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SIMPLICIAL MAPS WHICH STABILIZE TO NEAR-HOMEOMORPHISMS

by

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1. Introduction

Let $\mathscr{U}$ be an open cover of a space Y. Maps $f, g : X \to Y$ are *$\mathscr{U}$-close* if for each x in X, $f(x)$ and $g(x)$ lie in some member of $\mathscr{U}$. A map $f: X \to Y$ is a *near-homeomorphism* if it can be uniformly approximated by homeomorphisms – i.e., for every open cover $\mathscr{U}$ of Y there exists a homeomorphism $h : X \to Y$ such that f and h are $\mathscr{U}$-close. If $f \times \mathrm{id} : X \times Q \to Y \times Q$ is a near-homeomorphism, where $Q = \prod_1^\infty [0, 1]_i$ is the Hilbert cube, then *f stabilizes to a near-homeomorphism.*

The recognition of (stable) near-homeomorphisms, and their application in inverse limit calculations (see below), play an important role in the recent proof by Schori and West [7] that 2^I is homeomorphic to Q. It seems likely that techniques involving near-homeomorphisms will be useful in further investigations of hyperspaces.

Our main theorem (3.2) characterizes the stable near-homeomorphisms in the simplicial category as the surjections with compact and contractible point-inverses. The proof is by means of Q-factor decompositions, discussed in § 2.

Brown showed in [3] that if (X_i, f_i) is an inverse sequence such that each X_i is a copy of a compact metric space X and each f_i is a near-homeomorphism, then $\mathrm{Lim}\,(X_i, f_i)$ is homeomorphic to X. In § 4 we note some immediate applications using (3.2), and extend Brown's theorem to complete metric spaces.

2. Q-factor decompositions

A space X is a *Q-factor* if $X \times Q \simeq Q$. Note that if $X \times Y \simeq Q$, then $X \times Q \simeq X \times (X \times Y)^\infty \simeq (X \times Y)^\infty \simeq Q$, and X is a Q-factor. Every Q-factor is a compact metric AR; it is not known whether the converse is true. West [8] has shown that every compact contractible polyhedron is a Q-factor.

A closed subset A of X is a *Z-set in X* if for every nonempty open homotopically trivial (n-connected for all $n \geqq 0$) subset U of X, $U \setminus A$ is nonempty and homotopically trivial. Z-sets were introduced by Anderson [2], who showed that every homeomorphism between Z-sets in Q extends to a homeomorphism of Q. The endslice $W = \{0\} \times \prod_2^\infty [0, 1]_i \subset Q$ is a Z-set; in general, boundaries and collared sets are Z-sets. One useful technique for verifying the Z-set property is the following:

2.1. LEMMA (cf. [8], Lemma 2.2). *A closed subset A of a metric ANR, X is a Z-set in X if for each $\varepsilon > 0$ there exists a map $f : X \to X \setminus A$ with $d(f, \mathrm{id}) < \varepsilon$.*

PROOF. Clearly A is nowhere dense. Let U be open and homotopically trivial, and $g : S^n \to U \setminus A$ a map of the n-sphere. There exists an extension $\bar{g} : C^{n+1} \to U$ of g to the $(n+1)$-cell. As a metric ANR, X is locally equiconnected, and therefore has the property that for every open cover $\mathscr{V}$ there exists an open cover $\mathscr{W}$ such that maps into X which are $\mathscr{W}$-close are $\mathscr{V}$-homotopic (paths of the homotopy lie in members of $\mathscr{V}$) [6]. By the compactness of C^{n+1} there exists $\varepsilon > 0$ such that for any map f: $X \to X \setminus A$ with $d(f, \mathrm{id}) < \varepsilon, f\bar{g}(C^{n+1}) \subset U \setminus A$ and g is homotopic to $f \circ g$ in $U \setminus A$. This homotopy together with the map $f \circ \bar{g}$ provides an extension $\tilde{g} : C^{n+1} \to U \setminus A$ of g.

