STEVEN B. BANK

A general theorem concerning the growth of solutions of first-order algebraic differential equations

Compositio Mathematica, tome 25, nº 1 (1972), p. 61-70

<http://www.numdam.org/item?id=CM_1972__25_1_61_0>

COMPOSITIO MATHEMATICA, Vol. 25, Fasc. 1, 1972, pag. 61-70
Wolters-Noordhoff Publishing
Printed in the Netherlands

A GENERAL THEOREM CONCERNING THE GROWTH OF SOLUTIONS OF FIRST-ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS [1]

by

Steven B. Bank

1. Introduction

In this paper, we treat arbitrary first-order differential equations $\Omega(z, y, y') = 0$, where $\Omega(z, y, y') = \sum f_{kj}(z)y^k(y')^j$ is a polynomial in y and y' and where the coefficients $f_{kj}(z)$ are either (i) arbitrary entire functions, or (ii) arbitrary analytic functions in a finite disk $|z| < R_0$. Our main result (§ 3) provides a growth estimate for meromorphic solutions (on the plane in case (i) and on $|z| < R_0$ in case (ii)), of the equation $\Omega(z, y, y') = 0$. An important feature is that the estimate shows how the growth of a solution is affected separately by the growth of the coefficients of terms of maximum total degree in y and y', and by the growth of the other coefficients in the equation. The other quantities involved in the growth estimate for solutions are (a) a minimum modulus estimate, holding 'nearly everywhere' (see § 2), for one certain coefficient (the 'leading' coefficient) in the equation, and (b) the distribution of poles of the solution. The fact that the distribution of poles must be involved in the estimate for the growth of the solution, is indicated by the following phenomenon: In the special case where all coefficients $f_{kj}(z)$ are entire functions of finite order of growth, it follows from our main result here that for any entire solution, or more generally, for any meromorphic solution $y_0(z)$ whose sequence of poles has a finite exponent of convergence, the estimate $T(r, y_0) = O(\exp r^A)$ holds for some constant A as $r \to +\infty$. However, it is shown in [3], that no such uniform growth estimate exists for *arbitrary* meromorphic solutions of such equations, since for any preassigned function $\Phi(r)$ on $(0, +\infty)$, one can construct a meromorphic solution of such an equation, whose Nevanlinna characteristic dominates $\Phi(r)$ at a sequence of r tending to $+\infty$.

Concerning the other quantity (a) involved in our growth estimate for solutions, we note that the minimum modulus estimates in [8, p. 328]

[1] This research was supported in part by the National Science Foundation (GP 19590).

for entire functions of finite order, in [4, p. 97] for entire functions of infinite order, and in [9, p. 12] for analytic functions of finite order in the unit disk, all hold 'nearly everywhere' in the sense of § 2.

The proof of our main result here utilizes the Nevanlinna theory of meromorphic functions, and we emphasize that our main result will provide a growth estimate for solutions of *any* first-order algebraic differential equation provided that we possess growth estimates for the coefficients, the 'nearly everywhere' minimum modulus estimate for the leading coefficient, and an estimate on the distribution of poles of the solution. Thus, as very special cases, we obtain generalizations, to broader classes of equations, of the well-known theorems of Valiron ([10; p. 41] and [11; p. 294]) which assert that when the coefficients of Ω are polynomials, then all entire solutions and all analytic solutions in the unit disk are of finite order of growth. We also obtain alternate proofs of previous results (e.g. [1]) which were proved using the Valiron-Wiman theory for the plane. However, the only special case that we explicitly state as a corollary (§ 4), is an improvement of a previous result of the author [2; p. 369). This result treated the case where the coefficients of the terms of Ω having maximum total degree in y and y' are polynomials, while the other coefficients are arbitrary analytic functions of finite order in the unit disk. It was shown in [2], using the Valiron-Wiman theory for the unit disk [11; p. 299], that a meromorphic solution in the disk of such an equation, could not be written as the quotient of two analytic functions h/g, where h is of infinite order and g is of finite order in the disk, and where a technical condition concerning the growth of h away from the zeros of g is satisfied. (This technical condition was necessary because the Valiron-Wiman theory for the disk provides information only on a sequence of circles.) As a special case of our main result here, we show (§ 4) that this technical condition can be eliminated completely. (We remark here that equations of the type treated in § 4 and [2] can possess meromorphic solutions of infinite order in the disk (e.g. $(\sin \exp (1/(1-z)))^{-1}$)).

