COMPOSITIO MATHEMATICA

ALLEN B. ALTMAN
RAYMOND T. HOOBLER
STEVEN L. KLEIMAN

A note on the base change map for cohomology

Compositio Mathematica, tome 27, nº 1 (1973), p. 25-38
<http://www.numdam.org/item?id=CM_1973__27_1_25_0>

COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 1, 1973, pag. 25-38
Noordhoff International Publishing
Printed in the Netherlands

A NOTE ON THE BASE CHANGE MAP FOR COHOMOLOGY

Allen B. Altman[1], Raymond T. Hoobler and Steven L. Kleiman[2]

1. Introduction

Consider a commutative square of ringed spaces,

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{t} & S, \end{array}$$

and an $\mathcal{O}_X$-Module F. For each $n \geqq 0$ there is a canonical $\mathcal{O}_T$-homomorphism $\alpha^n(F) : t^*R^nf_*F \to R^nf'_*(g^*F)$; it is called the base change map if the square is cartesian. We prove that when the square is a cartesian square of schemes, f is a quasi-separated and quasi-compact morphism, t is a flat morphism and F is a quasi-coherent $\mathcal{O}_X$-Module, then $\alpha^n(F)$ is an isomorphism; simultaneously we deduce that the $\mathcal{O}_S$-Module R^nf_*F is quasi-coherent. The principal idea is to work carefully with the usual spectral sequence of Čech cohomology.

Both the quasi-coherence statement and the flat base change statement are made without proof in (EGA IV, 1.7.21). Both statements are proved in ([5] VI §2) using the method of hypercoverings developed in ([SGA 4] V ap.). Our proof is at the level of EGA III_1.

We include an example showing that the quasi-coherence statement is false without the assumption that f is quasi-separated and quasi-compact. It was inspired by the example in EGA (I, 6.7.3), which is, however, incorrect because the statement there that $M = M_0$ holds is false.

We also include the rudiments of the base change map because there is no adequate discussion in the literature. We use Godement's approach [2] to cohomology via the canonical flasque resolution $\mathscr{C}^\bullet(F)$ of a sheaf F. The heart of our discussion is a natural map $c^\bullet_g(G) : \mathscr{C}^\bullet(g_*G) \to g_*\mathscr{C}^\bullet(G)$ for each sheaf G on Y, which is essentially in [6]. Curiously, the bulk of the theory does not involve the bases S and T.

[1] The first author was supported in part by the United States Educational Foundation in Norway.

[2] The third author was supported in part by the Alfred P. Sloan Foundation and the National Science Foundation.

The authors wish to express their gratitude to the Mathematics Institute of Aarhus University for the generous hospitality extended to them.

2. The map $c_g^{\bullet}(G)$: $\mathscr{C}^{\bullet}(g_*G) \to g_*\mathscr{C}^{\bullet}(G)$ of canonical flasque resolutions

Let X be a ringed space and F an $\mathcal{O}_X$-Module. Let $\mathscr{C}^0(F)$ denote the sheaf of discontinuous sections of F; that is, for each open set U of X, we have $\mathscr{C}^0(F)(U) = \prod_{x \varepsilon U} F_x$. Obviously $\mathscr{C}^0(F)$ is a flasque sheaf and the natural map $\varepsilon(F) : F \to \mathscr{C}^0(F)$ is injective. Let $\mathscr{Z}^1(F)$ denote the cokernel of $\varepsilon(F)$ and define inductively $\mathscr{C}^n(F) = \mathscr{C}^0(\mathscr{Z}^n(F))$ and $\mathscr{Z}^{n+1}(F) = \mathscr{Z}^1(\mathscr{Z}^n(F))$. Clearly the $\mathscr{C}^n(F)$ form a resolution of F, which behaves functorially in F. It is called the *canonical flasque resolution* of F and denoted $\mathscr{C}^{\bullet}(F)$.

Let $g : Y \to X$ be a morphism of ringed spaces and G an $\mathcal{O}_Y$-Module. Let x be a point of X and y a point of $g^{-1}(x)$. For each open neighborhood V of x, there is a natural map from $G(g^{-1}(V))$ to G_y taking a section to its germ in G_y; shrinking V, we obtain a map $(g_*G)_x \to G_y$. Varying y, we obtain a map

$$(g_* G)_x \to \prod_{y \in g^{-1}(x)} G_y .$$

Finally varying x in an open set U of X, we obtain a map from $\prod_{x \in U}(g_* G)_x$ to

$$\prod_{x \in U} \Big(\prod_{y \in g^{-1}(x)} G_y \Big) = \prod_{y \in g^{-1}(U)} G_y ;$$

in other words, we have defined a map of sheaves

$$c_g^0(G) : \mathscr{C}^0(g_* G) \to g_*\mathscr{C}^0(G). \text{ Clearly } c_g^0(-)$$

is a natural transformation of functors.

Having defined $c_g^0(G)$, we shall extend it to a map of complexes in a purely formal way. Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & g_* G & \xrightarrow{\varepsilon(g_*G)} & \mathscr{C}^0(g_* G) & \longrightarrow & \mathscr{Z}^1(g_* G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle c_g^0(G)} & & \vdots\downarrow{\scriptstyle z_g^1(G)} & & \\ 0 & \longrightarrow & g_* G & \xrightarrow{g_*(\varepsilon(G))} & g_*\mathscr{C}^0(G) & \longrightarrow & g_*\mathscr{Z}^1(G). & & \end{array} \tag{2.1}$$

The left hand square is obviously commutative. Hence there is an induced map $z_g^1(G) : \mathscr{Z}^1(g_* G) \to g_*\mathscr{Z}^1(G)$. Clearly $z_g^1(-)$ is a natural transformation.

