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**COMPLETE INDEX SETS OF
 RECURSIVELY ENUMERABLE FAMILIES**

by

T. G. McLaughlin

1. Introduction

This paper is an adjunct to [2]. In [2, § 2], we remarked that the index set $G(\mathcal{F})$ of a recursively enumerable family \mathcal{F} of classes of r.e. sets can be \sum_n^0 complete for $1 \leq n \leq 5$ or \prod_n^0 complete for $1 \leq n \leq 4$. Here we shall give specific examples which verify that remark. All unexplained notation and background terminology in the present paper should be read according to the conventions laid down in [2, § 1]. We wish to take the opportunity to correct a minor technical error in [2]. The function ζ_0 referred to in the introductory section of [2] should not be alleged to be a *total* recursive function, but should rather be specified as a partial recursive function with $\delta\zeta_0 = \{e \mid W_e^{\mathcal{F}} \text{ is a nonempty family consisting of nonempty classes}\}$. The only point at which ζ_0 enters into a proof in [2] is at the beginning of the proof of Lemma A, where, under the alias of ζ , it is mistakenly treated as being defined for all arguments e . However, it is easily seen that even with its domain limited as indicated above, $\zeta (= \zeta_0)$ still permits the function ξ of [2, Lemma A] to be taken *total* recursive; none of the remaining discussion in [2] need then be modified or even reworded. (There are alternative ways of mending the error; but the way just indicated seems most direct.)

2. Complete index sets

Throughout this section, we let η denote a partial recursive function with the property that $(\forall x)[W_x \neq \emptyset \Rightarrow \eta(x) \in W_x]$; and we let μ denote a recursive function such that $(\forall x)[W_{\mu(x)} = \{x\}]$.

PROPOSITION 1. *If \mathcal{F} is an r.e. family of classes then its index set, $G(\mathcal{F})$, is \sum_5^0 .*

PROOF. Let $\mathcal{F} = \{W_e^c \mid e \in W_f\}$. Then

$$\begin{aligned} n \in G(\mathcal{F}) &\Leftrightarrow (\exists e)[e \in W_f \text{ and } W_e^c = W_n^c] \\ &\Leftrightarrow (\exists e)[e \in W_f \text{ and } (\forall j)[W_j \in W_e^c \Leftrightarrow W_j \in W_n^c]]. \end{aligned}$$

But $W_j \in W_k^c \Leftrightarrow (\exists r)[r \in W_k \text{ and } (\forall s)[s \in W_j \Leftrightarrow s \in W_r]]$. Hence, by means of the usual prenex transformation procedures, $W_j \in W_k^c$ is seen to be a \sum_3^0 predicate of j and k . It follows, again by the standard prenex operations, that $n \in G(\mathcal{F})$ is a \sum_5^0 predicate of n . Q.E.D.

PROPOSITION 2. Let $\mathcal{W} = \{W_e^c | e \in N\}$. Then \mathcal{W} is an r.e. family and $G(\mathcal{W})$ is recursive.

Since $G(\mathcal{W}) = N$, Proposition 2 is obvious.

PROPOSITION 3. \mathcal{W} and \emptyset are the only r.e. families \mathcal{F} for which $G(\mathcal{F})$ is recursive.

PROOF. The proposition is a precise analogue of a result of Rice's concerning *classes* ([3, Corollary B to Theorem 6]); and the proof follows the proof of Rice's result given in [1]. Thus, suppose \mathcal{F} is a family of r.e. classes such that $G(\mathcal{F})$ is recursive. Suppose further that neither \mathcal{F} nor $\mathcal{W} - \mathcal{F}$ is empty. Now, either $\emptyset \in \mathcal{F}$ or $\emptyset \in \mathcal{W} - \mathcal{F}$; let us first suppose that $\emptyset \in \mathcal{W} - \mathcal{F}$. Let Q be a fixed non-recursive r.e. subset of N ; and let W_e^c be some fixed element of \mathcal{F} . Let g be a recursive function such that $n \in Q \Rightarrow W_{g(n)} = W_e$ and $n \notin Q \Rightarrow W_{g(n)} = \emptyset$. Then, since $W_e \neq \emptyset$, we have $n \in Q \Leftrightarrow g(n) \in G(\mathcal{F})$. But therefore Q is recursive: contradiction. A similar contradiction arises if we assume $\emptyset \in \mathcal{F}$, since if $G(\mathcal{F})$ is recursive then so also is $G(\mathcal{W} - \mathcal{F})$. Hence either $\mathcal{F} = \mathcal{W}$ or $\mathcal{F} = \emptyset$. Q.E.D.