2. Notation

For $0 < R \leqq +\infty$, and a meromorphic function $g(z)$ in $|z| < R$, we will use the standard notation for the Nevanlinna functions $m(r, g)$, $N(r, g)$ and $T(r, g)$ (for $r < R$), introduced in [6, p.p. 6, 12]. We will also use the notation $n(r, g)$ for $r < R$, to denote the number of poles (counting multiplicity) of g in $|z| \leqq r$.

Following Hayman [5], we shall say that a certain property $P(r)$ holds 'n.e. in $[0, R)$' (*nearly everywhere* in $[0, R)$) if either,

(a) $R < +\infty$ and $P(r)$ holds for all r in $[0, R)$ except a set E such that $\int_E dr/(R-r) < +\infty$; or

(b) If $R = +\infty$ and $P(r)$ holds for all r in $[0, +\infty)$ except a set E such that $\int_E dr < +\infty$.

3.

We now state our main result:

THEOREM. *Let* $\Omega(z, y, y') = \sum f_{kj}(z) y^k (y')^j$ *be a polynomial in* y *and* y' *whose coefficients* $f_{kj}(z)$ *are analytic functions in* $|z| < R$, *where* $0 < R \leqq +\infty$. *Let* $p = \max \{k+j : f_{kj} \not\equiv 0\}$. *Let* $M_1(r)$ *and* $M_2(r)$ *be monotone nondecreasing functions on* $[0, R)$ *such that the following conditions hold n.e. in* $[0, R)$:

(A) $M_1(r) \geqq M_2(r) \geqq 1$.

(B) $|f_{kj}(z)| \leqq M_1(r)$ on $|z| = r$ if $k+j < p$.

(C) $|f_{kj}(z)| \leqq M_2(r)$ on $|z| = r$ if $k+j = p$.

Let $m = \max \{j : f_{p-j, j} \not\equiv 0\}$ *and let* $N(r)$ *be a monotone nonincreasing function on* $[0, R)$ *which satisfies* $N(r) > 0$ *on* $[0, R)$ *and for which the following condition holds n.e. in* $[0, R)$:

(D) $|f_{p-m, m}(z)| \geqq N(r)$ on $|z| = r$.

Let $y_0(z)$ *be a meromorphic function on* $|z| < R$ *such that*

$$\Omega(z, y_0(z), y_0'(z)) \equiv 0$$

and

$$T(r, y_0) \to +\infty \text{ as } r \to R.$$

Then:

(E) *If* $R = +\infty$, *then for any real number* $a > 1$, *there exist positive constants* K *and* r_0 *such that for all* $r > r_0$, *we have*

$$T(r, y_0) \leqq K[\log^+ M_1(ar) + (r^2 M_2(ar)/N(ar)) + \log r + r(n(ar, y_0)) + N(ar, y_0)]. \tag{1}$$

(F) *If* $R < +\infty$, *there exist positive constants* K *and* b, *with* $b < 1$, *such that if* $s(r) = R - b(R-r)$, *then for all* r *in* $[0, R)$, *we have*,

$$T(r, y_0) \leqq K(\log^+ M_1(s(r)) + (M_2(s(r))/N(s(r))) + \log((R-s(r))^{-1}) + n(s(r), y_0) + N(s(r), y_0)). \tag{2}$$

PROOF. We begin the proof with the following lemma:

LEMMA A. *Let* $I = \{(k, j) : k+j < p$ *and* $f_{kj} \not\equiv 0\}$, *and set* $q = 1 +$ (*cardinality of* I). *Then n.e. in* $[0, R)$, *the following holds*:

At any point z_0 *on* $|z| = r$ *for which*

(3) $$|y_0'(z_0)/y_0(z_0)| \geqq (m+1)M_2(r)/N(r),$$

we have

(4) $$|y_0(z_0)| \leqq q(M_1(r)/M_2(r))|y_0'(z_0)/(y_0(z_0)|^p.$$

PROOF. We assert that the lemma is valid for all r in $[0, R)$ for which the relations (A)−(D) hold (which of course is n.e. in $[0, R)$). To see this, let r be such a value, and let z_0 be a point on $|z| = r$ for which (3) holds. If $|y_0(z_0)| \leqq 1$, then in view of (3), (A) and the fact that $M_2(r)/N(r) \geqq 1$ (by (C) and (D)), we see that (4) holds, so we may assume that,

(5) $$|y_0(z_0)| > 1.$$

We may also assume $y_0(z_0) \neq \infty$ since otherwise both sides of (4) are $+\infty$. Now by dividing the relation, $\Omega(z_0, y_0(z_0), y_0'(z_0)) = 0$, through by $(y_0(z_0))^p$ (where p is as defined in the statement of the theorem), we can write this relation in the form,

(6) $$\Lambda(z_0) = -\sum_{(k,j)\in I} f_{kj}(z_0)(y_0'(z_0)/y_0(z_0))^j(y_0(z_0))^{k+j-p},$$

where,

(7) $$\Lambda(z_0) = \sum_{j=0}^{m} f_{p-j,j}(z_0)(y_0'(z_0)/y_0(z_0))^j.$$

Hence from (6),

(8) $$|\Lambda(z_0)| \leqq \sum_{(k,j)\in I} |f_{kj}(z_0)||y_0'(z_0)/y_0(z_0)|^j|y_0(z_0)|^{k+j-p}.$$

We now distinguish two cases.

CASE I. $m = 0$. Then $\Lambda(z_0) = f_{p-m,m}(z_0)$, so by (D),

(9) $$|\Lambda(z_0)| \geqq N(r) > 0.$$

In view of (8) and (9), I must be non-empty and it is clearly impossible that each term in the sum on the right side of (8) be $< (1/q)|\Lambda(z_0)|$. Hence for some (k, j) in I (depending on z_0, of course), we must have,

(10) $$|f_{kj}(z_0)||y_0'(z_0)/y_0(z_0)|^j|y_0(z_0)|^{k+j-p} \geqq (1/q)|\Lambda(z_0)|.$$

Since $k+j \leqq p-1$, it follows from (5) that $|y_0(z_0)|^{-1} \geqq |y_0(z_0)|^{k+j-p}$. Hence in view of (9) and (B), we obtain,

(11) $$q(M_1(r)/N(r))|y_0'(z_0)/y_0(z_0)|^j \geqq |y_0(z_0)|.$$

Using the estimate for $(1/N(r))$ given by (3), and noting that $j+1 \leqq p$ and that $|y_0'(z_0)/y_0(z_0)| \geqq 1$ (since $M_2(r) \geqq N(r)$ by (C) and (D)), we easily obtain (4) in this case.

CASE II. $m > 0$. We may write,

$$(12)\qquad \Lambda(z_0) = f_{p-m,m}(z_0)(y_0'(z_0)/y_0(z_0))^m(1+\sum_{j=0}^{m-1}\Psi_j(z_0)),$$

where

$$(13)\quad \Psi_j(z_0) = (f_{p-j,j}(z_0)/f_{p-m,m}(z_0))(y_0'(z_0)/y_0(z_0))^{j-m} \quad \text{for } j < m.$$

Since $j-m \leqq -1$, it easily follows from (C), (D) and (3), that $|\Psi_j(z_0)| \leqq (1/m+1))$ for $j < m$. Hence from (12),

$$|\Lambda(z_0)| \geqq (1/(m+1))|f_{p-m,m}(z_0)||y_0'(z_0)/y_0(z_0)|^m,$$

which in view of (D) and (3) (and the fact that $m \geqq 1$ in this case), yields

$$(14)\qquad |\Lambda(z_0)| \geqq M_2(r).$$

Since $M_2(r) \geqq 1$, we see from (8) and (14) that I must be non-empty and as in Case I, it is impossible that each term on the right side of (8) be $\langle(1/q)|\Lambda(z_0)|$. Hence for some (k,j) in I (depending on z_0), we must again have the relation (10). From this it follows using (B), (5) and (14) (and the fact that $j \leqq k+j < p$ and $|y_0'(z_0)/y_0(z_0)| \geqq 1$ by (3)) that (4) holds in this case too, thus proving the lemma.

