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A CLASS OF STARLIKE MAPPINGS OF THE UNIT DISK

by

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DEFINITION. Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ be univalent in the open unit disk D. We say $f \in S_\alpha (0 < \alpha \leqq 1)$ if

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha, \ z \in D \tag{1}$$

Note that if $f \in S_\alpha$ then $Re(zf'(z)/f(z)) > 0$ on D; hence f is a starlike function. Singh [4] and Wright [5] have derived certain properties of the class S_α. In this paper we extend their results as follows. First, the boundary behavior of $f \in S_\alpha$ is discussed. We then give the radius of convexity for the class S_α; for $\alpha = 1$ the radius has been given by Wright [5]. Finally, we give an invariance property for the class S_α.

THEOREM 1. *Let* $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in S_\alpha$. *Then f maps D onto a domain whose boundary is a rectifiable Jordan curve. Furthermore,* $a_n = o(1/n)$, *and this order is best possible.*

PROOF. It follows immediately from (1) that f' is bounded in D; hence $\partial f[D]$ is a rectifiable closed curve [3], and $a_n = o(1/n)$. Univalency of f on $\bar{D}$ is easily verified by a contradiction argument. Finally, let $\{k(n)\}$ be any sequence of positive numbers which converges to 0 as $n \to \infty$. Then there exists a subsequence $\{k(n_j)\}$, $n_1 \geqq 2$, such that $\sum_{j=1}^{\infty} k(n_j) \leqq \alpha$. Define

$$a_n = \begin{cases} \dfrac{k(n_j)}{n_j} & n = n_j, j = 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\sum_{n=2}^{\infty} (n+\alpha-1)|a_n| \leqq \sum_{n=2}^{\infty} n|a_n| = \sum_{j=1}^{\infty} n_j \frac{k(n_j)}{n_j} \leqq \alpha.$$

By a theorem of Merkes, Scott, and Robertson [1], $f(z) \equiv z+\sum_{n=2}^{\infty} a_n z^n \in S_\alpha$. Since $na_n = k(n)$ for infinitely many n, the proof is complete.

THEOREM 2. *If $f \in S_\alpha$ then f maps $|z| < r(\alpha)$ onto a convex domain, where*

$$r(\alpha) = \begin{cases} \dfrac{3-\sqrt{5}}{2\alpha} & \text{if } \alpha_0 \leqq \alpha \leqq 1 \\ \left[\dfrac{2(1+\alpha^2)-3\alpha-2(1-\alpha)\sqrt{\alpha^2+4\alpha+1}}{\alpha(4\alpha-5)}\right]^{\frac{1}{2}} & \text{if } 0 < \alpha \leqq \alpha_0 \end{cases} \tag{2}$$

and

$$\alpha_0 = \frac{3-\sqrt{5}+2\sqrt{3(7-3\sqrt{5})}}{2\sqrt{5}} \approx .589 \tag{3}$$

PROOF. Since $f \in S_\alpha$ there exists ϕ, $|\phi| \leqq \alpha$ in D, such that $zf'(z)/f(z) = 1+z\phi(z)$. Differentiation yields

$$1+\frac{zf''(z)}{f'(z)} = 1+z\phi(z)+z\left(\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right) \tag{4}$$

It is known [2] that

$$\left|\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right| \leqq \frac{\left(|z|+\dfrac{|\phi(z)|}{\alpha}\right)(\alpha-|z\phi(z)|)}{(1-|z\phi(z)|)(1-|z|^2)} \tag{5}$$

From (4) and (5) it follows that $Re(1+zf''(z)/f'(z)) \geqq 0$ provided

$$1-|z\phi(z)|-|z|\,\frac{\left(|z|+\dfrac{|\phi(z)|}{\alpha}\right)(\alpha-|z\phi(z)|)}{(1-|z\phi(z)|)(1-|z|^2)} \geqq 0 \tag{6}$$

We write $|z| = a$, $|\phi(z)| = x$, $t = ax$ and define $G(t) \equiv t^2(1-a^2+1/\alpha)-3t(1-a^2)+1-a^2-a^2\alpha$. Since (6) holds if and only if $G(t) \geqq 0$, we must determine the largest value of a for which $G(t) \geqq 0$ on $[0, a\alpha]$. Then f will map $|z| < a$ onto a convex domain. Note that $G(t)$ has its minimum where $t = t^* \equiv \frac{3}{2}(1-a^2)(1-a^2+1/\alpha)^{-1}$.

