . The higher derivatives are defined inductively by $S^{(d)} = (S^{(d-1)})^{(1)}$ ($d = 2, 3, \dots$). Our main theorem is as follows.

THEOREM. *Suppose $d > 4\kappa$. Then $S^{(d)}(\kappa)$ is empty.*

No special importance attaches to the quantity 4κ which could be somewhat reduced at the cost of further complications. But at the end of this paper we shall exhibit a sequence² for which $S^{(d)}(d)$ is not empty for $d = 1, 2, \dots$, and hence 4κ may not be replaced by $\kappa - \varepsilon$ where $\varepsilon > 0$. One shows easily by induction on d that a set S of real numbers for which $S^{(d)}$ is empty is at most countable and is nowhere dense. We therefore obtain the

COROLLARY. *The sets $S(\kappa)$ are at most countable and they are nowhere dense. The set $S(\infty)$ is at most countable.*

Let θ be irrational and let $\omega = \omega(\theta)$ be the sequence $\{\theta\}, \{2\theta\}, \dots$ where $\{\}$ denotes fractional parts. One can easily show (see Hecke [6], § 6) that the numbers $\{k\theta\}$ where k is an integer belong to $S(\infty)$. In answer to a question by Erdős and Szűsz, it was shown by Kesten [7] that the numbers $\{k\theta\}$ are the only elements of $S(\infty)$. Hence in this case the set $S(\infty)$ is known and is countable.

Now let I be a subinterval of U of the type $\alpha < \xi \leq \beta$ and put $D(n, I) = |Z(n, \beta) - Z(n, \alpha) - n(\beta - \alpha)|$. If $\omega = \omega(\theta)$, then $D(n, I)$ is bounded as a function of n if (Ostrowski [8]) and only if (Kesten [7]) I has length $l(I) = \beta - \alpha = \{k\theta\}$ where k is an integer. Hence in this example there are *continuum many intervals I* for which $D(n, I)$ remains bounded.

2. A proposition which implies the theorem

In what follows, U^0 will be the open interval $0 < \xi < 1$. All the numbers $\alpha, \beta, \gamma, \delta, \theta, \eta, \lambda, \mu, \alpha_i, \beta_i, \dots$ will be in U^0 . A *neighborhood* of a number α will by definition be an open interval containing α which is contained in U^0 . It will be convenient to extend the definition of the derivatives of a set S by putting $S^{(0)} = S$. By I, J, \dots we shall denote intervals of the type $a < n \leq b$ where the end points are integers with $0 \leq a < b$. Such an interval of length $l(I) = b - a$ contains precisely $l(I)$ integers.

The sequence ω will be fixed throughout. For α in U^0 we put

$$f(n, \alpha) = Z(n, \alpha) - n\alpha,$$

² In fact it is Van der Corput's sequence, as constructed in [3].

so that $D(n, \alpha) = |f(n, \alpha)|$. We write

$$g^+(I, \alpha) = \max_{n \in I} f(n, \alpha), \quad g^-(I, \alpha) = \min_{n \in I} f(n, \alpha)$$

and

$$h(I, \alpha) = g^+(I, \alpha) - g^-(I, \alpha).$$

PROPOSITION. *Suppose $d \geq 0$ and $\varepsilon > 0$. Let R be a set whose d -th derivative $R^{(d)}$ has a non-empty intersection with U^0 .*

Then there are $w = 2^d$ elements $\lambda_1, \dots, \lambda_w$ of R with neighborhoods L_1, \dots, L_w and a number p such that

$$(1) \quad w^{-1} \sum_{j=1}^w h(I, \mu_j) > \frac{1}{2}(d+1) - \varepsilon$$

for every interval I with $l(I) \geq p$ and every $\mu_1 \in L_1, \dots, \mu_w \in L_w$.