REMARK. The proof of Proposition 3 in fact shows that if $\mathcal{F} \neq \emptyset$ & $\emptyset \notin \mathcal{F}$ then every \sum_1^0 set is many-one reducible to $G(\mathcal{F})$. (Indeed, they are all one-one reducible to $G(\mathcal{F})$, since g can be taken one-one.)

PROPOSITION 4. Let $\mathcal{F}_0 = \{W_e^c | (\exists y)[y \in W_e \text{ and } W_y \neq \emptyset]\}$. Then \mathcal{F}_0 is an r.e. family and $G(\mathcal{F}_0)$ is \sum_1^0 complete.

PROOF. Clearly, we have

$$(\forall n)[n \in G(\mathcal{F}_0) \Leftrightarrow (\exists y)(\exists z)[y \in W_n \text{ and } z \in W_y]];$$

therefore $G(\mathcal{F}_0)$ is \sum_1^0 . *A fortiori*, \mathcal{F}_0 is an r.e. family. Next, it is easy to see that there exists a recursive function $\psi_0(x, y)$ such that

$$(\forall w)(\forall z)(\forall x)(\forall y)[\varphi_{\psi_0(f, z)}^3(w, x, y)$$

is defined $\Leftrightarrow T_1(f, z, w)$]. Let $\omega(x)$ be a recursive function such that

$$(\forall x)[W_{\omega(x)} = \bigcup_{z \in x} W_z].$$

Then, since $(\forall f)[W_f = \{z | (\exists w)T_1(f, z, w)\}]$, we have that

$$\omega(\zeta_1(\psi_0(f, n))) \in G(\mathcal{F}_0) \Leftrightarrow n \in W_f;$$

here and subsequently, ζ_1 is as in § 1 of [2]. Thus $G(\mathcal{F}_0)$ is Σ_1^0 complete. (Alternatively, note that $\emptyset \notin \mathcal{F} \neq \emptyset$ and use the remark following the proof of Proposition 3.) Q.E.D.

PROPOSITION 5. *Let $\mathcal{F}_1 = \{\emptyset, \{\emptyset\}\}$. Then \mathcal{F}_1 is an r.e. family and $G(\mathcal{F}_1)$ is \prod_1^0 complete.*

PROOF. Since every finite family of r.e. classes is r.e., \mathcal{F}_1 is an r.e. family. We can easily construct a recursive function $\psi_1(x, y)$ such that

$$(\forall w)(\forall z)(\forall x)(\forall y)[\varphi_{\psi_1(f, z)}^3(w, x, y) \text{ is defined} \Leftrightarrow T_1(f, z, x)].$$

Therefore $\eta(\zeta_1(\psi_1(f, n))) \in G(\mathcal{F}_1) \Leftrightarrow (\forall x) \neg T_1(f, n, x)$, so that if $G(\mathcal{F}_1)$ is \prod_1^0 then it is \prod_1^0 complete. (Alternatively, note that $\emptyset \notin \mathcal{W} - \mathcal{F} \neq \emptyset$ and use the remark following the proof of Proposition 3.) But, since $n \in G(\mathcal{F}_1) \Leftrightarrow (\forall x)(\forall y)[x \in W_n \Rightarrow y \notin W_x]$, we have that $G(\mathcal{F}_1)$ is indeed \prod_1^0 and therefore \prod_1^0 complete. Q.E.D.

REMARK. The alternative proofs of completeness indicated for Propositions 4 and 5 show that we could have taken $\mathcal{F}_0 = \{W_e^c \mid W_e \neq \emptyset\}$ for Proposition 4 and $\mathcal{F}_1 = \{\emptyset\}$ for Proposition 5. We prefer, however, the more involved choices of \mathcal{F}_0 and \mathcal{F}_1 since then the proofs can be given the common format shared by all the later proofs (with the single exception of our proof of Proposition 8).