2.2. DEFINITION. $\{X_\alpha\}$ is a *Q-factor decomposition* of a Hausdorff space X if:

i) $\{X_\alpha\}$ is a locally finite cover of X by Q-factors,
ii) $X_1, X_2 \in \{X_\alpha\}$ and $X_1 \cap X_2 \neq \phi$ imply $X_1 \cap X_2 \in \{X_\alpha\}$,
iii) $X_1, X_2 \in \{X_\alpha\}$ and $X_1 \subsetneqq X_2$ imply X_1 is a Z-set in X_2.

The spaces admitting Q-factor decompositions comprise a proper subclass of the class of locally compact metrizable ANR's, and include the locally compact polyhedra.

2.3. DEFINITION. Q-factor decompositions $\{X_\alpha\}$ and $\{Y_\alpha\}$ indexed by the same set are *similar* if $X_1 \cap X_2 \neq \phi$ is equivalent to $Y_1 \cap Y_2 \neq \phi$. $\{X_\alpha\}$ and $\{Y_\alpha\}$ are *isomorphic* if $X_1 \subset X_2$ is equivalent to $Y_1 \subset Y_2$.

Isomorphic decompositions are similar: if $X_1 \cap X_2 \neq \phi$, then $X_1 \cap X_2 = X_3 \in \{X_\alpha\}$, $X_3 \subset X_1$ and $X_3 \subset X_2$, therefore $Y_3 \subset Y_1$ and $Y_3 \subset Y_2$, and $Y_1 \cap Y_2 \supset Y_3 \neq \phi$.

For any space X, $\tau^n : X \times Q \to I^n$ will denote the projection onto the first n factors of Q.

2.4. THEOREM. *Let $\{X_\alpha\}$ and $\{Y_\alpha\}$ be isomorphic Q-factor decompositions of X and Y, respectively, and let a function $p : A \to Z^+$ from the indexing set into the positive integers be given. Then there exists a homeomorphism $H : X \times Q \to Y \times Q$ such that $H(X_\alpha \times Q) = Y_\alpha \times Q$ and $\tau^{p(\alpha)}/X_\alpha \times Q = \tau^{p(\alpha)} H/X_\alpha \times Q$ for each α.*

PROOF. Since $X_\alpha = X_\beta$ is equivalent to $Y_\alpha = Y_\beta$, and since $\{X_\alpha | X_\alpha = X_\beta\}$ is a finite collection for each X_β, there is no loss of generality in assuming that the isomorphic decompositions $\{X_\alpha\}$ and $\{Y_\alpha\}$ are faithfully indexed – i.e., $X_\alpha = X_\beta$ only if $\alpha = \beta$. For any subcollection $\{X_\alpha | \alpha \in B \subset A\}$ of $\{X_\alpha\}$, let $\mathrm{Min}\,\{X_\alpha | \alpha \in B\} = \{X_\alpha | \alpha \in B; \beta \in B \text{ with } X_\beta \subset X_\alpha \text{ implies } \alpha = \beta\}$, the collection of minimal elements. Inductively define $X^{(i)} = X^{(i-1)} \cup \mathrm{Min}\,\{X_\alpha | X_\alpha \notin X^{(i-1)}\}$, $i \geqq 0$, with $X^{(-1)} = \phi$. Then $\{X_\alpha\} = \bigcup X^{(i)}$; similarly $\{Y_\alpha\} = \bigcup Y^{(i)}$. It is easily seen that $X_\alpha \in X^{(i)}$ is equivalent to $Y_\alpha \in Y^{(i)}$. Since the indicator function $p : A \to Z^+$ can be redefined by setting $p'(\alpha) = \max\,\{p(\beta) | X_\alpha \subset X_\beta\}$, we may assume that $X_\alpha \subset X_\beta$ implies $p(\beta) \leqq p(\alpha)$.