Applying this with $\mu_1 = \lambda_1, \dots, \mu_t = \lambda_t$ we see that there is a λ_j with $h(I, \lambda_j) > \frac{1}{2}(d+1) - \varepsilon$. There are integers m, n with $f(m, \lambda_j) - f(n, \lambda_j) > \frac{1}{2}(d+1) - \varepsilon$, hence with

$$\max(D(m, \lambda_j), D(n, \lambda_j)) = \max(|f(m, \lambda_j)|, |f(n, \lambda_j)|) > \frac{1}{4}(d+1) - \frac{1}{2}\varepsilon.$$

This shows that $\lambda_j \notin S(\frac{1}{4}(d+1) - \varepsilon)$.

Now take $R = S(\frac{1}{4}(d+1) - \varepsilon)$. The assumption that an element α of U^0 lies in $R^{(d)}$ leads to the contradiction that $\lambda_j \in R$ and $\lambda_j \notin R$. Hence $R^{(d)} = S^{(d)}(\frac{1}{4}(d+1) - \varepsilon)$ is empty except for the possible elements 0 and 1. At any rate $S^{(d+1)}(\frac{1}{4}(d+1) - \varepsilon)$ is empty for $d \geq 0$, and hence $S^{(d)}(\frac{1}{4}d - \varepsilon)$ is empty for $d \geq 1$. It follows that $S^{(d)}(\kappa)$ is empty for $d > 4\kappa$.

Hence our proposition implies the theorem. The proposition will be proved by induction on d . Its generality is necessary to carry out this inductive proof.

3. The case $d = 0$

When $d = 0$ the hypotheses of the proposition are satisfied if R consists of a single element α in U^0 . In this case the conclusion must hold with $w = 2^0 = 1$ and with $\lambda_1 = \alpha$. Hence when $d = 0$ the proposition may be reformulated as follows.

LEMMA 1. *Suppose α is in U^0 and $\varepsilon > 0$. There is a neighborhood A of α and a number p such that*

$$(2) \quad h(I, \beta) > \frac{1}{2} - \varepsilon$$

for every β in A and every interval I with $l(I) \geq p$.

For α in U^0 put

$$c(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is irrational,} \\ 1/z & \text{if } \alpha = y/z \text{ with coprime positive integers } y, z. \end{cases}$$

Since $0 \leq c(\alpha) \leq \frac{1}{2}$, Lemma 1 is a consequence of

LEMMA 2. *The inequality (2) in Lemma 1 may be replaced by*

$$h(I, \beta) > 1 - c(\alpha) - \varepsilon.$$

PROOF. If $\alpha = y/z$, then given any real ψ there are integers m, n with $1 \leq n \leq z$ and $|n\alpha - m - \psi| \leq c(\alpha)/2$. Now suppose that α is irrational. Kronecker's Theorem implies that for every ψ there are positive integers m, n with $|n\alpha - m - \psi| < \varepsilon/8$. Find particular solutions m, n for $\psi = 0, \varepsilon/4, 2\varepsilon/4, \dots, [4\varepsilon^{-1}]\varepsilon/4$ (where $[]$ denotes the integer part), and denote the maximum of the numbers n so obtained by p . Then for every ψ there will be integers m, n with $1 \leq n \leq p$ and with $|n\alpha - m - \psi| < \varepsilon/4$. Hence for every α in U^0 there is a $p = p(\alpha, \varepsilon)$ such that for every ψ there are integers m, n with

$$1 \leq n \leq p \text{ and } |n\alpha - m - \psi| < \frac{1}{2}c(\alpha) + \frac{1}{4}\varepsilon.$$