Define inductively $c_g^n(G)$ as the composition, $c_g^0(\mathscr{Z}^n(G)) \circ \mathscr{C}^0(z_g^n(G))$, and $z_g^{n+1}(G)$ as $z_g^1(\mathscr{Z}^n(G))\circ \mathscr{Z}^1(z_g^n(G))$. Then, for each n, we have a commutative diagram with exact rows,

$$
(2.2)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{Z}^n(g_* G) & \longrightarrow & \mathscr{C}^0(\mathscr{Z}^n(g_* G)) & \longrightarrow & \mathscr{Z}^1(\mathscr{Z}^n(g_* G)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z_g^n(G)} & & \downarrow{\scriptstyle \mathscr{C}^0(z_g^n(G))} & & \downarrow{\scriptstyle \mathscr{Z}^1(z_g^n(G))} & & \\
0 & \longrightarrow & g_*\mathscr{Z}^n(G) & \longrightarrow & \mathscr{C}^0(g_*\mathscr{Z}^n(G)) & \longrightarrow & \mathscr{Z}^1(g_*\mathscr{Z}^n(G)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle c_g^0(\mathscr{Z}^n(G))} & & \downarrow{\scriptstyle z_g^1(\mathscr{Z}^n(G))} & & \\
0 & \longrightarrow & g_*\mathscr{Z}^n(G) & \longrightarrow & g_*\mathscr{C}^0(\mathscr{Z}^n(G)) & \longrightarrow & g_*\mathscr{Z}^1(\mathscr{Z}^n(G)), & &
\end{array}
$$

and the compositions in the middle column and right hand column are $c_g^n(G)$ and $z_g^{n+1}(G)$. Taken together, these diagrams show that the maps $c_g^n(G) : \mathscr{C}^n(g_* G) \to g_*\mathscr{C}^n(G)$ form a morphism of complexes.

Let $h : Z \to Y$ be a second morphism of ringed spaces and H an $\mathscr{O}_Z$-Module. We shall now verify that the diagram of complexes of sheaves,

$$
(2.3)\qquad
\begin{array}{ccc}
 & \mathscr{C}^\bullet(g_* h_* H) & \\
{\scriptstyle c_g^\bullet(h_* H)}\swarrow & & \searrow{\scriptstyle c_{(g\circ h)}^\bullet(H)} \\
g_*\mathscr{C}^\bullet(h_* H) & \xrightarrow[g_* c_h^\bullet(H)]{} & g_* h_* \mathscr{C}^\bullet(H)
\end{array}
$$

is commutative. For each $x \in X$, each $y \in g^{-1}(x)$ and each $z \in h^{-1}(y)$ the triangle,

$$
\begin{array}{ccc}
(g_* h_* H)_x & & \\
\downarrow & \searrow & \\
(h_* H)_y & \longrightarrow & H_z,
\end{array}
$$

is easily seen to be commutative. Taking products we obtain the formula,

$$
(2.4)\qquad c_{(g\circ h)}^n(H) = g_* c_h^n(H) \circ c_g^n(h_* H),
$$

in the case $n = 0$.

We establish formula (2.4) and the following formula,

$$
(2.5)\qquad z_{(g\circ h)}^n(H) = g_* z_h^n(H) \circ z_g^n(h_* H),
$$

together by induction on n. For $n \geqq 1$ we have a commutative diagram,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{Z}^n(g_* h_* H) & \longrightarrow & \mathscr{C}^0(\mathscr{Z}^n(g_* h_* H)) & \longrightarrow & \mathscr{Z}^1(\mathscr{Z}^n(g_* h_* H)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z_g^n(h_* H)} & & \downarrow{\scriptstyle c_g^n(h_* H)} & & \downarrow{\scriptstyle z_g^{n+1}(h_* H)} & & \\
0 & \longrightarrow & g_*\mathscr{Z}^n(h_* H) & \longrightarrow & g_*\mathscr{C}^0(\mathscr{Z}^n(h_* H)) & \longrightarrow & g_*\mathscr{Z}^1(\mathscr{Z}^n(h_* H)) & & \\
 & & \downarrow{\scriptstyle g_* z_h^n(H)} & & \downarrow{\scriptstyle g_* c_h^n(H)} & & \downarrow{\scriptstyle g_* z_h^{n+1}(H)} & & \\
0 & \longrightarrow & g_* h_*\mathscr{Z}^n(H) & \longrightarrow & g_* h_*\mathscr{C}^0(\mathscr{Z}^n(H)) & \longrightarrow & g_* h_*\mathscr{Z}^1(\mathscr{Z}^n(H)), & &
\end{array}
$$

with exact rows. If we set $\mathscr{Z}^0(G) = G$ and $z_g^0(G) = id$, then we also have this diagram for $n = 0$. Assume (2.4) and (2.5) hold for n. Then the compositions in the left hand column and the middle column are $z^n_{(g\circ h)}(H)$ and $c^n_{(g\circ h)}(H)$. Hence the composition in the right hand column must be $z^{n+1}_{(g\circ h)}(H)$; in other words, (2.5) holds for $n+1$.