For each α, let $\mathscr{H}_\alpha$ denote the non-empty collection of homeomorphisms of $X_\alpha \times Q$ onto $Y_\alpha \times Q$ of the form $h_\alpha = \tilde{h}_\alpha \times \mathrm{id}_\alpha$, where $\tilde{h}_\alpha : X_\alpha \times \prod \{I_i | i > p(\alpha)\} \to Y_\alpha \times \prod \{I_i | i > p(\alpha)\}$ and id_α is the identity map on $I^{p(\alpha)} = \prod_1^{p(\alpha)} I_i$. Suppose inductively that there exists a homeomorphism $H_i : \bigcup \{X_\alpha | X_\alpha \in X^{(i)}\} \times Q \to \bigcup \{Y_\alpha | Y_\alpha \in Y^{(i)}\} \times Q$ such that $H_i / X_\alpha \times Q$ is in $\mathscr{H}_\alpha$ for each $X_\alpha \in X^{(i)}$. Consider $X_\beta \in X^{(i+1)} \setminus X^{(i)} = \mathrm{Min}\,\{X_\alpha | X_\alpha \notin X^{(i)}\}$, and set $\dot{X}_\beta = \bigcup \{X_\alpha | X_\alpha \subsetneqq X_\beta\}$. Then $\dot{X}_\beta$, as a finite union of Z-sets, is a Z-set in X_β (it may be empty), and $\dot{X}_\beta = X_\beta \cap (\bigcup \{X_\alpha | X_\alpha \in X^{(i)}\})$. Similarly for $\dot{Y}_\beta$; note that $H_i(\dot{X}_\beta \times Q) = \dot{Y}_\beta \times Q$. Since $p(\beta) \leqq p(\alpha)$ for each $X_\alpha \subset X_\beta$, an application of Anderson's homeomorphism extension theorem to $X_\beta \times \prod \{I_i | i > p(\beta)\}$ and $Y_\beta \times \prod \{I_i | i > p(\beta)\}$ shows there exists $h_\beta \in \mathscr{H}_\beta$ such that $h_\beta / \dot{X}_\beta \times Q = H_i / \dot{X}_\beta \times Q$. For distinct elements X_α and X_β of $X^{(i+1)} \setminus X^{(i)}$, either $X_\alpha \cap X_\beta = \phi$ or $X_\alpha \cap X_\beta \in X^{(i)}$. Since $\{X_\alpha\}$ is a locally finite closed cover of X, we may define $H_{i+1} : \bigcup \{X_\alpha | X_\alpha \in X^{(i+1)}\} \times Q \to \bigcup \{Y_\alpha | Y_\alpha \in Y^{(i+1)}\} \times Q$ by requiring that H_{i+1} extend H_i and $H_{i+1} / X_\beta \times Q = h_\beta$ for each $X_\beta \in X^{(i+1)} \setminus X^{(i)}$. Then $H : X \times Q \to Y \times Q$ defined by $H / X_\alpha \times Q = H_i / X_\alpha \times Q$ for $X_\alpha \in X^{(i)}$, $i \geqq 0$, is the desired homeomorphism.

In [5] we obtain an extension of (2.4) to similar Q-factor decompositions, in which the requirement $H(X_\alpha \times Q) = Y_\alpha \times Q$ is replaced by $H(X_\alpha \times Q) \subset \mathrm{St}\,(Y_\alpha) \times Q$. This result promises to be useful in recognizing stable near-homeomorphisms in situations where Theorem 3.2 (see below) does not apply.

3. Stable near-homeomorphisms

In this section we shall be dealing with simplicial maps between locally finite complexes. A map $f : K \to L$ is *compact* or *contractible* if $f^{-1}(x)$ is compact or contractible for each x in L.