Let A be the neighborhood of α consisting of numbers β in U^0 with $|\beta - \alpha|p < \varepsilon/4$. For every β in A and every ψ there are integers m, n with $1 \leq n \leq p$ and $|n\beta - m - \psi| < \frac{1}{2}c(\alpha) + \frac{1}{2}\varepsilon$. Since this is true for every ψ , there will also be integers m, n with $1 \leq n \leq p$ and $0 < n\beta - m - \psi < c(\alpha) + \varepsilon$. It is clear that the interval $1 \leq n \leq p$ may be replaced by any interval I with $l(I) \geq p$. Thus for every such interval I and every ψ and for every β in A there are integers m, n with

$$n \in I \text{ and } 0 < n\beta - m - \psi < c(\alpha) + \varepsilon.$$

Now choose integers m, n with

$$n \in I \text{ and } 0 < n\beta - m + g^-(I, \beta) < c(\alpha) + \varepsilon.$$

We have $Z(n, \beta) = f(n, \beta) + n\beta \geq g^-(I, \beta) + n\beta > m$, whence $Z(n, \beta) \geq m + 1$. This implies that

$$\begin{aligned} h(I, \beta) &= g^+(I, \beta) - g^-(I, \beta) \\ &\geq Z(n, \beta) - n\beta - g^-(I, \beta) \\ &> m + 1 - n\beta + (n\beta - m - c(\alpha) - \varepsilon) \\ &= 1 - c(\alpha) - \varepsilon. \end{aligned}$$

4. Variations on Lemma 2

Write

$$f(n, \alpha, \beta) = f(n, \beta) - f(n, \alpha) = Z(n, \beta) - Z(n, \alpha) - n(\beta - \alpha).$$

LEMMA 3. *Suppose $\varepsilon > 0, q \geq 1$ and*

$$(3) \quad 0 < |\alpha - \beta| < \varepsilon/(8q).$$

Then there is a p and there are neighborhoods A of α and B of β such that for every $\gamma \in A$ and $\delta \in B$ and for every interval I with $l(I) \geq p$ there are two subintervals J and J' with $l(J) = l(J') = q$ such that

$$(4) \quad f(n, \gamma, \delta) - f(n', \gamma, \delta) > 1 - \varepsilon$$

for every $n \in J$ and every $n' \in J'$.

PROOF. We may assume that $\alpha < \beta$. Put $p_0 = [(\beta - \alpha)^{-1}]$. Every number in U has a distance less than $\varepsilon/8$ from at least one of the numbers $\beta - \alpha, 2(\beta - \alpha), \dots, p_0(\beta - \alpha)$. Thus for every ψ there are integers m, n with $1 \leq n \leq p_0$ and $|n(\beta - \alpha) - m - \psi| < \varepsilon/8$. Let A, B be disjoint neighborhoods of α, β such that elements γ of A and δ of B satisfy

$$(5) \quad 16|\gamma - \alpha| \max(q, p_0) < \varepsilon \text{ and } 16|\delta - \beta| \max(q, p_0) < \varepsilon,$$

respectively. For every $\gamma \in A$ and $\delta \in B$ and for every ψ there are integers m, n with $1 \leq n \leq p_0$ and $|n(\delta - \gamma) - m - \psi| < \varepsilon/4$, and similarly there are numbers m, n with $1 \leq n \leq p_0$ and $0 < n(\delta - \gamma) - m - \psi < \varepsilon/2$. Here the interval $1 \leq n \leq p_0$ may be replaced by any interval I_0 with $l(I_0) \geq p_0$.

Now suppose that $\gamma \in A, \delta \in B$ and $l(I_0) \geq p_0$. Let n'_0 be the integer in I_0 with

$$f(n'_0, \gamma, \delta) = \min_{n \in I_0} f(n, \gamma, \delta).$$

Choose integers m, n_0 with

$$n_0 \in I_0 \text{ and } 0 < n_0(\delta - \gamma) - m + f(n'_0, \gamma, \delta) < \varepsilon/2.$$