Consider the following diagram of sheaves

$$\begin{array}{ccccc}
\mathscr{C}^0(\mathscr{Z}^n(g_*h_*H)) & & & & \\
\downarrow & \searrow & & & \\
\mathscr{C}^0(g_*\mathscr{Z}^n(h_*H)) & \longrightarrow & \mathscr{C}^0(g_*h_*\mathscr{Z}^n(H)) & & \\
\downarrow & & \downarrow & \searrow & \\
g_*\mathscr{C}^0(\mathscr{Z}^n(h_*H)) & \longrightarrow & g_*\mathscr{C}^0(h_*\mathscr{Z}^n(H)) & \longrightarrow & g_*h_*\mathscr{C}^0(\mathscr{Z}^n(H)).
\end{array}$$

The upper triangle is commutative by (2.5), which we are assuming holds for n, and by the functoriality of $\mathscr{C}^0$; the square is commutative by the naturality of c_g^0; and the lower triangle is commutative by (2.4) for $n = 0$. Hence (2.4) holds for $n+1$. Thus (2.3) is commutative.

3. The natural map $h_g^n(F)$: $H^n(X, F) \to H^n(Y, g^*F)$

Let $g : Y \to X$ be a morphism of ringed spaces, F an $\mathscr{O}_X$-Module and $\rho_g(F) : F \to g_*g^*F$ the adjoint of the identity map of g^*F. Then composing $c_g^\bullet(g^*F)$ with $\mathscr{C}^\bullet(\rho_g(F))$ we obtain a map of complexes of sheaves,

$$\theta_g^\bullet(F) : \mathscr{C}^\bullet(F) \to g_*\mathscr{C}^\bullet(g^*F).$$

Applying the functor $\Gamma(X, -)$ and taking cohomology, we clearly obtain a map from $H^n(X, F)$ to $H^n(Y, g^*F)$; we shall denote it by $h_g^n(F)$.

For $n = 0$, we obviously have a commutative square,

$$\text{(3.1)} \qquad \begin{array}{ccc}
\Gamma(X, F) & \xrightarrow{\Gamma(X,\, \rho_g(F))} & \Gamma(Y, g^*F) \\
\downarrow \simeq & & \downarrow \simeq \\
H^0(X, F) & \xrightarrow{h_g^0(F)} & H^0(Y, g^*F),
\end{array}$$

where the vertical maps are induced by $\varepsilon(F)$ and $\varepsilon(g^*F)$. For each n, the map $h_g^n(-)$ is clearly a natural transformation.

Assume g is flat. Then it is easy to verify that a short exact sequence $0 \to F' \to F \to F'' \to 0$ of $\mathscr{O}_X$-Modules gives rise to a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{C}^\bullet(F') & \longrightarrow & \mathscr{C}^\bullet(F) & \longrightarrow & \mathscr{C}^\bullet(F'') & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \theta_g^\bullet(F')} & & \downarrow {\scriptstyle \theta_g^\bullet(F)} & & \downarrow {\scriptstyle \theta_g^\bullet(F'')} & & \\
0 & \longrightarrow & g_*\mathscr{C}^\bullet(g^*F') & \longrightarrow & g_*\mathscr{C}^\bullet(g^*F) & \longrightarrow & g_*\mathscr{C}^\bullet(g^*F'') & \longrightarrow & 0.
\end{array}$$

Hence the $H^n(Y, g^*F)$ form a cohomological functor in F and the h_g^n form a morphism of cohomological functors. Moreover since $H^n(X, -)$ is effaceable for each $n > 0$, the h_g^n form the unique morphism of cohomological functions extending $\Gamma(X, \rho_g(F))$.

Let $h : Z \to Y$ be a second morphism of ringed spaces. We shall identify the functors h^*g^* and $(g \circ h)^*$. Then we have a diagram of complexes of sheaves,

$$\begin{array}{ccccc}
\mathscr{C}^\bullet(F) & & & & \\
\downarrow {\scriptstyle \mathscr{C}^\bullet(\rho_g)} & \searrow {\scriptstyle \mathscr{C}^\bullet(\rho_{g\circ h})} & & & \\
\mathscr{C}^\bullet(g_*g^*F) & \xrightarrow{\mathscr{C}^\bullet(g_*\rho_h)} & \mathscr{C}^\bullet(g_*h_*(g\circ h)^*F) & & \\
\downarrow {\scriptstyle c_g^\bullet} & & \downarrow {\scriptstyle c_g^\bullet} & \searrow {\scriptstyle c^\bullet_{(g\circ h)}} & \\
g_*\mathscr{C}^\bullet(g^*F) & \xrightarrow[g_*\mathscr{C}^\bullet(\rho_h)]{} & g_*\mathscr{C}^\bullet(h_*(g\circ h)^*F) & \xrightarrow[g_*(c_h^\bullet)]{} & g_*h_*\mathscr{C}^\bullet((g\circ h)^*F).
\end{array}$$

It follows formally from the theory of adjoints that the composition,

$$F \xrightarrow{\rho_g} g_*g^*F \xrightarrow{g_*\rho_h} g_*h_*(g\circ h)^*F,$$

is equal to the map,

$$\rho_{(g\circ h)} : F \to g_*h_*(g\circ h)^*F;$$

so, since $\mathscr{C}^\bullet$ is a functor, the upper triangle is commutative. The square is commutative by the naturality of $c_g^\bullet$. The commutativity of the lower triangle results from (2.3) applied with $H = (g\circ h)^*F$. Applying $\Gamma(X, -)$ and taking cohomology, we obtain a commutative diagram,

$$\begin{array}{ccc}
H^n(X, F) & & \\
\downarrow {\scriptstyle h_g^n(F)} & \searrow {\scriptstyle h^n_{(g\circ h)}(F)} & \\
H^n(Y, g^*F) & \xrightarrow[h_h^n(g^*F)]{} & H^n(Z, (g\circ h)^*F),
\end{array} \tag{3.2}$$

of cohomology groups for each integer n.