PROPOSITION 6. *Let $\mathcal{F}_2 = \{W_e^c \mid \emptyset \in W_e^c\}$. Then \mathcal{F}_2 is an r.e. family and $G(\mathcal{F}_2)$ is Σ_2^0 complete.*

PROOF. \mathcal{F}_2 is r.e., since $\mathcal{F}_2 = \{W_e^c \mid (\exists f)[W_e^c = W_f^c \cup \{\emptyset\}]\}$. Next, since $n \in G(\mathcal{F}_2) \Leftrightarrow (\exists j)[j \in W_n \text{ and } (\forall z)(z \notin W_j)]$, it is easily seen by routine prenex-form manipulation that $G(\mathcal{F}_2)$ is Σ_2^0 . To show that $G(\mathcal{F}_2)$ is Σ_2^0 complete, we need only construct a recursive function $\psi_2(x, y)$ with the property that

$$(\forall f)(\forall n)[(\exists w)(\forall z) \neg T_2(f, n, w, z) \Leftrightarrow (\exists v)(\exists u)(\delta\varphi_{\psi_2(f, n)}^3(v, u, y) = \emptyset)];$$

for then we have that $n \in$ the f -th Σ_2^0 set $\Leftrightarrow \omega(\zeta_1(\psi_2(f, n))) \in G(\mathcal{F}_2)$. To obtain ψ_2 , we first construct an auxiliary partial recursive function v by stages, thus:

$$v_{f, n}^s(v, u, y) \simeq \begin{cases} 0, & \text{if } (\forall w \leq v)(\exists z \leq s)T_2(f, n, w, z), \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

and we set

$$v_{f, n}^s =_{df} \bigcup_{s=0}^{\infty} v_{f, n}^s.$$

Clearly, $v_{f, n}$ is partial recursive uniformly in the parameters f and n ; so let $\psi_2(x, y)$ be a recursive function such that

$$(\forall f)(\forall n)[\varphi_{\psi_2(f,n)}^3 \simeq v_{f,n}].$$

From the definition of $v_{f,n}$, it is easy to see that, for each pair $\langle f, n \rangle$,

$$(\exists w)(\forall z) \neg T_2(f, n, w, z) \Leftrightarrow (\exists v)(\exists u)(\delta\varphi_{\psi_2(f,n)}^3(v, u, y) = \emptyset).$$

Hence $G(\mathcal{F}_2)$ is \sum_2^0 complete. Q.E.D.

PROPOSITION 7. *Let $\mathcal{F}_3 = \{\{N\}\}$. Then \mathcal{F}_3 is an r.e. family and $G(\mathcal{F}_3)$ is \prod_2^0 complete.*

PROOF. \mathcal{F}_3 is r.e. since it is a finite family of r.e. classes. Since

$$n \in G(\mathcal{F}_3) \Leftrightarrow [W_n \neq \emptyset \text{ and } (\forall z)[z \in W_n \Rightarrow (\forall k)(k \in W_z)]],$$

and since $W_n \neq \emptyset$ is a \sum_1^0 predicate of n , we see by the usual prenex transformation procedures that $G(\mathcal{F}_3)$ is \prod_2^0 . We shall construct a recursive function $\psi_3(x, y)$ with the property:

$$(\forall w)(\forall z)(\forall x)(\forall y)[\varphi_{\psi_3(f,z)}^3(w, x, y) \text{ is defined} \Leftrightarrow (\exists u)T_2(f, z, y, u)].$$

Since

$$\begin{aligned} z \in \text{the } f\text{-th } \prod_2^0 \text{ set} &\Leftrightarrow (\forall y)(\exists u)T_2(f, z, y, u) \\ &\Leftrightarrow (\forall w)(\forall x)[\delta\varphi_{\psi_3(f,z)}^3(w, x, y) = N], \end{aligned}$$

the function $\eta\zeta_1\psi_3$ will then witness the \prod_2^0 completeness of $G(\mathcal{F}_3)$. The required function ψ_3 is very simply obtainable via a stage-by-stage construction of an auxiliary function τ : at stage s we set

$$\tau_{f,z}^s = \{\langle w, x, y, 0 \rangle \mid (\exists u \leq s)T_2(f, z, y, u)\},$$

for all pairs $\langle f, z \rangle$; then we take $\tau_{f,z} = \bigcup_{s=0}^{\infty} \tau_{f,z}^s$. Obviously, the construction of $\tau_{f,z}$ is effective uniformly in the parameters f and z ; i.e., there is a recursive function ψ_3 such that $(\forall f)(\forall z)[\tau_{f,z} \simeq \varphi_{\psi_3(f,z)}^3]$. ψ_3 as so specified is plainly an indexing function of the kind that we require, and hence $G(\mathcal{F}_3)$ is \prod_2^0 complete. Q.E.D.