We have

$$Z(n_0, \delta) - Z(n_0, \gamma) = f(n_0, \gamma, \delta) + n_0(\delta - \gamma) \geq f(n'_0, \gamma, \delta) + n_0(\delta - \gamma) > m,$$

whence $Z(n_0, \delta) - Z(n_0, \gamma) \geq m + 1$. This implies that

$$(6) \quad \begin{aligned} f(n_0, \gamma, \delta) - f(n'_0, \gamma, \delta) &\geq m + 1 - n_0(\delta - \gamma) - f(n'_0, \gamma, \delta) \\ &> m + 1 - m - \frac{1}{2}\varepsilon \\ &= 1 - \frac{1}{2}\varepsilon. \end{aligned}$$

Since $\alpha < \beta$ and since A, B are disjoint, any elements $\gamma \in A$ and $\delta \in B$ have $\gamma < \delta$. Moreover by (3) and (5) they satisfy

$$(7) \quad 0 < q(\delta - \gamma) < \varepsilon/4.$$

Put $p = p_0 + 2q$ and let I be an interval with $l(I) \geq p$. The interval I_0 obtained from I by removing intervals of length q from both ends has

$l(I_0) \geq p_0$. Hence for every $\gamma \in A$ and $\delta \in B$ there are integers n_0, n'_0 in I_0 with (6). Let J and J' be the intervals

$$n_0 < n \leq n_0 + q \quad \text{and} \quad n'_0 - q < n' \leq n'_0,$$

respectively. For every n in J and every n' in J' one has

$$\begin{aligned} f(n, \gamma, \delta) - f(n_0, \gamma, \delta) &\geq -(\delta - \gamma)(n - n_0) \geq -q(\delta - \gamma) > -\varepsilon/4, \\ f(n', \gamma, \delta) - f(n'_0, \gamma, \delta) &< \varepsilon/4 \end{aligned}$$

by (7). These inequalities in conjunction with (6) yield (4). Since J and J' have length q and are contained in I , the lemma follows.

Write

$$g^+(J, \alpha, \beta) = \max_{n \in J} f(n, \alpha, \beta), \quad g^-(J, \alpha, \beta) = \min_{n \in J} f(n, \alpha, \beta).$$

The statement in Lemma 3 that (4) holds for $n \in J$ and $n' \in J'$ may now be expressed by

$$g^-(J, \gamma, \delta) - g^+(J', \gamma, \delta) > 1 - \varepsilon.$$

We shall need the function

$$h(J, J', \alpha, \beta) = \max(g^-(J, \alpha, \beta) - g^+(J', \alpha, \beta), g^-(J', \alpha, \beta) - g^+(J, \alpha, \beta)).$$

LEMMA 4. Suppose $\theta_1, \dots, \theta_t$ belong to the derivative $R^{(1)}$ of some set R . Let D_1, \dots, D_t be neighborhoods of $\theta_1, \dots, \theta_t$, respectively. Suppose $\varepsilon > 0$ and $q \geq 1$.

Then there is an r and there are elements $\alpha_1, \beta_1, \dots, \alpha_t, \beta_t$ of R with neighborhoods $A_1, B_1, \dots, A_t, B_t$ satisfying $A_i \subseteq D_i$, $B_i \subseteq D_i$ ($i = 1, \dots, t$) and with the following property. For $\gamma_1 \in A_1$, $\delta_1 \in B_1, \dots, \gamma_t \in A_t$, $\delta_t \in B_t$ and for intervals I, I' with $l(I) \geq r$, $l(I') \geq r$ there are subintervals $J \subseteq I$ and $J' \subseteq I'$ with

$$l(J) = l(J') = q$$

and with

$$(8) \quad h(J, J', \gamma_i, \delta_i) > 1 - \varepsilon \quad (i = 1, \dots, t).$$

We shall apply this lemma only in the special case when $I = I'$. The general formulation is necessary to carry out a proof by induction on t .