Let Y' denote the ringed space $(Y, g^{-1}\mathcal{O}_X)$ where g^{-1} denotes the (left) adjoint of g_* in the category of abelian sheaves. Then since the map $\mathcal{O}_X \to g_*\mathcal{O}_Y$ can be factored as $\mathcal{O}_X \to g_*g^{-1}\mathcal{O}_X \to g_*\mathcal{O}_Y$, the morphism g can be factored as $Y \xrightarrow{g''} Y' \xrightarrow{g'} X$. Now g' is clearly flat since, for each

$y \in Y$, the ring $\mathcal{O}_{Y',y}$ is equal to $\mathcal{O}_{X,g(y)}$ (see EGA 0_{I}, 3.7.2) and in fact $g'^*(F)$ is clearly equal to $g^{-1}(F)$. Hence

$$h^n_{g'}(F) : H^n(X, F) \to H^n(Y, g^{-1}(F))$$

is the unique extension of the canonical map $\Gamma(X, F) \to \Gamma(Y, g^{-1}(F))$ to cohomology. Since g'' is the identity map on topological spaces, $c^\bullet_{g''}$ is the identity map. Hence

$$h^n_{g''}(g^{-1}(F)) : H^n(Y, g^{-1}(F)) \to H^n(Y, g^*F)$$

is the map induced by the canonical map,

$$g^{-1}(F) \to g^*(F) = g^{-1}(F) \otimes_{g^{-1}(\mathcal{O}_X)} \mathcal{O}_Y .$$

Thus $h^n_{g''}(g^{-1}(F))$ and $h^n_{g'}(F)$ are intrinsic; that is, they do not depend on the construction of a map like $c^\bullet_g$. Now the commutativity of (3.2) expresses $h^n_g(F)$ as the composition

$$h^n_g(F) = h^n_{g''}(g^{-1}(F)) \circ h^n_{g'}(F). \tag{3.3}$$

In (EGA O_{III}, 12.1.3.5), this formula is taken as the definition of $h^n_g(F)$.

4. The spectral sequence of Čech cohomology

Let $g : Y \to X$ be a morphism of ringed spaces and F an $\mathcal{O}_X$-Module. Let $\mathcal{U} = (U_i)$ be an open covering of X and set $g^{-1}\mathcal{U} = (g^{-1}(U_i))$. Let $\check{C}^\bullet(\mathcal{U}, F)$ denote the Čech complex of F with respect to $\mathcal{U}$; its formation is clearly functorial in F. Thus applying $\check{C}^\bullet(\mathcal{U}, -)$ to $\theta^\bullet_g(F)$, we obtain a map of double complexes

$$\check{C}^\bullet(\mathcal{U}, \theta^\bullet_g(F)) : \check{C}^\bullet(\mathcal{U}, \mathscr{C}^\bullet(F)) \to \check{C}^\bullet(\mathcal{U}, g_*\mathscr{C}^\bullet(g^*F)) = \check{C}^\bullet(g^{-1}\mathcal{U}, \mathscr{C}^\bullet(g^*F)) \tag{4.1}$$

It is clearly natural in F. Take the H^p_{I}-cohomology in (4.1). Since the Čech cohomology of a flasque sheaf is zero ([2], II. 5.2.3), we obtain zero in both double complexes for $p > 0$. For $p = 0$, we obtain the map,

$$\Gamma(X, \theta^\bullet_g(F)) : \Gamma(X, \mathscr{C}^\bullet(F)) \to \Gamma(Y, \mathscr{C}^\bullet(g^*F)).$$

Thus the map on the limits of the spectral sequences is

$$h^n_g(F) : H^n(X, F) \to H^n(Y, g^*F).$$

For any sheaf G, let $\mathscr{H}^n(G)$ denote the n^{th} cohomology object of $\mathscr{C}^\bullet(G)$ in the category of presheaves; thus for each open set U, we have $\mathscr{H}^n(G)(U) = H^n(U, G)$. Since the functor $G \mapsto \check{C}^\bullet(\mathcal{U}, G)$ is exact on the category of presheaves, taking the H^q_{II}-cohomology in (4.1) yields a map of spectral sequences (starting at the E_1-level),

$$\begin{array}{ccc} \check{C}^p(\mathscr{U},\mathscr{H}^q(\theta^\bullet_g(F))) : \check{C}^p(\mathscr{U},\mathscr{H}^q(F)) & \longrightarrow & \check{C}^p(g^{-1}\mathscr{U},\mathscr{H}^q(g^*F)) \\ \Downarrow & & \Downarrow \\ h^n_g(F) : H^n(X,F) & \longrightarrow & H^n(Y, g^*F). \end{array} \tag{4.2}$$

The $E_1^{p,q}$-terms $\check{C}^p(\mathscr{U},\mathscr{H}^q(F))$ (resp. $\check{C}^p(g^{-1}\mathscr{U},\mathscr{H}^q(g^*F)))$ are by definition direct products of terms $H^q(U, F)$ (resp. $H^q(g^{-1}U, g^*F)$) where U is an intersection of $(p+1)$ members of $\mathscr{U}$. It is evident that the map $\check{C}^p(\mathscr{U},\mathscr{H}^q(\theta^\bullet_g(F)))$ is the product of the maps

$$h^q_{g|g^{-1}U}(F|U) : H^q(U, F) \to H^q(g^{-1}U, g^*F).$$

5. Quasi-coherence of R^nf_*F

Let $f: X \to S$ be a morphism of ringed spaces and F an $\mathcal{O}_X$-Module. Then R^nf_*F is equal to the sheaf associated to the presheaf $U \mapsto H^n(f^{-1}U, F)$ on S. Moreover, the map,

$$H^n(X, F) \to \Gamma(S, R^nf_*F), \tag{5.1}$$

from the global sections of the presheaf to those of its associated sheaf is equal to the edge homomorphism of the Leray spectral sequence $H^p(S, R^qf_*F) \Rightarrow H^n(X, F)$, (see EGA 0_{III}, 12.2.5). Assume that S is an affine scheme and that R^nf_*F is quasi-coherent. Then the Leray spectral sequence degenerates by (EGA III, 1.3.1). Therefore (5.1) is an isomorphism in this case. On the other hand, the proof below that, under suitable hyphoteses, R^nf_*F is quasi-coherent yields that (5.1) is an isomorphism directly.

