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SUBSPACES OF L_p WHICH EMBED INTO l_p

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Abstract

If X is a subspace of L_p $(2 < p < \infty)$ and no subspace of X is isomorphic to l_2, then X is isomorphic to a subspace of l_p. This main result combines with known facts to yield that a separable $\mathscr{L}_p$ space $(1 < p < \infty)$ either contains a subspace isomorphic to l_2 or is isomorphic to l_p.

1. Introduction

Kadec and Pelczynski [5] proved that for $2 < p < \infty$, a subspace X of L_p $(= L_p[0, 1])$ is either isomorphic to l_2 or contains a smaller subspace isomorphic to l_p. Thus if X contains no subspace isomorphic to l_2, then X is modeled rather closely on l_p. The main result here establishes that such an X embeds into l_p.

The principal application of the main theorem is that for $1 < p < \infty$, a separable $\mathscr{L}_p$ space which does not contain a copy of l_2 is isomorphic to l_p.

For $1 < p < 2$, a weaker version of the main result is proved: if X is a subspace of L_p which has an unconditional finite dimensional decomposition, then X embeds into l_p provided there is a constant K so that every normalized basic sequence in X is K-equivalent to the unit vector basis for l_p. Of course, the hypothesis that X have an unconditional finite dimensional decomposition is probably superfluous, but Example 1 shows that the condition on basic sequences in X cannot be weakened.

Finally, in the last section a non-separable version of the main result is proved; namely, if X is a subspace of $L_p(\mu)(2 < p < \infty)$ for some measure μ, then either X contains a subspace isomorphic to l_2 or X is isomorphic to a subspace of $l_p(\Gamma)$ for some set Γ.

Our terminology is that of [4]. All spaces are assumed to be infinite dimensional, unless expressly specified finite dimensional. For $x \in L_1$, $\int x$ denotes

$$\int_0^1 x(t)\,\mathrm{d}t.$$

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2. The main result

The technique for proving the main result is similar to the blocking technique used in [4].

We first mention some notion. If $A \subseteq [0, 1]$ is a measurable set, $|A|$ denotes the Lebesgue measure of A and $\sim A$ is the complement of A in $[0, 1]$. If $1 \leqq p < \infty$ and $\varepsilon > 0$, $M_p(\varepsilon) = \{f \in L_p : |\{t : |f(t)| \geqq \varepsilon ||f||\}| \geqq \varepsilon\}$. The $M_p(\varepsilon)$ sets were used by Kadec and Pelczynski in [5] to study subspaces of L_p for $2 < p < \infty$. We shall use the fact (cf. [5, Theorem 1]) that if $f \in M_p(\varepsilon) (2 < p < \infty)$, then $||f||_2 \geqq \varepsilon^{\frac{3}{2}} ||f||_p$.

If for each n, X_n is a Banach space with norm $||\cdot||_n$, then $(\sum X_n)_{l_p}$ denotes the Banach space of sequences (x_n), with $x_n \in X_n$ for each n such that $||(x_n)|| = (\sum ||x_n||_n^p)^{1/p} < \infty$.

When we are working in L_p, the sequence (h_i) shall represent the normalized Haar functions in L_p. These functions form an unconditional basis in L_p for $1 < p < \infty$ ([10]).

The following lemma is well known (cf., e.g. [12, p. 209] for the $p < 2$ case; the proof for $p > 2$ is similar).