PROOF. Suppose at first that $t = 1$. Since θ_1 is a limit point of R , there are elements α_1, β_1 of R which belong to D_1 and which have

$$0 < |\alpha_1 - \beta_1| < \varepsilon/(8q).$$

By Lemma 3 there is a p and there are neighborhoods A_1, B_1 of α_1, β_1 such that for every $\gamma_1 \in A_1$, $\delta_1 \in B_1$ and for every I with $l(I) \geq p$ there are subintervals J_1, J_2 of length q with

$$(9) \quad g^-(J_1, \gamma_1, \delta_1) - g^+(J_2, \gamma_1, \delta_1) > 1 - \varepsilon.$$

We may shrink the neighborhoods A_1, B_1 , if necessary, to get $A_1 \subseteq D_1, B_1 \subseteq D_1$. If an interval I' also has length $l(I') \geq p$, then I' has subintervals J'_1, J'_2 of length q with

$$(10) \quad g^-(J'_1, \gamma_1, \delta_1) - g^+(J'_2, \gamma_1, \delta_1) > 1 - \varepsilon.$$

By adding (9) and (10) we see that either

$$g^-(J_1, \gamma_1, \delta_1) - g^+(J'_2, \gamma_1, \delta_1) > 1 - \varepsilon$$

or

$$g^-(J'_1, \gamma_1, \delta_1) - g^+(J_2, \gamma_1, \delta_1) > 1 - \varepsilon.$$

In the first case we take $J = J_1, J' = J'_2$, and in the second case we take $J = J_2, J' = J'_1$. The inequality (8) is then true for $i = 1$. Hence when $t = 1$, Lemma 4 is true with $r = r^{(1)} = p$.

The induction from $t-1$ to t goes as follows. Construct $\alpha_1, \beta_1, \dots, \alpha_{t-1}, \beta_{t-1}, A_1, B_1, \dots, A_{t-1}, B_{t-1}$ and $r^{(t-1)}$ such that (8) holds (under the conditions stated in the lemma) for $i = 1, \dots, t-1$. By the case $t = 1$ we can find α_t, β_t in R with neighborhoods A_t, B_t contained in D_t and a number $\bar{r}^{(1)}$ such that for every $\gamma_t \in A_t, \delta_t \in B_t$ and for intervals I, I' with $l(I) \geq \bar{r}^{(1)}, l(I') \geq \bar{r}^{(1)}$ there are subintervals $I_0 \subseteq I, I'_0 \subseteq I'$ with $l(I_0) = l(I'_0) = r^{(t-1)}$ such that

$$(11) \quad h(I_0, I'_0, \gamma_t, \delta_t) > 1 - \varepsilon.$$

By our construction of $\alpha_1, \beta_1, \dots, \alpha_{t-1}, \beta_{t-1}, A_1, B_1, \dots, A_{t-1}, B_{t-1}$ and $r^{(t-1)}$ there are subintervals $J \subseteq I_0, J' \subseteq I'_0$ with $l(J) = l(J') = q$ such that (8) holds for $i = 1, \dots, t-1$. Now in view of (11) and since $h(J, J', \gamma_t, \delta_t) \geq h(I_0, I'_0, \gamma_t, \delta_t)$, the inequality (8) holds for $i = 1, \dots, t$. This shows that Lemma 4 is true with $r = \bar{r}^{(1)}$.

5. An inequality

LEMMA 5. *Suppose α, β belong to U^0 and suppose that J, J' are subintervals of an interval I . Then*

$$(12) \quad h(I, \alpha) + h(I, \beta) \geq h(J, J', \alpha, \beta) \\ + \frac{1}{2}(h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta)).$$

PROOF. We may assume without loss of generality that

$$h(J, J', \alpha, \beta) = g^-(J, \alpha, \beta) - g^+(J', \alpha, \beta).$$

Then we have $f(n, \alpha, \beta) - f(n', \alpha, \beta) \geq h(J, J', \alpha, \beta)$, i.e.