Continuing with the proof of the main result, we now define for $r \in [0, R)$, the set

$$(15)\quad D_r = \{z : |z| = r \text{ and } |y_0'(z)/y_0(z)| < (m+1)M_2(r)/N(r)\}.$$

Let C_r denote the complement of D_r with respect to the circle $|z| = r$. Now by lemma A, n.e. in $[0, R)$ if z belongs to C_r then (4) holds for z. Since $M_2(r) \geqq 1$, we therefore obtain, n.e. in $[0, R)$ for $z \in C_r$,

$$(16)\qquad \log^+ |y_0(z)| \leqq \log^+ (qM_1(r)) + p\log^+ |y_0'(z)/y_0(z)|.$$

Writing $z = re^{i\theta}$, and letting $C_r^* = \{\theta : 0 \leqq \theta \leqq 2\pi \text{ and } re^{i\theta} \in C_r\}$, we clearly have from (16), that n.e. in $[0, R)$,

$$(17)\quad (1/2\pi)\int_{C_r^*} \log^+ |y_0(re^{i\theta})|d\theta \leqq \log^+ (qM_1(r)) + pm(r, y_0'/y_0).$$

Now by the Nevanlinna theory [7; p. 245, 246], there exists a constant $K_1 > 0$ such that n.e. in $[0, R)$,

$$(18)\qquad m(r, y_0'/y_0) \leqq K_1(\log T(r, y_0) + L(r)),$$

where $L(r) = \log r$ if $R = +\infty$, while $L(r) = \log((R-r)^{-1})$, if $R < +\infty$. Since $T(r, y_0) \to +\infty$ as $r \to R$ by assumption, clearly n.e. in $[0, R)$, we have,

$$(19)\qquad \log^+ q + pK_1 \log T(r, y_0) \leqq (1/2)T(r, y_0).$$

Now let $D_r^* = \{\theta : 0 \leqq \theta \leqq 2\pi \text{ and } re^{i\theta} \in D_r\}$. Adding

$$(1/2\pi)\int_{D_r^*} \log^+ |y_0(re^{i\theta})|d\theta + N(r, y_0)$$

to both sides of (17), and noting that the left side then becomes $T(r, y_0)$, and then using (18) and (19), we see that n.e. in $[0, R)$, we have,

$$T(r, y_0) \leqq 2\log^+ M_1(r) + 2pK_1 L(r) + (1/\pi)\int_{D_r^*} \log^+ |y_0(re^{i\theta})|d\theta + 2N(r, y_0). \tag{20}$$

In the following lemma, we estimate the integrand appearing on the right side of (20).

LEMMA B. *There exist positive constants* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *and* α_5 *such that n.e. in* $[0, R)$, *we have for* $z \in D_r$,

$$|y_0(z)| \leqq (qM_1(r)) \exp [(\alpha_1 + \alpha_2 r + \alpha_3 r^2)M_2(r)/N(r) + \alpha_4 rn(r, y_0) + \alpha_5 |\log r|]. \tag{21}$$

PROOF. Let ε be a number in $(0, R)$ such that $y_0(z)$ has no zeros or poles for $0 < |z| \leqq \varepsilon$. Set $\alpha_2 = 2\pi(m+1)$, $\alpha_3 = (m+1)/\varepsilon$ and $\alpha_4 = 2/\varepsilon$. By Jensen's formula [5; p. 166], there is a constant $\lambda > 0$ such that for $0 < r < R$,

$$T(r, 1/y_0) = T(r, y_0) + h(r), \text{ where } |h(r)| \leqq \lambda. \tag{22}$$

Set $\alpha_1 = \max\{p(m+1), \lambda\}$ and $\alpha_5 = n(0, y_0) + n(0, 1/y_0)$. We assert that for these choices of the constants, (21) is valid for all r in $[\varepsilon, R)$ for which the relations (A)-(D) (and hence also Lemma A) hold. To prove this, let r be such a value. We distinguish two cases.