LEMMA 1: *If $(x_n)_{n=1}^\infty$ is a normalized unconditional basic sequence in L_p with unconditional basis constant λ, then*

(a) if $2 < p < \infty$, $(\sum_{n=1}^k |\alpha_n|^p)^{1/p} \leqq \lambda || \sum_{n=1}^k \alpha_n x_n ||_p$

(b) if $1 \leqq p < 2$, $|| \sum_{n=1}^k \alpha_n x_n ||_p \leqq \lambda (\sum_{n=1}^k |\alpha_n|^p)^{1/p}$

LEMMA 2: *If Y is a subspace of $L_p (p > 2)$ containing no subspace isomorphic to l_2, then for any $\delta > 0$, there exists n such that if $y = \sum \alpha_i h_i \in Y$ and $||y|| \leqq 1$, then*

$$|| \sum_n^\infty \alpha_i h_i ||_2 \leqq \delta$$

PROOF: We need only show that the inclusion mapping of Y into L_2 is a compact operator. Suppose not. Then there exists a sequence $(y_n) \subseteq Y$, such that y_n converges weakly to 0 in L_p, but $||y_n||_2$ does not converge to 0. We may assume $||y_n||_2 \geqq \varepsilon > 0$ for all n and some ε. But then in the L_p norm a subsequence (y_{n_i}) is equivalent to the unit vector basis for l_p by [5, Corollary 5], while some subsequence of (y_{n_i}) is, in K_2, equivalent to the unit vector basis of l_2 (cf. [1]), a contradiction.

THEOREM 1: *If Y is a subspace of L_p $(2 < p < \infty)$ such that no subspace of Y is isomorphic to l_2, then Y is isomorphic to a subspace of l_p.*

PROOF: By lemma 1 we need only produce a blocking, $X_n = [h_i]_{i=p_n}^{p_{n+1}-1}$ of the Haar functions and a constant $C < \infty$ such that if $\sum y_n \in Y$ with $y_n \in X_n$ for each n, then

$$\|\sum_1^\infty y_n\| \leqq C(\sum \|y_n\|^p)^{1/p}.$$

Y is then isomorphic to $\{(y_n) \in (\sum X_n)_{l_p} : \sum y_n \in Y\}$ and hence isomorphic to a subspace of l_p.

We first make the simple observation,

(*) If $E \subseteq L_p$, $\dim E < \infty$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $A \subseteq [0, 1]$ with $|A| < \delta$, then

$$\left(\int_A |x|^p\right)^{1/p} < \varepsilon\|x\|_p \text{ for all } x \in E.$$

Let $\lambda = \lambda(p)$ be the unconditional basis constant of the Haar functions in L_p. Set $0 < \varepsilon_1 < (6\lambda)^{-1}$ and $p_1 = 1$. Inductively we choose $\varepsilon_n > 0$, $\delta_n > 0$, and integers $p_1 < p_2 < p_3 < \cdots$ to satisfy $\sum \varepsilon_n < (6\lambda)^{-1}$ and

(1) if $x \in [h_i]_{i=1}^{p_n-1}$ and $|A| < \varepsilon_n$, then

$$\left(\int_A |x|^p\right)^{1/p} < \varepsilon_{n-1}\|x\|$$

(2) $\delta_n < (2^{-n-1}\varepsilon_n)^{\frac{3}{2}}2^{-n-3}$ (so that $\sum_1^\infty (2^{-n-1}\varepsilon_n)^{-\frac{3}{2}}\delta_n \leqq \frac{1}{4}$)

(3) if $y = \sum \alpha_i h_i \in Y$, then

$$\|\sum_{p_{n+1}}^\infty \alpha_i h_i\|_2 \leqq \delta_n\|y\|_p.$$

We verify that the blocking $X_n = [h_i]_{p_n}^{p_{n+1}-1}$ works. Assume $y = \sum y_n \in Y$ with $y_n \in X_n$ for each n. Then, assuming without loss of generality that

$$\|\sum_1^\infty y_{2n}\| \geqq \|\sum_1^\infty y_{2n-1}\|,$$

we have

$$\|\sum_1^\infty y_n\| \leqq 2\|\sum_1^\infty y_{2n}\| \tag{4}$$

Let $I = \{n : y_{2n} \in M_p(2^{-2n}\varepsilon_{2n-1})\}$

$J = \{n : y_{2n} \notin M_p(2^{-2n}\varepsilon_{2n-1})\}$

Recalling that $y_{2n} \in [h_i]_{p_{2n}}^{p_{2n+1}-1}$ we have that if $n \in I$, then