$$(13) \quad f(n, \beta) - f(n, \alpha) - f(n', \beta) + f(n', \alpha) \geq h(J, J', \alpha, \beta)$$

for every $n \in J$ and every $n' \in J'$. Let $m_\alpha, n_\alpha, m_\beta, n_\beta$ be integers in J with

$$\begin{aligned} f(m_\alpha, \alpha) &= g^+(J, \alpha), & f(n_\alpha, \alpha) &= g^-(J, \alpha), \\ f(m_\beta, \beta) &= g^+(J, \beta), & f(n_\beta, \beta) &= g^-(J, \beta). \end{aligned}$$

Then

$$(14) \quad f(m_\alpha, \alpha) - f(n_\alpha, \alpha) = h(J, \alpha),$$

$$(15) \quad f(m_\beta, \beta) - f(n_\beta, \beta) = h(J, \beta).$$

Similarly, there are elements $m'_\alpha, n'_\alpha, m'_\beta, n'_\beta$ of J' such that

$$(16) \quad f(m'_\alpha, \alpha) - f(n'_\alpha, \alpha) = h(J', \alpha),$$

$$(17) \quad f(m'_\beta, \beta) - f(n'_\beta, \beta) = h(J', \beta).$$

Applying (13) with $n = m_\alpha, n' = m'_\beta$ we obtain

$$f(m_\alpha, \beta) - f(m_\alpha, \alpha) - f(m'_\beta, \beta) + f(m'_\beta, \alpha) \geq h(J, J', \alpha, \beta).$$

Applying (13) with $n = n_\beta, n' = n'_\alpha$ we obtain

$$f(n_\beta, \beta) - f(n_\beta, \alpha) - f(n'_\alpha, \beta) + f(n'_\alpha, \alpha) \geq h(J, J', \alpha, \beta).$$

Adding these two inequalities and the four equations (14), (15), (16), (17) we get

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \geq 2h(J, J', \alpha, \beta) + h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta),$$

where

$$\varphi_1 = f(m'_\alpha, \alpha) - f(n_\alpha, \alpha), \quad \varphi_2 = f(m'_\beta, \alpha) - f(n_\beta, \alpha),$$

$$\varphi_3 = f(m_\beta, \beta) - f(n'_\beta, \beta), \quad \varphi_4 = f(m_\alpha, \beta) - f(n'_\alpha, \beta).$$

Since $h(I, \alpha) \geq \varphi_1, h(I, \alpha) \geq \varphi_2, h(I, \beta) \geq \varphi_3, h(I, \beta) \geq \varphi_4$, the lemma follows.

6. Proof of the proposition

Lemma 1 shows the truth of the proposition when $d = 0$. From here on we shall have $d \geq 1$, and we shall assume the truth of the proposition for $d-1$ and proceed to prove it for d .

By this assumption we see that if $\varepsilon > 0$ and if $R^{(d)}$ and U^0 have a non-empty intersection, then there are $t = 2^{d-1}$ elements $\theta_1, \dots, \theta_t$ of $R^{(1)}$ with neighborhoods D_1, \dots, D_t and a number $p^{(d-1)}$ such that

$$(18) \quad t^{-1} \sum_{j=1}^t h(I, \eta_j) > \frac{1}{2}d - \frac{1}{2}\varepsilon$$

for $\eta_1 \in D_1, \dots, \eta_t \in D_t$ and every interval I with $l(I) \geq p^{(d-1)}$. We now apply Lemma 4 with these particular $\theta_1, \dots, \theta_t, D_1, \dots, D_t$ and with

$q = p^{(d-1)}$. We construct elements $\alpha_1, \beta_1, \dots, \alpha_t, \beta_t$ of R with neighborhoods $A_1, B_1, \dots, A_t, B_t$ and

$$(19) \quad r = r(\theta_1, \dots, \theta_t; D_1, \dots, D_t; p^{(d-1)})$$

with the properties enunciated in that lemma.