CASE I. C_r is not empty.

Let z_0 be an element of D_r, say $z_0 = re^{i\theta_0}$. Let θ_1 be the infimum of the set of all θ in $[\theta_0, \theta_0 + 2\pi)$ for which $re^{i\theta}$ belongs to C_r, and let $z_1 = re^{i\theta_1}$. It easily follows from the definitions of D_r and C_r that

$$|y_0'(z_1)/y_0(z_1)| = (m+1)M_2(r)/N(r), \quad (\text{so } \theta_0 < \theta_1 < \theta_0 + 2\pi), \tag{23}$$

and that for $\theta_0 \leqq \theta < \theta_1$, $re^{i\theta}$ belongs to D_r. Hence if we let Γ denote the arc $z = re^{-i\varphi}$, $-\theta_1 \leqq \varphi \leqq -\theta_0$, then

$$|y_0'(z)/y_0(z)| \leqq (m+1)M_2(r)/N(r) \text{ on } \Gamma. \tag{24}$$

Thus clearly y_0 has no zeros or poles on the arc Γ, and it easily follows that y_0 is analytic and nowhere zero on some simply-connected neighborhood of Γ. Hence there exists an analytic branch g of $\log y_0$ on this neighborhood. Since $g' = y_0'/y_0$, we have,

$$(25)\qquad g(z_0)-g(z_1)=\int_\Gamma (y_0'(\zeta)/y_0(\zeta))d\zeta.$$

Taking the exponential of (25) and using (24), we obtain,

$$(26)\qquad |y_0(z_0)| \leqq |y_0(z_1)| \exp(2\pi r(m+1)M_2(r)/N(r)).$$

But by (23) and Lemma A (applied to z_1), we have, (noting that $M_2(r) \geqq 1$),

$$(27)\qquad |y_0(z_1)| \leqq qM_1(r)((m+1)M_2(r)/N(r))^p.$$

Using the inequality $t^p \leqq \exp(pt)$, it is clear from (26) and (27) that the conclusion (21) holds for $z = z_0$. Since z_0 was an arbitrary element of D_r, we have the desired conclusion in Case I.

CASE II. C_r is empty.

In this case, assume that (21) fails to hold for some point $z_0 = re^{i\theta_0}$ in D_r. Now let $z_1 = re^{i\theta_1}$ (where $\theta_0 < \theta_1 < \theta_0+2\pi$) be any point on $|z| = r$ distinct from z_0, and let Γ be the arc $\zeta = re^{-i\varphi}$, $-\theta_1 \leqq \varphi \leqq -\theta_0$. Since all points on $|z| = r$ belong to D_r in Case II, (24) holds on Γ. Hence y_0 is analytic and nowhere zero on Γ as in Case I, so there is a branch g of $\log y_0$ in a neighborhood of Γ. Thus (25) and hence (26) hold. From the assumption that (21) fails to hold for z_0, and the fact that $qM_1(r) \geqq 1$ and $\alpha_2 = 2\pi(m+1)$, it follows from (26) that,

$$(28)\qquad |y_0(z_1)| > \exp[(\alpha_1+\alpha_3 r^2)M_2(r)/N(r)+\alpha_4 rn(r, y_0)+\alpha_5|\log r|].$$

Of course, by assumption, (28) also holds for $z_1 = z_0$, so that (28) holds for all points z_1 on $|z| = r$. It follows that,

$$(29)\qquad m(r, y_0) > (\alpha_1+\alpha_3 r^2)M_2(r)/N(r)+\alpha_4 rn(r, y_0)+\alpha_5|\log r|,$$

and that

$$(30)\qquad m(r, 1/y_0) = 0.$$

Now since all points on $|z| = r$ are points of D_r, clearly y_0 has no zeros or poles on $|z| = r$. Hence by the argument principle,

$$(31)\qquad n(r, 1/y_0)-n(r, y_0) = (1/2\pi i)\int_{|z|=r} (y_0'(z)/(y_0(z))dz.$$