PROPOSITION 8. *Let $\mathcal{F}_4 = \{\{A\} \mid A \text{ is a cofinite subset of } N\}$. Then \mathcal{F}_4 is an r.e. family and $G(\mathcal{F}_4)$ is \sum_3^0 complete.*

PROOF. The class *COF* of cofinite subsets of N is r.e.; hence \mathcal{F}_4 is r.e. since $\mathcal{F}_4 = \{W_{\mu(e)}^c \mid W_e \in \text{COF}\}$. Now, it is shown in [4] that the set $C = \{e \mid W_e \in \text{COF}\}$ is \sum_3^0 complete. But hence $G(\mathcal{F}_4)$ is also \sum_3^0 complete, provided that it is \sum_3^0 at all; for if A is a \sum_3^0 subset of N and β is a recursive function such that $n \in A \Rightarrow \beta(n) \in C$ and $n \notin A \Rightarrow \beta(n) \notin C$, then $n \in A \Rightarrow \mu(\beta(n)) \in G(\mathcal{F}_4)$ and $n \notin A \Rightarrow \mu(\beta(n)) \notin G(\mathcal{F}_4)$. To see that $G(\mathcal{F}_4)$ is \sum_3^0 , we first note that the predicate $(\forall z)[z \in W_n \Rightarrow W_z = W_e]$ is a \prod_2^0 predicate of n and e , and we then apply the standard prenex

operations to the right-hand side of the equivalence $n \in G(\mathcal{F}_4) \Leftrightarrow [W_n \neq \emptyset \text{ and } (\exists e)[e \in C \text{ and } (\forall z)[z \in W_n \Rightarrow W_z = W_x]]]$. Thus, $G(\mathcal{F}_4)$ is \prod_3^0 complete. Q.E.D.

PROPOSITION 9. *Let $\mathcal{F}_5 = \{W_e^c \mid \{W_j \mid W_j \text{ is finite}\} \subseteq W_e^c\}$. Then \mathcal{F}_5 is an r.e. family and $G(\mathcal{F}_5)$ is \prod_3^0 complete.*

PROOF. The class $\{W_j \mid W_j \text{ is finite}\}$ is r.e.; hence, since

$$\mathcal{F}_5 = \{W_e^c \mid (\exists k)[W_e^c = W_k^c \cup \{W_j \mid W_j \text{ is finite}\}]\},$$

it is easily deduced that \mathcal{F}_5 is an r.e. family. To show that $G(\mathcal{F}_5)$ is \prod_3^0 , we make use of a 'canonical enumeration' of the class $\{W_j \mid W_j \text{ is finite}\}$. The particular enumeration that we shall apply is defined (as in [5, p. 70]) as follows:

$$D_0 \stackrel{df}{=} \emptyset; D_{n+1} \stackrel{df}{=} \{k_1, \dots, k_l\}, \text{ where } n+1 = \sum_{m=1}^l 2^{km}$$

with $k_1 < k_2 < \dots < k_m$. It is easily verified that the predicate $D_j \subseteq W_k$ is a \sum_1^0 predicate of j and k , and that the predicate $x \in D_j$ is a recursive predicate of x and j . Now, we have

$$\begin{aligned} n \in G(\mathcal{F}_5) &\Leftrightarrow (\forall j)(\exists m)[m \in W_n \text{ and } D_j = W_m] \\ &\Leftrightarrow (\forall j)(\exists m)[m \in W_n \text{ and } D_j \subseteq W_m \text{ and } W_m \subseteq D_j]; \end{aligned}$$

hence, by routine prenex manipulations we obtain a \prod_3^0 predicate form for $G(\mathcal{F}_5)$. We shall construct a recursive function $\psi_4(x, y)$ such that, for every pair of numbers $\langle f, n \rangle$, we have

$$(\forall z)(\exists w)(\forall y) \neg T_3(f, n, z, w, y) \Leftrightarrow (\forall j)(\exists l)(\exists k)[\delta\varphi_{\psi_4(f, n)}^3(l, k, u) = D_j].$$