Now suppose that $l(I) = r$ and let $\gamma_1, \delta_1, \dots, \gamma_t, \delta_t$ be elements of $A_1, B_1, \dots, A_t, B_t$, respectively. There are subintervals J, J' of I with $l(J) = l(J') = p^{(d-1)}$ such that

$$h(J, J', \gamma_i, \delta_i) > 1 - \varepsilon \quad (i = 1, \dots, t).$$

Hence by Lemma 5 we have

$$(20) \quad h(I, \gamma_i) + h(I, \delta_i) > (1 - \varepsilon) + \frac{1}{2}(h(J, \gamma_i) + h(J, \delta_i) + h(J', \gamma_i) + h(J', \delta_i)) \quad (i = 1, \dots, t).$$

Now γ_j lies in D_j since $\gamma_j \in A_j$ and $A_j \subseteq D_j$ ($j = 1, \dots, t$). We therefore may apply (18) with $\eta_1 = \gamma_1, \dots, \eta_t = \gamma_t$, and we obtain

$$\sum_{j=1}^t h(J, \gamma_j) > t(\frac{1}{2}d - \frac{1}{2}\varepsilon).$$

More generally, each of the four quantities

$$\chi_1 = \sum_{j=1}^t h(J, \gamma_j), \chi_2 = \sum_{j=1}^t h(J, \delta_j), \chi_3 = \sum_{j=1}^t h(J', \gamma_j), \chi_4 = \sum_{j=1}^t h(J', \delta_j)$$

exceeds $t(\frac{1}{2}d - \frac{1}{2}\varepsilon)$. Taking the sum of the inequalities (20) with $i = 1, \dots, t$ and dividing by $2t$ we obtain

$$(21) \quad (2t)^{-1} \left(\sum_{i=1}^t h(I, \gamma_i) + \sum_{i=1}^t h(I, \delta_i) \right) > (\frac{1}{2} - \frac{1}{2}\varepsilon) + (4t)^{-1} (\chi_1 + \chi_2 + \chi_3 + \chi_4) > \frac{1}{2}(d+1) - \varepsilon.$$

The $w = 2t = 2 \cdot 2^{d-1} = 2^d$ quantities $\lambda_1 = \alpha_1, \dots, \lambda_t = \alpha_t, \lambda_{t+1} = \beta_1, \dots, \lambda_{2t} = \beta_t$ and their respective neighborhoods $L_1 = A_1, \dots, L_t = A_t, L_{t+1} = B_1, \dots, L_{2t} = B_t$ and $p = r$ where r is given by (19) have the desired properties stated in the proposition. Namely, (21) shows that (1) is true for every interval I with $l(I) \geq p$ and arbitrary elements μ_1, \dots, μ_w in L_1, \dots, L_w .

7. An example

Let R_0 be the set consisting of 0, and for integers $d \geq 1$ let R_d be the set consisting of 0 and of the numbers

$$(22) \quad 2^{-g_1} + \dots + 2^{-g_t}$$

where t, g_1, \dots, g_t are integers with

$$(23) \quad 1 \leq t \leq d \text{ and } 1 \leq g_1 < g_2 < \dots < g_t.$$

LEMMA 6. For every $d \geq 1$,

$$R_d^{(1)} = R_{d-1}.$$

PROOF. It is clear that $R_1^{(1)} = R_0$. We now proceed by induction on d and assume that $d \geq 2$ and that $R_{d-1}^{(1)} = R_{d-2}$. Since the relation $R_{d-1} \subseteq R_d^{(1)}$ is rather obvious, it will remain for us to show that $R_d^{(1)} \subseteq R_{d-1}$.