Since all points of $|z| = r$ are in D_r, we thus have,

$$(32)\qquad n(r, 1/y_0) \leqq n(r, y_0)+(m+1)rM_2(r)/N(r).$$

Now for $\varepsilon \leqq t \leqq r$, we have $n(t, y_0)-n(0, y_0) \leqq n(r, y_0)$ and $(1/t) \leqq (1/\varepsilon)$. Hence it follows from the definition of $N(r, y_0)$ that

(33) $$|N(r, y_0)| \leqq (r/\varepsilon)n(r, y_0)+n(0, y_0)|\log r|.$$

Similarly, we have,

(34) $$|N(r, 1/y_0)| \leqq (r/\varepsilon)n(r, 1/y_0)+n(0, 1/y_0)|\log r|.$$

Now by (22) and (30), we have,

(35) $$m(r, y_0) \leqq |N(r, y_0)|+N(r, 1/y_0)|+\lambda.$$

Using the estimates (33), (34) and (32) and the fact that $\lambda \leqq \lambda M_2(r)/N(r)$ (by (C) and (D) of the hypothesis of the main theorem), we easily obtain from (35) an inequality which is in direct contradiction to (29). This contradiction proves the desired conclusion in Case II, and hence the lemma is proved.

In view of (20), Lemma B and the definition of $L(r)$, we have shown the following:

(a) If $R = +\infty$, then there exists a positive constant K, such that n.e. in $[0, +\infty)$, the inequality (1) holds with $a = 1$, and clearly both sides of (1) are monotone nondecreasing on $(0, +\infty)$.

(b) If $R < +\infty$, then there exists a positive constant K, such that n.e. in $[0, R)$, the inequality (2) holds with $s(r) = r$, and clearly both sides of (2) are monotone nondecreasing on $[0, R)$.

Hence it is clear that the proof of the main result will be complete when we establish the following lemma:

LEMMA C. (a) *If $R = +\infty$ and $g(r)$ and $h(r)$ are monotone nondecreasing functions on $(O, +\infty)$ such that n.e. in $[O, +\infty)$, $g(r) \leqq h(r)$, then for any real number $a > 1$, there exists $r_0 > 0$ such that $g(r) \leqq h(ar)$ for all $r > r_0$.*

(b) *If $R < +\infty$ and $g(r)$ and $h(r)$ are monotone nondecreasing functions on $[0, R)$ such that n.e. in $[0, R)$, $g(r) \leqq h(r)$, then there is a positive constant b, with $b < 1$, such that if $s(r) = R-b(R-r)$ then $g(r) \leqq h(s(r))$ for all r in $[0, R)$.*

PROOF. (a). By assumption, there is a set E with $\int_E dr = \sigma < +\infty$ such that for $r \in [0, +\infty)$ and $r \notin E$, we have $g(r) \leqq h(r)$. For any $r > 0$, the linear measure of $[r, r+\sigma+1]$ is $\sigma+1$ and hence this interval cannot be contained in E. Thus there exists s in $[r, r+\sigma+1]$ such that $s \notin E$ and hence $g(s) \leqq h(s)$. Thus we have $g(r) \leqq h(r+\sigma+1)$ by the monotonicity of g and h. Hence if $a > 1$ and we take r_0 to be a real number which is greater than $(\sigma+1)/(a-1)$, we have $g(r) \leqq h(ar)$ for $r > r_0$, proving Part (a).