$$\begin{aligned}||y_{2n}||_p &\leqq (2^{-2n}\varepsilon_{2n-1})^{-\frac{3}{2}}||y_{2n}||_2 \\ &\leqq (2^{-2n}\varepsilon_{2n-1})^{-\frac{3}{2}}||\sum_{i=2n}^{\infty} y_i||_2 \\ &\leqq (2^{-2n}\varepsilon_{2n-1})^{-\frac{3}{2}}\delta_{2n-1}||\sum_{1}^{\infty} y_n||_p \quad \text{(by (3))}.\end{aligned}$$

So

$$\begin{aligned}||\sum_{n\in I} y_{2n}||_p &\leqq \Big(\sum_{1}^{\infty}(2^{-2n}\varepsilon_{2n-1})^{-\frac{3}{2}}\delta_{2n-1}\Big)||\sum_{1}^{\infty} y_n||_p \\ &\leqq \tfrac{1}{4}||\sum_{1}^{\infty} y_n||_p.\end{aligned}$$

By (4),

$$||\sum_{1}^{\infty} y_n||_p \leqq 2\Big(\tfrac{1}{4}||\sum_{1}^{\infty} y_n||_p + ||\sum_{n\in J} y_{2n}||_p\Big)$$

or

$$||\sum_{1}^{\infty} y_n|| \leqq 4||\sum_{n\in J} y_{2n}|| \tag{5}$$

In what follows we shall use the notation $\sum'$ to mean summation over only those indices in J. Thus

$$\sum_{i=n+1}^{\infty}{}' \text{ means } \sum_{\substack{i\in J\\ i\geqslant n+1}}.$$

$\cup'$ shall have a similar meaning.

For $n \in J$, there exists $A_{2n} \subseteqq [0, 1]$ with $|A_{2n}| \leqq 2^{-2n}\varepsilon_{2n-1}$ so that

$$\Big(\int_{\sim A_{2n}} |y_{2n}|^p\Big)^{1/p} \leqq 2^{-2n}\varepsilon_{2n-1}||y_{2n}||.$$

Let

$$B = \bigcup_{n=1}^{\infty} A_{2n} \text{ and } B_n = A_{2n} \sim \bigcup_{i=n+1}^{\infty}{}' A_{2i} \text{ for } n \in J.$$

Then,

$$||\sum_{n=1}^{\infty}{}' y_{2n}|| \leqq \Big(\int_B |\textstyle\sum' y_{2n}|^p\Big)^{1/p} + \Big(\int_{\sim B} |\textstyle\sum' y_{2n}|^p\Big)^{1/p} \tag{6}$$

$$\Big(\int_{\sim B} |\textstyle\sum' y_{2n}|^p\Big)^{1/p} \leqq \displaystyle\sum_{n=1}^{\infty}{}' \Big(\int_{\sim B} |y_{2n}|^p\Big)^{1/p} \tag{7}$$

$$\leqq \sum_{n=1}^{\infty}{}' 2^{-2n}\varepsilon_{2n-1}||y_{2n}|| \leqq \sum_{n=1}^{\infty}{}' 2^{-2n}\Big(\sum_{1}^{\infty}{}' ||y_{2i}||^p\Big)^{1/p} \leqq \Big(\sum_{1}^{\infty}{}' ||y_{2i}||^p\Big)^{1/p}.$$

Also, if $1/p+1/q = 1$,

$$\left(\int_B |\sum_n{}' y_{2n}|^p\right)^{1/p} = \left(\sum_n{}' \int_{B_n} |\sum_i{}' y_{2i}|^p\right)^{1/p}$$

$$\leqq 3^{1/q}\left[\sum_n{}' \int_{B_n} \left(|\sum_{i=1}^{n-1}{}' y_{2i}|^p + |y_{2n}|^p + |\sum_{i=n+1}^{\infty}{}' y_{2i}|^p\right)\right]^{1/p}$$