Let ξ be the limit of a sequence of distinct numbers $\eta(1), \eta(2), \dots$ of R_d ; we have to show that ξ lies in R_{d-1} . We clearly may assume that none of the numbers $\eta(n)$ is 0. Let $t(n), g_1(n), \dots, g_{t(n)}(n)$ be the numbers t, g_1, \dots, g_t in (22) which belong to $\eta(n)$. In view of (23) there are only finitely many numbers in R_d for which g_t lies under a given upper bound, and hence $g_{t(n)}(n)$ must tend to infinity. Therefore ξ is also the limit of the sequence $\hat{\eta}(1), \hat{\eta}(2), \dots$ where

$$\hat{\eta}(n) = \eta(n) - 2^{-g_{t(n)}(n)}.$$

The numbers $\hat{\eta}(n)$ lie in R_{d-1} . If infinitely many of them are equal, then their limit ξ is in R_{d-1} . If infinitely many among them are distinct, then we know by induction that their limit ξ is in R_{d-2} , hence a fortiori in R_{d-1} .

We now construct a sequence $\omega_0 = \{\xi_1, \xi_2, \dots\}$ as follows. We put $\xi_1 = 0$, and if $k \geq 0$ and if ξ_1, \dots, ξ_{2^k} have already been constructed, then we define $\xi_{2^{k+1}}, \dots, \xi_{2^{k+1}}$ by

$$(24) \quad \xi_{2^k+t} = \xi_t + \frac{1}{2^{k+1}} \quad (t = 1, \dots, 2^k).$$

Thus $\omega_0 = \{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \dots\}$. In what follows, the sets $S(\kappa)$ will be defined in terms of this sequence ω_0 .

LEMMA 7. For every integer $d \geq 0$,

$$R_d \subseteq S(d).$$

Repeated application of Lemma 6 shows that $R_d^{(d)}$ consists of 0, and we obtain the

COROLLARY. The sets $S^{(d)}(d)$ are non-empty for $d = 0, 1, 2, \dots$.

PROOF OF LEMMA 7. The assertion is true for $d = 0$ since $S(0)$ contains 0. Assuming the truth of the lemma for $d-1$ we now proceed to prove it for d . It will suffice to show that every element η of R_d of the type

$$\eta = 2^{-g_1} + \dots + 2^{-g_a}$$

lies in $S(d)$. Put $\hat{\eta} = 2^{-g_1} + \dots + 2^{-g_{a-1}}$. We know by our inductive hypothesis that

$$(25) \quad |Z(n, \hat{\eta}) - n\hat{\eta}| \leq d-1 \quad (n = 1, 2, \dots).$$

The first 2^{g_a} elements of ω_0 are the numbers $j2^{-g_a}$ ($j = 0, 1, \dots, 2^{g_a} - 1$) in some order. Hence there is precisely one t_0 with $1 \leq t_0 \leq 2^{g_a}$ and $\xi_{t_0} = \hat{\eta}$. The other elements ξ_t with $1 \leq t \leq 2^{g_a}$ lie outside the interval I given by

$$\hat{\eta} \leq \xi < \eta = \hat{\eta} + 2^{-g_a}.$$

Now if $t' = t + m2^{g_a}$ where $1 \leq t \leq 2^{g_a}$ and where m is a nonnegative integer, then

$$\xi_t \leq \xi_{t'} < \xi_t + 2^{-g_a-1} + 2^{-g_a-2} + \dots = \xi_t + 2^{-g_a}$$

by repeated application of (24). Therefore $\xi_{t'}$ lies in I precisely if $t \equiv t_0 \pmod{2^{g_a}}$. This implies that

$$n2^{-g_a} - 1 < Z(n, \eta) - Z(n, \hat{\eta}) < n2^{-g_a} + 1 \quad (n = 1, 2, \dots),$$

hence that

$$\begin{aligned} & |Z(n, \eta) - Z(n, \hat{\eta}) - n(\eta - \hat{\eta})| \\ & = |Z(n, \eta) - Z(n, \hat{\eta}) - n2^{-g_a}| < 1 \quad (n = 1, 2, \dots). \end{aligned}$$

Combining this inequality with (25) we obtain $|Z(n, \eta) - n\eta| < d$, which shows that η lies in $S(d)$.

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