(b). By assumption, there is a set E with $\int_E dr/(R-r) = \sigma < +\infty$ such that for $r \in [0, R)$ and $r \notin E$, we have $g(r) \leqq h(r)$. Now if we set

$\varphi(r) = (R-r)(1-e^{-(\sigma+1)})$, then for any $r \in [0, R)$, it is easily verified that the interval $J_r = [r, r+\varphi(r)]$ lies in $[0, R)$, and the integral $\int_{J_r} dt/(R-t) = \sigma+1$. Hence J_r cannot be contained in E so there exists t in $[r, r+\varphi(r)]$ such that $g(t) \leqq h(t)$. By the monotonicity of g and h, we obtain $g(r) \leqq h(r+\varphi(r))$. If we set $b = e^{-(\sigma+1)}$, then $r+\varphi(r) = s(r)$ which proves Part (b). As remarked above, this establishes the main theorem.

4.

COROLLARY. *Let $\Omega(z, y, y')$ be a polynomial in y and y' whose coefficients of terms of maximum total degree in y and y' are polynomials while the other coefficients are analytic functions of finite order in the unit disk. Then a meromorphic solution $y_0(z)$ in the unit disk, of the equation $\Omega(z, y, y') = 0$, whose sequence of poles has a finite convergence exponent* [9; *p.* 7], *must be of finite order of growth in the disk. Hence a meromorphic solution in the disk cannot be written as the quotient of analytic functions h/g, where h is of infinite order and g is of finite order in the disk.*

PROOF. To prove the first assertion, we note that if y_0 is of bounded characteristic in $|z| < 1$ then y_0 is of finite order [6; p. 140]. If $T(r, y_0) \to +\infty$ as $r \to 1$, we can apply the previous theorem with $R = 1$, taking $M_1(r)$ to be a function of the form $\exp((1-r)^{-\Delta})$ (where $\Delta > 0$), taking $M_2(r)$ to be a suitable constant function, and taking $N(r)$ to be a function of the form $K_1(1-r)^d$ (see [2; p. 373]). If the poles of y_0 have a finite convergence exponent, it follows from [6; p. 139] that on $[0, 1)$, $N(r, y_0)$ and $n(r, y_0)$ are each $\leqq K_2(1-r)^{-\alpha}$ for some $K_2 > 0$ and $\alpha > 0$. Since (in the notation of (2)), $1-s(r) = b(1-r)$, it follows easily from (2) that for $r \in [0, 1)$, $T(r, y_0) \leqq K_3(1-r)^{-\lambda}$ where $K_3 > 0$ and $\lambda > 0$ are fixed constants independent of r. Hence y_0 is of finite order in the disk. To prove the second assertion, if $y_0 = h/g$ where g is of finite order in the disk, then by [6; p. 139], the sequence of poles of y_0 has a finite convergence exponent. Hence by the first assertion y_0 is of finite order and thus $h = gy_0$ is of finite order also. This concludes the proof.

BIBLIOGRAPHY

S. BANK

[1] On the growth of solutions of algebraic differential equations whose coefficients are arbitrary entire functions, Nagoya Math. J. 39 (1970), 107–117.

[2] A result concerning meromorphic solutions in the unit disk of algebraic differential equations, Compositio Math. 22 (1970), 367–381.

[3] A note on algebraic differential equations whose coefficients are entire functions of finite order. (To appear).

O. BLUMENTHAL

[4] Principes de la théorie des fonctions entières d'ordre infini, Gauthier-Villars, Paris, 1910.

W. HAYMAN

[5] Picard values of meromorphic functions and their derivatives, Ann. of Math. 70 (1959), 9–42.

R. NEVANLINNA

[6] Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929.

[7] Analytic functions, Springer-Verlag, New York, 1970 (Engl. Transl.)

S. SAKS AND A. ZYGMUND

[8] Analytic functions, Monografie Mat. (Engl. Transl.), Tom. 28, Warsaw, 1952.

M. TSUJI

[9] Canonical product for a meromorphic function in a unit circle, J. Math. Soc. Japan 8 (1956), 7–21.

G. VALIRON

[10] Sur les fonctions entières verifiant une classe d'équations différentielles, Bull. Soc. Math. France, 51 (1923), 33–45.

[11] Fonctions analytiques et équations différentielles, J. Math. Pures et appl. 31 (1952), 292–303.

(Oblatum 23–IV–1971)

Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801
U.S.A.