3.1. LEMMA. *Let $f : K \to L$ be a compact contractible simplicial surjection, and let $\mathscr{U}$ be an open cover of L. Then there exist isomorphic Q-*

factor decompositions $\{K_\alpha\}$ *of* K *and* $\{L_\alpha\}$ *of* L *such that* $\{L_\alpha\}$ *refines* $\mathscr{U}$ *and* $K_\alpha = f^{-1}(L_\alpha)$ *for each* α.

PROOF. It is well-known that there exist subdivisions K_* of K and L_* of L such that $f: K_* \to L_*$ is simplicial and the cover by vertex stars of L_* refines $\mathscr{U}$. For notational convenience assume that $K = K_*$ and $L = L_*$. We show that the dual structures on K and L described by Cohen [4] are the desired Q-factor decompositions.

Let L' be the standard barycentric subdivision of L, and let K' be a barycentric subdivision of K chosen so that $f: K' \to L'$ is simplicial. The barycenter of a simplex σ is denoted by $\hat{\sigma}$. If $\sigma_0 \subset \cdots \subset \sigma_q$, then $\hat{\sigma}_0 \cdots \hat{\sigma}_q$ is the simplex spanned by the barycenters. If α is a simplex of L, then $D(\alpha, L)$, the dual to α in L, and its subcomplex $\dot{D}(\alpha, L)$ are defined by $D(\alpha, L) = \{\hat{\sigma}_0 \cdots \hat{\sigma}_q | \alpha \subset \sigma_0 \subset \cdots \subset \sigma_q\}$, $\dot{D}(\alpha, L) = \{\hat{\sigma}_0 \cdots \hat{\sigma}_q | \alpha \subsetneqq \sigma_0 \subset \cdots \subset \sigma_q\}$. $D(\alpha, f)$, the dual to α with respect to f, is a subcomplex of K' defined by $D(\alpha, f) = \{\hat{\tau}_0 \cdots \hat{\tau}_q | \alpha \subset f(\tau_0), \tau_0 \subset \cdots \subset \tau_q\}$; similarly for $\dot{D}(\alpha, f)$. Each dual $D(\alpha, L)$ is a finite subcomplex of L', and since f is a compact surjection each $D(\alpha, f)$ is also finite and nonempty. Clearly $D(\alpha, L)$ is the join $\hat{\alpha}\dot{D}(\alpha, L)$. It is known [4] that $D(\alpha, f) = f^{-1}D(\alpha, L)$, $\dot{D}(\alpha, f) = f^{-1}\dot{D}(\alpha, L)$, and $D(\alpha, f)$ collapses to $f^{-1}(\hat{\alpha})$.

Set $\{K_\alpha\} = \{D(\alpha, f)\}$ and $\{L_\alpha\} = \{D(\alpha, L)\}$, where α runs through all the simplexes of L. Then $\{K_\alpha\}$ and $\{L_\alpha\}$ are isomorphic locally finite covers of K and L, and $\{L_\alpha\}$ refines $\mathscr{U}$. Each dual $D(\alpha, L)$ is contractible, and since f is contractible each dual $D(\alpha, f)$ is contractible. It follows from West's theorem (see § 2) that each dual is a Q-factor. If $D(\alpha, L) \cap D(\beta, L) \neq \phi$, then α and β span a simplex γ and $D(\alpha, L) \cap D(\beta, L) = D(\gamma, L)$. In this case $D(\alpha, f) \cap D(\beta, f) = f^{-1}D(\alpha, L) \cap f^{-1}D(\beta, L) = f^{-1}(D(\alpha, L) \cap D(\beta, L)) = f^{-1}D(\gamma, L) = D(\gamma, f)$. If $D(\beta, L) \subsetneqq D(\alpha, L)$, then $\alpha \subsetneqq \beta$, $D(\beta, L) \subset \dot{D}(\alpha, L)$, and $D(\beta, f) \subset \dot{D}(\alpha, f)$. Since $D(\alpha, L) = \hat{\alpha}\dot{D}(\alpha, L)$ there exists for each $\varepsilon > 0$ a map $r: D(\alpha, L) \to D(\alpha, L) \setminus \dot{D}(\alpha, L)$ with $d(r, \mathrm{id}) < \varepsilon$, and such that x and $r(x)$ have the same carrier in L. Since $f: K \to L$ is simplicial, the map r can be lifted to a map $\bar{r}: D(\alpha, f) \to D(\alpha, f) \setminus \dot{D}(\alpha, f)$ such that $f \circ \bar{r} = r \circ f$ on $D(\alpha, f)$ and $d(\bar{r}, \mathrm{id}) < \varepsilon$. It follows from (2.1) that $\dot{D}(\alpha, L)$ and $\dot{D}(\alpha, f)$, and therefore $D(\beta, L)$ and $D(\beta, f)$, are Z-sets in $D(\alpha, L)$ and $D(\alpha, f)$, respectively. This completes the proof that $\{K_\alpha\}$ and $\{L_\alpha\}$ are Q-factor decompositions.