$$\leqq 3\left[\left(\sum_{n=2}^{\infty}{}' \int_{B_n} |\sum_{i=1}^{n-1}{}' y_{2i}|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty}{}' \int_{B_n} |y_{2n}|^p\right)^{1/p}\right.$$

$$\left. + \left(\sum_{n=1}^{\infty}{}' \int_{B_n} \sum_{n+1}^{\infty}{}' |y_{2i}|^p\right)^{1/p}\right]$$

$$\leqq 3\left(\sum_{n=1}^{\infty}{}' ||y_{2n}||^p\right)^{1/p} + 3\sum_{n=2}^{\infty}{}' \left(\int_{B_n} |\sum_{i=1}^{n-1}{}' y_{2i}|^p\right)^{1/p}$$

$$+3\sum_{n=1}^{\infty}{}' \left(\int_{B_n} |\sum_{i=n+1}^{\infty}{}' y_{2i}|^p\right)^{1/p} \equiv M.$$

Since
$$\sum_{i=1}^{n-1}{}' y_{2i} \in [h_i]_{i=1}^{p_{2n-1}-1}$$

and
$$|B_n| \leqq |A_{2n}| \leqq 2^{-2n}\varepsilon_{2n-1} < \varepsilon_{2n-1},$$

we have from (1) that

$$\left(\int_{B_n} |\sum_{i=1}^{n-1}{}' y_{2i}|^p\right)^{1/p} \leqq \varepsilon_{2n-2} ||\sum_{i=1}^{n-1}{}' y_{2i}||$$

$$\leqq \varepsilon_{2n-2}\lambda ||\sum_{i=1}^{\infty}{}' y_{2i}||$$

Thus

$$M \leqq 3\left(\sum_{n=1}^{\infty}{}' ||y_{2n}||^p\right)^{1/p} + 3\lambda \sum_{n=2}^{\infty}{}' \varepsilon_{2n-2} ||\sum_{i=1}^{\infty}{}' y_{2i}|| + 3\sum_{n=1}^{\infty}{}' \sum_{i=n+1}^{\infty}{}' \left(\int_{\sim A_{2i}} |y_{2i}|^p\right)^{1/p}$$

$$\leqq 3\left(\sum_{n=1}^{\infty}{}' ||y_{2n}||^p\right)^{1/p} + 3\lambda(6\lambda)^{-1} ||\sum_{i=1}^{\infty}{}' y_{2i}|| + 3\sum_{n=1}^{\infty}{}' \sum_{i=n+1}^{\infty}{}' 2^{-2i}\varepsilon_{2i-1} ||y_{2i}||$$

$$\leqq 3\left(\sum_{n=1}^{\infty}{}' ||y_{2n}||^p\right)^{1/p} + 2^{-1} ||\sum_{i=1}^{\infty}{}' y_{2i}|| + 3\sum_{n=1}^{\infty}{}' 2^{-2n}\left(\sum_{i=1}^{\infty}{}' ||y_{2i}||^p\right)^{1/p}$$

$$\leqq 6\left(\sum_{n=1}^{\infty}{}' ||y_{2n}||^p\right)^{1/p} + 2^{-1} ||\sum_{i=1}^{\infty}{}' y_{2i}||$$

From (6) and (7)

$$\|\sum_{n=1}^{\infty}{}' y_{2n}\| \leqq 7(\sum_{n=1}^{\infty}{}' \|y_{2n}\|^p)^{1/p} + 2^{-1}\|\sum_{i=1}^{\infty}{}' y_{2i}\| \text{ or}$$

$$\|\sum_{n=1}^{\infty}{}' y_{2n}\| \leqq 14(\sum_{n=1}^{\infty}{}' \|y_{2n}\|^p)^{1/p}$$

whence by (5),

$$\|\sum_{i=1}^{\infty} y_n\| \leqq 56(\sum_{n=1}^{\infty} \|y_n\|^p)^{1/p}.$$

COROLLARY 1: *Suppose X is a separable $\mathscr{L}_p$ space* $(1 < p < \infty, p \neq 2)$ *and no subspace of X is isomorphic to l_2. Then X is isomorphic to l_p.*

PROOF: It is known (cf. [7]) that X is isomorphic to a complemented subspace of L_p. If $p > 2$, by Theorem 1, X is isomorphic to a subspace of l_p. Since X is $\mathscr{L}_p$, Corollary 1 of [4] yields that X is isomorphic to l_p.