It is then obviously the case that

$$n \in \text{the } f\text{-th } \prod_3^0 \text{ set} \Leftrightarrow \omega(\zeta_1(\psi_4(f, n))) \in G(\mathcal{F}_5),$$

where ω is as in the proof of Proposition 4. Thus, the existence of such a function ψ_4 implies \prod_3^0 completeness of $G(\mathcal{F}_5)$. In order to specify ψ_4 , we shall define an auxiliary partial recursive function $\bar{\tau}$ by stages, as follows:

$$\bar{\tau}_{f, n}(j, k, u) \simeq \begin{cases} 0, & \text{if } u \in D_j \text{ or if } (\exists y \leq s)T_3(f, n, j, k, y), \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

$$\bar{\tau}_{f, n} = \bigcup_{s=0}^{\infty} \bar{\tau}_{f, n}^s.$$

It is obvious that the definition of $\bar{\tau}_{f, n}$ is effective uniformly in the parameters f and n ; hence, there is a recursive function $\psi_4(x, y)$ such that

$$(\forall f)(\forall n)[\varphi_{\psi_4(f,n)}^3 \simeq \bar{\tau}_{f,n}].$$

But then $\delta\varphi_{\psi_4(f,n)}^3(j, k, u)$, for a given pair $\langle j, k \rangle$, is either N or D_j , according as $(\exists y)T_3(f, n, j, k, y)$ or $(\forall y)\neg T_3(f, n, j, k, y)$. Thus, ψ_4 has the required property, and so $G(\mathcal{F}_5)$ is \prod_3^0 complete. Q.E.D.

Before stating Proposition 10 we remind the reader that p_n denotes the n -th prime number in order of magnitude, starting with $p_0 = 2$. We shall denote by P_n the set $\{p_n^m | m \in N - \{0\}\}$ of positive powers of p_n .

PROPOSITION 10. *Let $\mathcal{F}_6 = \{W_e^c | (\exists n)[\{W_j | W_j \text{ is a finite subset of } P_n\} \subseteq W_e^c]\}$. Then \mathcal{F}_6 is an r.e. family and $G(\mathcal{F}_6)$ is \sum_4^0 complete.*

PROOF. It is easily demonstrated that there is a recursive function $\chi(x, y)$, with $\chi(n, y)$ one-to-one for each n , such that $(\forall n)[\{D_{\chi(n,y)} | y \in N\} = \{W_j | W_j \text{ is a finite subset of } P_k\}]$. Hence \mathcal{F}_6 is r.e., since

$$\mathcal{F}_6 = \{W_e^c | (\exists f)(\exists n)[W_e^c = W_f^c \cup \{D_{\chi(n,y)} | y \in N\}]\}.$$

Next, we observe that

$$n \in G(\mathcal{F}_6) \Leftrightarrow (\exists k)(\forall j)(\exists m)[m \in W_n \text{ and } D_{\chi(k,j)} = W_m].$$

Therefore $G(\mathcal{F}_6)$ is \sum_4^0 . To show that $G(\mathcal{F}_6)$ is \sum_4^0 complete, we need only make a slight modification of our above proof that $G(\mathcal{F}_5)$ is \prod_3^0 complete. Thus, we begin by defining

$$\zeta_{f,n}^s(k, j, y, z) \simeq \begin{cases} 0, & \text{if } z \in D_{\chi(k,j)} \text{ or if } (\exists w \leq s)T_4(f, n, k, j, y, w), \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

and we then set

$$\zeta_{f,n} = a_f \bigcup_{s=0}^{\infty} \zeta_{f,n}^s,$$

for all pairs $\langle f, n \rangle$. Clearly, $\zeta_{f,n}$ is partial recursive uniformly in f and n ; so there is a recursive function $\xi(x, y)$ such that

$$(\forall f)(\forall n)[\varphi_{\xi(f,n)}^4 \simeq \zeta_{f,n}].$$

Let $\zeta_2(f, n)$ be a recursive function such that

$$(\forall f)(\forall n)[\varphi_{\zeta_2(f,n)}^3(\pi_2(k, j), y, z) \simeq \varphi_{\xi(f,n)}^4(k, j, y, z)];$$

π_2 is here a recursive pairing function as in [2, § 1]. Then, for all 5-tuples $\langle f, n, k, j, y \rangle$, we have $\delta\varphi_{\zeta_2(f,n)}^3(\pi_2(k, j), y, z) =$ either N or $D_{\chi(k,j)}$ according as $(\exists w)T_4(f, n, k, j, y, w)$ or $(\forall w)\neg T_4(f, n, k, j, y, w)$. It follows that, for every pair of numbers $\langle f, n \rangle$, we have

$$\begin{aligned}
& (\exists k)(\forall j)(\exists y)(\forall w) \neg T_4(f, n, k, j, y, w) \\
& \Leftrightarrow [(\forall k)(\forall j)(\forall p)(\forall q)(\forall y)(p \neq k \\
& \Rightarrow \delta\varphi_{\zeta_2(f, n)}^3(\pi_2(p, q), y, z) \neq D_{\chi(k, j)}) \text{ and} \\
& (\exists k)(\forall j)(\exists y)[\delta\varphi_{\zeta_2(f, n)}^3(\pi_2(k, j), y, z) = D_{\chi(k, j)}]].
\end{aligned}$$