3.2. THEOREM. *A simplicial map* $f: K \to L$ *stabilizes to a near-homeomorphism if and only if* f *is a compact contractible surjection.*

PROOF. Suppose f is a compact contractible surjection. Let $\mathscr{W}$ be an open cover of $L \times Q$. There exists an open cover $\mathscr{U}$ of L and a function $m: \mathscr{U} \to Z^+$ such that for (x_1, q_1) and (x_2, q_2) in $L \times Q$ with $\{x_1, x_2\} \subset U \in \mathscr{U}$ and $\tau^{m(U)}(q_1) = \tau^{m(U)}(q_2)$, $\{(x_1, q_1), (x_2, q_2)\} \subset W \in \mathscr{W}$. By (3.1) there exist isomorphic Q-factor decompositions $\{K_\alpha\}$ of K and $\{L_\alpha\}$ of L such that $\{L_\alpha\}$ refines $\mathscr{U}$ and $K_\alpha = f^{-1}(L_\alpha)$. Define $p : A \to Z^+$ by $p(\alpha) = \min\{m(U)|L_\alpha \subset U \in \mathscr{U}\}$. By (2.4) there exists a homeomorphism $H : K \times Q \to L \times Q$ such that $H(K_\alpha \times Q) = L_\alpha \times Q$ and $\tau^{p(\alpha)}/K_\alpha \times Q = \tau^{p(\alpha)}H/K_\alpha \times Q$ for each α. Clearly H and $f \times \mathrm{id}$ are $\mathscr{W}$-close.

Conversely, suppose that $f \times \mathrm{id}$ is a near-homeomorphism. Since the image of $f \times \mathrm{id}$ must be dense in $L \times Q$, f is surjective. Consider a point x in L and the inverse $f^{-1}(x) \subset K$. Since there exists a homeomorphism of $K \times Q$ onto $L \times Q$ taking $f^{-1}(x) \times Q$ into a compact neighborhood of $\{x\} \times Q$, $f^{-1}(x)$ is compact. (The same argument shows that the inverse image of every compact set is compact.) Since f is simplicial $f^{-1}(x)$ is polyhedral and therefore a retract of some neighborhood U in K. Using compactness of the inverse image of a compact neighborhood of x, we obtain a neighborhood V of x such that $f^{-1}(V) \subset U$. Then there exists a contractible neighborhood W of x and a homeomorphism $H : K \times Q \to L \times Q$ such that $H(f^{-1}(x) \times Q) \subset W \times Q \subset H(U \times Q)$. Thus $f^{-1}(x) \times Q$ is contractible in the neighborhood $U \times Q$ which retracts onto it, and therefore $f^{-1}(x)$ is contractible.

A non-piecewise linear map $f : K \to L$ which stabilizes to a near-homeomorphism may not be contractible (although it follows from the proof above that point-inverses must have the shape of a point). For example, it is easily seen that there exists a map $f : I^2 \to I$ such that $f^{-1}(t)$ is an arc if $t \neq \frac{1}{2}$, $f^{-1}(\frac{1}{2})$ is a topologist's sine curve containing $I \times \{0, 1\}$, and f is the uniform limit of piecewise-linear maps satisfying the conditions of (3.2). Hence f itself stabilizes to a near-homeomorphism.