(5.2) LEMMA. *Let A be a ring, X a quasi-separated and quasi-compact A-scheme, F a quasi-coherent $\mathcal{O}_X$-Module and B a flat A-algebra. Let Y denote the fibered product $X \otimes_A B$ and $g : Y \to X$ the projection. Then for each integer $n \geqq 0$, the canonical map induced by $h^n_g(F)$,*

$$h^n_g(F)^\# : H^n(X, F) \otimes_A B \to H^n(Y, g^*F), \tag{5.3}$$

is an isomorphism.

PROOF. The proof proceeds by induction on n. Since (3.1) is commutative, the map $h^0_g(F)$ is equal to

$$\Gamma(X, \rho_g(F))^\# : \Gamma(X, F) \otimes_A B \to \Gamma(Y, g^*F).$$

The latter map is an isomorphism by (EGA I, 1.7.7 (i), 6.7.1, and 9.3.3); alternatively this fact can be proved directly using the ideas in the proof of (EGA I, 6.7.1 or 9.3.2).

Assume the assertion holds for each integer $q < n$ for some $n > 0$. Let $\mathscr{U}$ be a finite affine open covering of X and consider the map of spectral sequences,

$$(5.4)\quad E_1^{p,q} = \check{C}^p(\mathscr{U}, \mathscr{H}^q(F)) \otimes_A B \xrightarrow{u_1^{p,q}} \check{C}^p(g^{-1}\mathscr{U}, \mathscr{H}^q(g^*F)) = F_1^{p,q},$$

induced by (4.2).

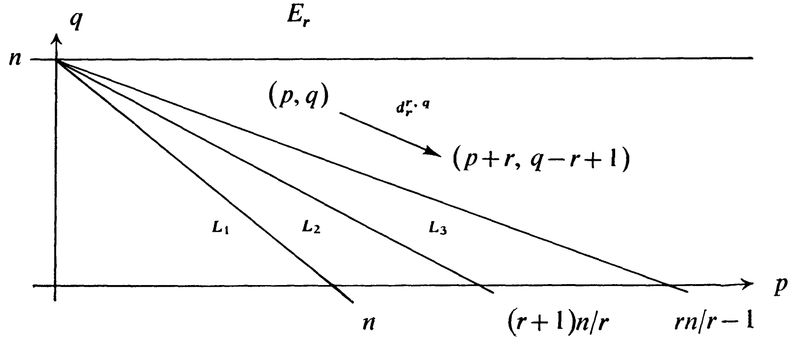
The term $\check{C}^p(\mathscr{U}, \mathscr{H}^q(F)) \otimes_A B$(resp. $\check{C}^p(g^{-1}\mathscr{U}, \mathscr{H}^q(g^*F)))$ is a finite direct sum of terms $H^q(U, F)$(resp. $H^q(g^{-1}U, g^*F))$ where U is an intersection of $(p+1)$ members of $\mathscr{U}$. If $p = 0$ holds, then both U and $g^{-1}U$ are affine. So for $q > 0$, both $H^q(U, F)$ and $H^q(g^{-1}U, g^*F)$ are zero by (EGA III, 1.3.1). Hence $\check{C}^0(\mathscr{U}, \mathscr{H}^q(F))$ and $\check{C}^0(g^{-1}\mathscr{U}, \mathscr{H}^q(g^*F))$ are both zero for each $q > 0$. In order words, we have

$$(5.5)\qquad E_1^{0,q} = F_1^{0,q} = 0 \text{ for each } q > 0.$$

If $p > 0$ holds, then since U is quasi-separated and quasi-compact, the map $H^q(U, F) \otimes_A B \to H^q(g^{-1}U, g^*F)$ is an isomorphism for $q < n$ by induction. Consequently $u_1^{p,q} : E_1^{p,q} \to F_1^{p,q}$ is an isomorphism for each $q < n$.

For $r \geqq 2$, we cannot *a priori* conclude that $u_r^{p,q} : E_r^{p,q} \to F_r^{p,q}$ is an isomorphism for each pair (p, q) with $q < n$ because we do not have enough information about the various differentials $d_{r-1}^{p,q}$. However, we are going to prove that $u_r^{p,q}$ is an isomorphism when $p+q = n$ holds for each $r \geqq 2$ by induction on r.