Suppose $p < 2$. X^* is isomorphic to a complemented subspace of L_q $(1/p+1/q = 1)$. Now X^* contains no subspace isomorphic to l_2, because Kadec and Pelczynski [5] proved that every subspace of L_q isomorphic to l_2 is complemented in L_q. Therefore X^* is isomorphic to l_q by the first part of the proof, whence X is isomorphic to l_p.

3. Subspaces of $L_p(1 < p < 2)$

We first show that there is a subspace X of L_p $(1 < p < 2)$ such that every normalized basic sequence in X contains a subsequence which is equivalent to the unit vector basis of l_p, and yet X is not isomorphic to a subspace of l_p.

EXAMPLE 1: for $0 < \lambda < 1$, let X_λ be the subspace of $(l_p \oplus l_2)_{l_p}$ spanned by $(\lambda e_n + \delta_n)$, where (e_n) (respectively, (δ_n)) is the unit vector basis for l_p (respectively, l_2). Since $p < 2$, X_λ is isomorphic to l_p.

Let $X = (\sum X_{1/m})_{l_p}$. Certainly X is isometric to a subspace of L_p (since as is well known, l_2 is isometric to a subspace of L_p). By construction, there is for each $M < \infty$ a normalized basic sequence (x_n) in X (namely, $x_n = m^{-1}(1+m^p)^{1/p}[m^{-1}e_n+\delta_n]$ in $X_{1/m}$ for m suitably large) such that no subsequence (x_{n_k}) of (x_n) is M-equivalent to the unit vector basis of l_p (i.e., if $T: [x_{n_k}] \to l_p$ is the linear extension of $x_{n_k} \to e_k$ then $\|T\| \cdot \|T^{-1}\| \geqq M$). Thus by a result of Pelczynski's [11], X is not isomorphic to a subspace of l_p. That X has the other desired property follows from:

PROPOSITION 1: *Suppose (X_n) is a sequence of subspaces of L_p* $(1 < p < 2)$ *and each X_n is isomorphic to l_p. Then every normalized basic* *sequence in* $X = (\sum X_n)_{l_p}$ *contains a subsequence which is equivalent to the unit vector basis of* l_p.

PROOF: Suppose $(x_n) \subseteq X$, $||x_n|| = 1$, (x_n) basic.

For each n, let P_n be the natural projection of X onto X_n. Assume first that

$$(^*) \text{ for each } n, \lim_{m\to\infty} ||P_n x_m|| = 0.$$

Then a standard gliding hump argument shows that (x_n) contains a subsequence equivalent to the unit vector basis of l_p.

If (*) is false, we may assume by passing to a subsequence of (x_m), that there is n and $\varepsilon > 0$ so that $||P_n x_m|| \geqq \varepsilon$ for $m = 1, 2, \cdots$ Now $x_m \to 0$ weakly by reflexivity of X, hence $P_n x_m \to 0$ weakly, whence by a result of Bessaga and Pelczynski [1], some subsequence of $(P_n x_m)_{m=1}^{\infty}$ is equivalent to the unit vector basis of l_p. So assume $(P_n x_m)_{m=1}^{\infty}$ is itself equivalent to the unit vector basis of l_p.

Obviously X is isometric to a subspace of L_p and L_p has an unconditional basis, so there is a subsequence (x_{m_k}) of (x_m) which is an unconditional basic sequence. As was noted in Section 2, this means that $\sum \alpha_k x_{m_k}$ converges whenever $\sum |\alpha_k|^p < \infty$.