Therefore if we define $\psi_5(x, y)$ by $\psi_5 = \omega_{\zeta_1} \zeta_2$, we obtain the equivalence: $n \in$ the f -th \sum_4^0 set $\Leftrightarrow \psi_5(f, n) \in G(\mathcal{F}_6)$. Thus $G(\mathcal{F}_6)$ is \sum_4^0 complete. Q.E.D.

PROPOSITION 11. *Let $\mathcal{F}_7 = \{W_e^c \mid \{W_j \mid W_j \text{ is cofinite}\} \subseteq W_e^c\}$. Then \mathcal{F}_7 is an r.e. family and $G(\mathcal{F}_7)$ is \prod_4^0 complete.*

PROOF. The class $\{W_j \mid W_j \text{ is cofinite}\}$ is, of course, r.e.; say, $\{W_j \mid W_j \text{ is cofinite}\} = W_{e_0}^c$. Hence \mathcal{F}_7 is an r.e. family, since

$$\bar{\mathcal{F}}_7 = \{W_e^c \mid (\exists f)[W_e^c = W_f^c \cup W_{e_0}^c]\}.$$

Next, observe that

$$n \in G(\mathcal{F}_7) \Leftrightarrow (\forall h)[W_h \text{ is cofinite} \Rightarrow (\exists l)[l \in W_n \text{ and } (W_l = W_h)]].$$

But $W_l = W_h$ is a \prod_2^0 predicate of l and h ; and the assertion that W_h is cofinite is ([4]) a \sum_3^0 predicate of h . Hence, by the usual prenex operations (carried out with several alternations of priority between the antecedent and the consequent inside the main quantifier), we see that $n \in G(\mathcal{F}_7)$ is a \prod_4^0 predicate of n . To show \prod_4^0 completeness of $G(\mathcal{F}_7)$, we construct a recursive function $\psi_6(x, y)$ with the property that

$$\begin{aligned}
& (\forall f)(\forall n)[(\forall u)(\exists v)(\forall w)(\exists z)T_4(f, n, u, v, w, z) \\
& \Leftrightarrow (\forall j)(\exists l)(\exists k)[\delta\varphi_{\psi_6(f, n)}^3(l, k, y) = N - D_l]].
\end{aligned}$$

To this end we define a partial recursive function $\bar{\zeta}$ as follows:

$$\bar{\zeta}_{f, n}^s(l, k, y) \simeq \begin{cases} 0, & \text{if } (\forall w \leq y)(\exists z \leq s)T_4(f, n, l, k, w, z) \text{ and} \\ y \in N - D_l, & \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

$$\bar{\zeta}_{f, n} =_{df} \bigcup_{s=0}^{\infty} \bar{\zeta}_{f, n}^s.$$

$\bar{\zeta}_{f, n}$ is partial recursive uniformly in f and n ; so let $\psi_6(x, y)$ be a recursive function such that

$$(\forall f)(\forall n)[\varphi_{\psi_6(f, n)}^3 \simeq \bar{\zeta}_{f, n}].$$

Now, it is easily seen from the definition of $\bar{\zeta}$ that for each pair $\langle l, k \rangle$ we have either

$$\delta\varphi_{\psi_6(f,n)}^3(l, k, y) = N - D_l \text{ or } \delta\varphi_{\psi_6(f,n)}^3(l, k, y) = a$$

finite set, according as $(\forall w)(\exists z)T_4(f, n, l, k, w, z)$ or $\neg(\forall w)(\exists z)T_4(f, n, l, k, w, z)$. It readily follows that for every pair $\langle f, n \rangle$ we have

$$\begin{aligned} & (\forall u)(\exists v)(\forall w)(\exists z)T_4(f, n, u, v, w, z) \\ & \Leftrightarrow (\forall j)(\exists l)(\exists k)[\delta\varphi_{\psi_6(f,n)}^3(l, k, y) = N - D_j]. \end{aligned}$$

Hence $n \in$ the f -th \prod_4^0 set $\Leftrightarrow \omega(\zeta_1(\psi_6(f, n))) \in G(\mathcal{F}_7)$, and $G(\mathcal{F}_7)$ is \prod_4^0 complete. Q.E.D.