Assume that $u_r^{p,q}$ is an isomorphism for all pairs (p, q) with $q < ((1-r)/r)p+n$. (Notice that this implies $q < n$.) Since the slope of each



$L_1 : q = -p+n$
$L_2 : q = [-r/(r+1)]p+n$
$L_3 : q = [(1-r)/r]p+n$

differential in $E_r^{p,q}$ and $F_r^{p,q}$ is $(1-r)/r$, it follows that $u_{r+1}^{p,q}$ is also an isomorphism for each pair (p, q) with $q < ((1-r)/r)p+n$. In particular,

$u_{r+1}^{p,q}$ is an isomorphism for each pair (p, q) with $q < ((-r)/(r+1))p+n$. Hence by induction, $u_r^{p,q}$ is an isomorphism for each $r \geqq 1$ for each pair (p, q) with $p+q \leqq n$ and $q < n$. However by (5.5), $E_r^{0,n}$ and $F_r^{0,n}$ are both zero for each $r \geqq 1$. Hence the map $u_\infty^{p,q} : E_\infty^{p,q} \to F_\infty^{p,q}$ is an isomorphism for each pair (p, q) with $p+q = n$. Since B is flat over A, the functor $-\otimes_A B$ commutes with cohomology; hence $h_g^n(F)^\#$ is equal to the map on the limits of the spectral sequences. Therefore $h_g^n(F)^\#$ is an isomorphism.

(5.6) THEOREM. *Let $f: X \to S$ be a quasi-separated, quasi-compact morphism of schemes and F a quasi-coherent $\mathcal{O}_X$-Module. Then for each $n \geqq 0$, the sheaf $R^n f_* F$ is quasi-coherent.*

PROOF. The assertion is local on S, so we may assume S is affine. Set $A = \Gamma(S, \mathcal{O}_S)$ and let h be an element of A. Then A_h is a flat A-algebra and the fibered product $X \otimes_A A_h$ is equal to $f^{-1}(S_h)$. Let g denote the inclusion of $f^{-1}(S_h)$ in X. Then by (5.2), the canonical map,

$$h_g^n(F)^\# : H^n(X, F) \otimes_A A_h \to H^n(f^{-1}(S_h), F),$$

is an isomorphism. Therefore the presheaf defined by $S_h \to H^n(f^{-1}(S_h), F)$ is a quasi-coherent sheaf by (EGA I, 1.3.7). However, $R^n f_* F$ is equal to the sheaf associated to this presheaf. Thus, $R^n f_* F$ is quasi-coherent.

6. The base change map

Consider a commutative diagram of ringed spaces

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{t} & S \end{array}$$

Then form the composition,

$$(6.1) \qquad H^n(X, F) \xrightarrow{h_g^n(F)} H^n(Y, g^*F) \to \Gamma(T, R^n f'_*(g^*F)),$$

where the second arrow is the map (5.1) from the global sections of the presheaf $V \mapsto H^n(f'^{-1}(V), g^*F)$ on T to those of its associated sheaf. Take an open subset U of S, replace X, Y and T by the inverse images of U and form the corresponding maps of cohomology groups,

$$(6.2) \qquad H^n(f^{-1}(U), F) \to \Gamma(t^{-1}(U), R^n f'_*(g^*F)).$$

Now, the $h_g^n(F)$ were defined as the maps of cohomology groups induced by the maps $\theta_g^\bullet(F)$ of complexes of sheaves. It is evident that the formation of $\theta_g^\bullet(F)$ commutes with restriction. Therefore the formation of $h_g^n(F)$ commutes with restriction. Hence as U runs through the open sets of S, the maps (6.2) form a morphism of presheaves. Passing to associated sheaves, we obtain a map

$$\beta^n(f, f', t, g, F) : R^n f_* F \to t_* R^n f'_*(g^* F).$$

The adjoint of $\beta^n(f, f', t, g, F)$ with respect to t is denoted $\alpha^n(f, f', t, g, F)$ or $\alpha^n(F)$ for short.

For $n = 0$, we clearly have a commutative diagram,

$$\begin{array}{ccc} t^* f_* F & \longrightarrow & f'_* g^* F \\ \simeq \downarrow & & \simeq \downarrow \\ t^* R^0 f_* F & \xrightarrow{\alpha^0(F)} & R^0 f'_*(g^* F), \end{array} \tag{6.3}$$

where the top map is the adjoint of

$$f_*(\rho_g(F)) : f_* F \to f_* g_* g^* F = t_* f'_* g^* F$$

with respect to t and the vertical maps are induced by $\varepsilon(F)$ and $\varepsilon(g^* F)$. For each n, the map $\alpha^n(-)$ is clearly a natural transformation. Assume in addition that t and g are flat. Then both the $t^* R^n f_* F$ and the $R^n f'_*(g^* F)$ form cohomological functors in F and it is easy to verify that the $\alpha^n(F) : t^* R^n f_* F \to R^n f'_*(g^* F)$ form a morphism of cohomological functors in F because the $h_g^n(F)$ do. Since $t^* R^n f_* F$ is effaceable for each $n > 0$, the $\alpha^n(F)$ form the unique extension of the adjoint of $f_*(\rho_g(F))$ with respect to t to the higher direct images.

Let U be an open subset of S and W its preimage in X. Give each its induced ringed-space structure. Let $i : U \to S$ and $j : W \to X$ denote the inclusions. Then the $(R^n f_* F)|U$ and the $R^n(f|W)_*(F|W)$ both form universal cohomological functors in F, and so $\alpha^n(f, f|W, i, j, F)$ is the unique extension of $\alpha^0(f, f|W, i, j, F)$ to the higher direct images. Now, for each open subset V of W, the map $\Gamma(V, \rho_j(F))$ is clearly the identity map of $\Gamma(V, F)$. Hence, by (6.3), $\alpha^0(f, f|W, i, j, F)$ is an isomorphism. Therefore its extensions are the isomorphisms

$$\alpha^n(f, f|W, i, j, F) : (R^n f_* F)|U \xrightarrow{\sim} R^n(f|W)_*(F|W). \tag{6.4}$$