4. Inverse limit applications

Brown's theorem (see § 1) and Theorem (3.2) imply that if (K_i, f_i) is an inverse sequence of finite complexes with simplicial contractible surjections as bonding maps, then $\mathrm{Lim}\,(K_i, f_i) \times Q$ is homeomorphic to $K_i \times Q$. Since a dendron is an inverse limit of finite trees with elementary collapses as bonding maps, this technique provides a quick proof of the fact, announced in [1] and demonstrated in [8], that every dendron is a Q-factor. Let $J^\infty = \prod_1^\infty [-1, 1]_i$, and let J^∞/R be the quotient space obtained by identifying (x_i) with $(-x_i)$. Schori and Barit have recently used the same technique to show that J^∞/R is a Q-factor.

The following extension of Brown's theorem to complete metric spaces permits the application of (3.2) in the non-compact case.

4.1. THEOREM. *If (X_i, f_i) is an inverse sequence of copies of a complete metric space X with near-homeomorphisms as bonding maps, then* $\mathrm{Lim}\,(X_i, f_i)$ *is homeomorphic to X.*

PROOF. We inductively choose homeomorphisms $h_i : X_{i+1} \to X_i$, $i \geqq 1$, such that Lim (X_i, f_i) is homeomorphic to Lim (X_i, h_i). For $i < j$ let $f_{ij} = f_i \circ \cdots \circ f_{j-1}$ and $h_{ij} = h_i \circ \cdots \circ h_{j-1}$ be compositions of the bonding maps, and let $f_{i\infty} : \text{Lim}\,(X_i, f_i) \to X_i$ and $h_{i\infty} : \text{Lim}\,(X_i, h_i) \to X_i$ be the projections. Suppose that $h_1, \cdots, h_{j-1}$ have been chosen. Then there exists an open cover $\mathscr{U}_j$ of X_j such that mesh $f_{ij}(\mathscr{U}_j) < 2^{-j}$ and mesh $h_{ij}(\mathscr{U}_j) < 2^{-j}$ for $1 \leqq i < j$. Choose a homeomorphism $h_j : X_{j+1} \to X_j$ such that f_j and h_j are $\mathscr{U}_j$-close.

A straight-forward verification shows there exists a map $F : \text{Lim}\,(X_i, f_i) \to \text{Lim}\,(X_i, h_i)$ such that $h_{i\infty} F(x) = \lim_{n\to\infty} h_{in} f_{n\infty}(x)$ for each i. Likewise there exists a map $H : \text{Lim}\,(X_i, h_i) \to \text{Lim}\,(X_i, f_i)$ such that $f_{i\infty} H(x) = \lim_{n\to\infty} f_{in} h_{n\infty}(x)$. We show that $H \circ F$ and $F \circ H$ are the identity maps. Let $1 \leqq i < n$ and $x \in \text{Lim}\,(X_i, f_i)$ be given. Then $d(f_{i\infty} HF(x), f_{in} h_{n\infty} F(x)) < 2^{-n+1}$, and for each $m > n$, $d(f_{i\infty}(x), f_{in} h_{nm} f_{m\infty}(x)) < 2^{-n+1}$. Since $h_{n\infty} F(x) = \lim_{m\to\infty} h_{nm} f_{m\infty}(x)$, there exists $m > n$ such that $d(f_{in} h_{n\infty} F(x), f_{in} h_{nm} f_{m\infty}(x)) < 2^{-n}$. Thus $d(f_{i\infty} HF(x), f_{i\infty}(x)) < 3 \cdot 2^{-n+1}$, and since n was arbitrary $H \circ F = \text{id}$. Similarly $F \circ H = \text{id}$.

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