CASE A ($a\alpha \leqq t^*$). $G(a\alpha) = (1-a^2)(a^2\alpha^2-3a\alpha+1) \geqq 0$ provided $a \leqq (3-\sqrt{5})/2\alpha$. Since $G(t)$ is decreasing on $[0, a\alpha]$, f maps $|z| < (3-\sqrt{5})/2\alpha$ onto a convex domain provided $a\alpha = (3-\sqrt{5})/2 \leqq t^*$. This restraint requires that $\alpha \in [\alpha_0, 1]$ where α_0 is given by (3). The function $f_\alpha(z) = ze^{\alpha z}$, $\alpha_0 \leqq \alpha \leqq 1$, shows that the number $r(\alpha) = (3-\sqrt{5})/2\alpha$ is sharp.

CASE B ($t^* \leqq a\alpha$). We assume $0 < \alpha < \alpha_0$. The minimum value of $G(t)$ on $[0, a\alpha]$ is $G(t^*)$; and $G(t^*) \geqq 0$ if

$$a^4(-5+4\alpha)+a^2\left(6-\frac{4}{\alpha}-4\alpha\right)-5+\frac{4}{\alpha} \geqq 0. \tag{7}$$

Since (7) holds for $0 \leqq a \leqq r(\alpha)$, where $r(\alpha)$ is given by (2), it follows that f maps $|z| < r(\alpha)$ onto a convex domain if $t^* \leqq \alpha r(\alpha)$. A tedious calculation shows this restraint to be satisfied for $0 < \alpha < \alpha_0$. We now construct a function to show that $r(\alpha)$ is best possible. Fix α in $(0, \alpha_0)$ and let $a = r(\alpha)$. Set $\beta \equiv [2-3\alpha-\alpha a^2(3-2\alpha)][2a(\alpha-1)^2]^{-1}$. Define f by $f(z) \equiv z \exp [\alpha \int_0^z (\beta-t)/(1-\beta t)\,dt]$. By a theorem of Wright [5] $f \in S_\alpha$ provided $-1 \leqq \beta \leqq 1$. For the present suppose this has been done. Now, f will not be convex in $|z| < r$, $r > a$, if $1+zf''(z)/f'(z) = 0$ at $z = a$, or, equivalently, if β is a root of $P(s)$, where

$$P(s) = s^2[a^2(\alpha-1)^2]+s[a(3\alpha-2)-\alpha a^3(3-2\alpha)]+1-4\alpha a^2+\alpha^2 a^4.$$

Since $a = r(\alpha)$ is a root of the left-hand side of (7), the definition of β implies that $P(\beta) = 0$. In fact $P(s)$ has a double root at $s = \beta$. It follows that $\beta^2 \leqq 1$ provided

$$a^2 \geqq \frac{(1+\alpha)^2-\sqrt{(1+\alpha)^4-4\alpha^2}}{2\alpha^2}. \tag{8}$$

Now, $a = r(\alpha)$ is the only root of the left-hand side of (7) which lies in [0, 1]. Thus, (8) holds since substitution of its right-hand side for a^2 in the left-hand side of (7) preserves the inequality in (7). Hence, $-1 \leqq \beta \leqq 1$, and the proof is complete.

THEOREM 3. *If* $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in S_\alpha$ *then for each* λ, $0 < \lambda < 1$, $h_\lambda(z) \equiv z+\sum_{n=2}^{\infty} \lambda a_n z^n \in S_\alpha$.

PROOF Since $h_\lambda(z) = \lambda f(z)+(1-\lambda)z$, we have

$$\frac{zh_\lambda'(z)}{h_\lambda(z)}-1 = \left[\frac{zf'(z)}{f(z)}-1\right]\left[1+\frac{1-\lambda}{\lambda}\frac{z}{f(z)}\right]^{-1}. \tag{9}$$

By a theorem of Wright [5], there exists ϕ, $|\phi| \leqq 1$ in D, such that $f(z) = z\exp[\alpha \int_0^z \phi(t)dt]$. Thus $Re(f(z)/z) > 0$ and so $|1+z(1-\lambda)(\lambda f(z))^{-1}| > 1$ for $z \in D$. It follows from (9) that $h_\lambda \in S_\alpha$.

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