Consider a second commutative square of ringed spaces,

$$\begin{array}{ccc} Z & \xrightarrow{h} & Y \\ f'' \downarrow & & \downarrow f' \\ R & \xrightarrow{r} & T. \end{array}$$

Then the commutativity of (3.2) yields, by passing to associated sheaves, the commutativity of the triangle,

$$\begin{array}{ccc} R^nf_*F & & \\ \downarrow & \searrow & \\ t_*R^nf'_*(g^*F) & \longrightarrow & t_*r_*R^nf''_*((g\circ h)^*F). \end{array}$$

Therefore taking adjoints, we obtain the following commutative triangle:

$$(6.5)\qquad \begin{array}{ccc} (t\circ r)^*R^nf_*F & & \\ {\scriptstyle r^*(\alpha^n(F))}\downarrow & \searrow{\scriptstyle \alpha^n(F)} & \\ r^*R^nf'_*(g^*F) & \xrightarrow[\alpha^n(g^*F)]{} & R^nf''_*((g\circ h)^*F). \end{array}$$

This triangle expresses the compatibility of the base change map with composition.

Let U be an open subset of S, and V an open subset of $t^{-1}U$, and let $i: U \to S$ and $j: V \to T$ denote the inclusions. Then we have $i\circ t' = t\circ j$ where $t': U \to V$ is induced by t. So, applying on the one hand (6.5) to $i\circ t'$ and (6.4) to i and on the other hand, (6.5) to $t\circ j$ and (6.4) to j, we obtain a commutative diagram,

$$(6.6)\qquad \begin{array}{ccc} (t^*(R^nf_*F))|V & \xrightarrow{\sim} & t'^*(R^n(f|f^{-1}U)_*(F|f^{-1}U) \\ \downarrow & & \downarrow \\ (R^nf'_*(g^*F))|V & \xrightarrow{\sim} & R^n(f'|f'^{-1}V)_*(g^*Ff'^{-1}V). \end{array}$$

The horizontal maps are isomorphisms by (6.4).

This diagram expresses the local nature of the base change map; the restriction of the base change map to an open set V contained in the preimage of an open set U is equal to the base change map of the restricted sheaf with respect to the induced map from V to U.

(6.7) THEOREM. *Let $f: X \to S$ be a quasi-separated, quasi-compact morphism of schemes and F a quasi-coherent $\mathcal{O}_X$-Module. Let $t: T \to S$ be a flat morphism of schemes and set $Y = Xx_ST$ with projections f' and g to T and X. Then the base change map,*

$$\alpha^n(F): t^*R^nf_*F \to R^nf'_*(g^*F),$$

is an isomorphism for each $n \geqq 0$.

PROOF. By (6.6), the assertion is local on both S and T; so we may assume S and T are affine. Set $A = \Gamma(S, \mathcal{O}_S)$ and $B = \Gamma(T, \mathcal{O}_T)$. By (5.6), the sheaves $R^n f_* F$ and $R^n f'_*(g^* F)$ are quasi-coherent. Therefore the maps (5.1), $H^n(X, F) \to \Gamma(S, R^n f_* F)$ and $H^n(Y, g^* F) \to \Gamma(T, R^n f'_*(g^* F))$, are isomorphisms. Hence by (EGA I, 1.7.7(i)), we have $\Gamma(T, t^* R^n f_* F) = H^n(X, F) \otimes_A B$. Thus $\Gamma(T, \alpha^n(F))$ is equal to the map,

$$h^n(F)^\# : H^n(X, F) \otimes_A B \to H^n(Y, g^* F),$$

of (5.3) and so it is an isomorphism. Hence $\alpha^n(F)$ is an isomorphism.

Alternately we could note that the map of stalks, $\alpha^n(F)_\tau$, is an isomorphism for each point $\tau \in T$ because it is the direct limit of the isomorphisms of (5.3),

$$H^n(f^{-1}U, F) \otimes_{\Gamma(U, \mathcal{O}_S)} \Gamma(V, \mathcal{O}_T) \to H^n(f'^{-1}V, g^* F),$$

as U runs through the affine neighborhoods of $t(\tau)$ and V runs through the affine neighborhoods of τ contained in $t^{-1}U$.

(6.8) Examples. Let k be a field, $k[T]$ a polynomial ring in one variable over k. Let A denote the subring of $\Pi_{i \in N} k[T]$ consisting of those sequences (f_i) such that $f_n = f_{n+1}$ holds for $n \gg 0$. Let I denote the ideal of A consisting of those sequences (f_i) such that $f_n = 0$ holds for $n \gg 0$. Set $S = \text{Spec}(A)$ and set $U = S - V(I)$. Let $j : U \to S$ denote the inclusion. We shall show that the canonical map,

(6.9) $\Gamma(S, j_* \mathcal{O}_U) \otimes_A A_g \to \Gamma(S_g, j_* \mathcal{O}_U)$, with $g = (T, T, T, \cdots)$,

is not surjective; thus $j_* \mathcal{O}_U$ is not quasi-coherent.