PROPOSITION 12. *Let $\mathcal{F}_8 = \{W_e^c \mid (\exists n)[\{W_j \mid W_j \text{ is a relatively cofinite subset of } P_n\} \subseteq W_e^c]\}$. Then \mathcal{F}_8 is an r.e. family and $G(\mathcal{F}_8)$ is \sum_5^0 complete.*

PROOF. Let $\bar{\chi}(x, y)$ be a recursive function with $\bar{\chi}(n, y)$ one-to-one for each number n , such that $(\forall n)[\{P_n - D_{\bar{\chi}(n, y)} \mid y \in N\} = \{W_j \mid W_j \text{ is a relatively cofinite subset of } P_n\}]$. It follows that \mathcal{F}_8 is r.e., since

$$\mathcal{F}_8 = \{W_e^c \mid (\exists f)(\exists n)[W_e^c = W_f^c \cup \{P_n - D_{\bar{\chi}(n, y)} \mid y \in N\}]\}.$$

Next, since $n \in G(\mathcal{F}_8) \Leftrightarrow (\exists k)(\forall j)(\exists m)[m \in W_n \text{ and } P_k - D_{\bar{\chi}(k, j)} = W_m]$, and since the predicate $P_k - D_{\bar{\chi}(k, j)} = W_m$ is a \prod_2^0 predicate of k, j and m , we have that $G(\mathcal{F}_8)$ is \sum_5^0 . In the same way in which we showed $G(\mathcal{F}_6)$ to be \sum_4^0 complete by modifying our proof of the \prod_3^0 completeness of $G(\mathcal{F}_5)$, we shall now show $G(\mathcal{F}_8)$ to be \sum_5^0 complete by making appropriate alterations in our proof of the \prod_4^0 completeness of $G(\mathcal{F}_7)$. First, we define a partial recursive function γ by stipulating that

$$\begin{aligned} \gamma_{f,n}^s(l, k, j, y) & \simeq \begin{cases} 0, & \text{if } y \in P_l - D_{\bar{\chi}(l, k)} \text{ and} \\ & (\forall w \leq y)(\exists z \leq s)T_5(f, n, l, k, j, w, z), \\ \text{undefined,} & \text{otherwise;} \end{cases} \\ \gamma_{f,n} & =_{df} \bigcup_{s=0}^{\infty} \gamma_{f,n}^s. \end{aligned}$$

$\gamma_{f,n}$, as so defined, is partial recursive uniformly in f and n ; hence there is a recursive function $\beta(x, y)$ such that $(\forall f)(\forall n)[\gamma_{f,n} \simeq \varphi_{\beta(f,n)}^4]$. Let ζ_3 be a recursive function such that $(\forall f)(\forall n)[\varphi_{\zeta_3(f,n)}^3(\pi_2(l, k), j, y) \simeq \varphi_{\beta(f,n)}^4(l, k, j, y)]$. Then, for all 5-tuples $\langle f, n, l, k, j \rangle$, we have that $\delta\varphi_{\zeta_3(f,n)}^3(\pi_2(l, k), j, y) =$ either $P_l - D_{\bar{\chi}(l, k)}$ or a finite set, according as $(\forall w)(\exists z)T_5(f, n, l, k, j, w, z)$ or not. It follows that, for every pair $\langle f, n \rangle$ of numbers, we have

$$\begin{aligned} & (\exists l)(\forall k)(\exists j)(\forall w)(\exists z)T_5(f, n, l, k, j, w, z) \\ & \Leftrightarrow (\exists l)(\forall k)(\exists j)[\delta\varphi_{\zeta_3(f,n)}^3(\pi_2(l, k), j, y) = P_l - D_{\bar{\chi}(l, k)}]. \end{aligned}$$

So, if we set $\psi_7 = \omega_{df}^{\zeta_1} \zeta_3$ then

$$n \in \text{the } f\text{-th } \sum_5^0 \text{ set} \Leftrightarrow \psi_7(f, n) \in G(\mathcal{F}_8).$$

Thus $G(\mathcal{F}_8)$ is \sum_5^0 complete. Q.E.D.

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