Let e_n denote the element of I that coincides with the zero sequence except for a 1 in the nth place. Clearly, the elements e_n generate I. So, we have $U = \cup S_{e_n}$. Hence, for any element $f = (f_i)$ of A, we have $U \cap S_f = \cup S_{fe_n}$. Moreover, A_{fe_n} is clearly equal to $k[T]_{f_n}$. Since $e_n \cdot e_m = 0$ holds for $n \neq m$, we have $S_{fe_n} \cap S_{fe_m} = \phi$. Therefore, we have

$$\Gamma(U \cap S_f, \mathcal{O}_S) = \prod_{i \in N} k[T]_{f_i};$$

equivalently, we have

$$\Gamma(S_f, j_* \mathcal{O}_U) = \prod_{i \in N} k[T]_{f_i}.$$

In particular, for $f = 1$, we have

$$\Gamma(S, j_* \mathcal{O}_U) = \prod_{i \in N} k[T].$$

Clearly $\Gamma(S, j_* \mathcal{O}_U) \otimes_A A_g$ consists of all sequences of the form (g_i/T^m) with $g_i \in k[T]$ and m fixed. On the other hand, the element $h = (1/T^i)$ is in $\Gamma(S_g, j_* \mathcal{O}_U)$ and it obviously does not have the form (g_i/T^m). Thus h is not in the image of (6.9).

In the above example, the morphism j is quasi-separated, being an embedding, but it is obviously not quasi-compact. We now construct from it a morphism $u : X \to S$ that is quasi-compact but not quasi-separated such that $R^1 u_* \mathcal{O}_X$ is not quasi-coherent.

Let S_1, S_2 be two copies of S. Let X denote the scheme obtained by identifying S_1 and S_2 along U. Let $u : X \to S$ denote the morphism that is equal to the identity on each S_i. Then u is quasi-compact but not quasi-separated (EGA I, 6.3.10). Let $j_i : S_i \to X$, for $i = 1, 2$, and $j_3 : U \to X$ denote the inclusions.

Consider the (augmented) Čech resolution of the covering $\{S_1, S_2\}$ of X([2], II, 5.2.1):

$$0 \to \mathcal{O}_X \to j_{1*}\mathcal{O}_{S_1} \oplus j_{2*}\mathcal{O}_{S_2} \to j_{3*}\mathcal{O}_U \to 0.$$

It yields an exact sequence,

$$(6.10) \quad 0 \to u_*\mathcal{O}_X \to u_* j_{1*}\mathcal{O}_{S_1} \oplus u_* j_{2*}\mathcal{O}_{S_2} \to u_* j_{3*}\mathcal{O}_U \to R^1 u_*\mathcal{O}_X \to R^1 u_*(j_{1*}\mathcal{O}_{S_1}) \oplus R^1 u_*(j_{2*}\mathcal{O}_{S_2}).$$

For $i = 1, 2$, the exact sequence of terms of low degree of the Leray spectral sequence,

$$R^p u_*(R^q j_{i*}\mathcal{O}_{S_i}) \Rightarrow R^{p+q}(u \circ j_i)_*\mathcal{O}_{S_i},$$

begins with the exact sequence,

$$0 \to R^1 u_*(j_{i*}\mathcal{O}_{S_i}) \to R^1(u \circ j_i)_*\mathcal{O}_{S_i}.$$

So, since $u \circ j_i$ is equal to the identity of S, we have $R^1 u_*(j_{i*}\mathcal{O}_{S_i}) = 0$ and $u_* j_{i*}\mathcal{O}_{S_i} = \mathcal{O}_S$. Since the maps $\Gamma(S_f, \mathcal{O}_S) \to \Gamma(U \cap S_f, \mathcal{O}_S)$ are injective for each $f \in A$, it is evident that $u_*\mathcal{O}_X = \mathcal{O}_S$ holds. Since $u \circ j_3$ is equal to the inclusion j of U in S, we have $u_* j_{3*}\mathcal{O}_U = j_*\mathcal{O}_U$. So, (6.10) is equal to the exact sequence,

$$0 \to \mathcal{O}_S \xrightarrow{w} \mathcal{O}_S \oplus \mathcal{O}_S \to j_*\mathcal{O}_U \to R^1 u_*\mathcal{O}_X \to 0.$$

Since $\mathcal{O}_S$ and $\mathcal{O}_S \oplus \mathcal{O}_S$ are quasi-coherent, the cokernel of w is quasi-coherent (EGA I, 2.2.7i). So, since $j_*\mathcal{O}_U$ is not quasi-coherent, $R^1 u_*\mathcal{O}_X$ is not quasi-coherent.

BIBLIOGRAPHY

M. Artin, A. Grothendieck and J.-L. Verdier

[1] *Théorie des Topos et Cohomologie Etale des Schémas*, Tome 2. Séminaire de Géométrie Algébrique du Bois Marie 1963/64, Lecture Notes in Math., V. 270. Springer-Verlag (1972). (cited SGA 4).

R. Godement

[2] *Topologie algébrique et théorie des faisceaux*. Hermann, Paris (1958).

A. GROTHENDIECK and J. A. DIEUDONNÉ
[3] *Eléments de géométrie algébrique I.* Springer-Verlag (1971). (cited EGA $0_{\rm I}$, EGA I).

A. GROTHENDIECK and J. A. DIEUDONNÉ
[4] *Eléments de géométrie algébrique III, IV.* Publ. Math. IHES 11, 20. (cited EGA $0_{\rm III}$, EGA III, EGA IV), Paris (1961, 1964).

M. HAKIM
[5] *Topos annelés et schémas relatifs.* Springer-Verlag (1972).

R. HOOBLER
[6] *Non-abelian sheaf cohomology by derived functors.* Category Theory, Homology Theory and their Applications III, Lecture Notes in Math., V.99, pp. 313–365, Springer-Verlag (1969).

(Oblatum: 8-II-1973)

A. B. Altman
Department of Mathematics, University of California, San Diego, La Jolla, California 92037, U.S.A.

R. T. Hoobler
Department of Mathematics, Rice University, Houston, Texas 77001, U.S.A.

S. L. Kleiman
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02138, U.S.A.