On the other hand, if $\sum \alpha_k x_{m_k}$ converges, so does $\sum \alpha_k P_n x_{m_k}$, and hence $\sum |\alpha_k|^p < \infty$. Therefore (x_{m_k}) is equivalent to the unit vector basis of l_p.

REMARK: The version of Proposition 1 for $2 \leqq p < \infty$ is an immediate consequence of the results of [5].

EXAMPLE 2: J. Lindenstrauss has shown that Proposition 1 is false if the assumption that each X_n is a subspace of L_p is dropped. We wish to thank Professor Lindenstrauss for permission to reproduce here his example.

For $n = 1, 2, \cdots$ define an equivalent norm $|\cdot|_n$ on l_p by

$$|(\alpha_i)|_n = \max \{(\sum |\alpha_i|^p)^{1/p}, \sup_{k_1<k_2<\cdots<k_n} \sum_{i=1}^{n} |\alpha_{k_i}|\}.$$

Let $X_n = (l_p, |\cdot|_n)$ and set $X = (\sum X_n)_{l_p}$.

For each n, let $(e_i^n)_{i=1}^{\infty}$ be the unit vector basis of X_n. Pick any sequence (α_j) of positive scalars with $\sum \alpha_j^p = 1$. Choose integers $n_1 < n_2 < \cdots$ so that $\alpha_k n_k^{1-1/p} \to \infty$ as $k \to \infty$ and define

$$y_k = \sum_{j=1}^{\infty} \alpha_j e_k^j.$$

It is clear that (y_k) is a normalized basic sequence in X and (y_k) is 1-equivalent to each of its subsequences. Thus we need to show only that

(y_k) is not equivalent to the unit vector basis of l_p. But one sees by projecting

$$\sum_{i=1}^{n_k} y_i \text{ onto } X_{n_k} \text{ that } \|\sum_{i=1}^{n_k} y_i\| \geqq \alpha_k n_k$$

while the norm of the sum of the first n_k unit vectors in l_p is $n_k^{1/p}$. Since $\alpha_k n_k^{1-1/p} \to \infty$, this completes the proof.

We turn now to the main result of this section. We need a preliminary lemma which is probably known. Before stating this lemma, we recall that the Rademacher functions (r_n) on [0, 1] are defined by $r_n(t) = \text{sign}\,[\sin 2^{(n-1)}2\pi t]$.

LEMMA 2: *Suppose $1 \leqq \lambda < \infty$ and $\delta > 0$. Then there is a constant $K = K(\lambda, \delta)$ so that if (x_n) is a normalized unconditional basic sequence in L_p $(1 \leqq p \leqq 2)$ with unconditional constant $\leqq \lambda$ and there exists a pairwise disjoint sequence (B_n) of measurable subsets of* [0, 1] *satisfying*

$$\int_{B_n} |x_n|^p \geqq \delta^p \; (n = 1, 2, 3, \cdots),$$

then (x_n) is K-equivalent to the unit vector basis in l_p.

PROOF: As was observed in Section 2,

$$\|\sum \alpha_n x_n\| \leqq \lambda(\sum |\alpha_n|^p)^{1/p} \text{ for any choice}$$

of scalars. To get a similar inequality going the other way, we use the standard technique of integrating against the Rademacher functions (r_n). (For a more revealing proof, see remark 2 below.)

We have

$$\|\sum \alpha_n x_n\|^p \geqq \lambda^{-p} \int_0^1 \|\sum \alpha_n x_n r_n(t)\|^p \, dt$$

$$= \lambda^{-p} \int_0^1 \int_0^1 |\sum \alpha_n x_n(s) r_n(t)|^p \, ds \, dt$$

$$= \lambda^{-p} \int_0^1 \int_0^1 |\sum \alpha_n x_n(s) r_n(t)|^p \, dt \, ds \text{ (by Fubini's Theorem)}$$

$$\geqq \lambda^{-p} \sum_{j=1}^{\infty} \int_{B_j} \int_0^1 |\sum_{n=1}^{\infty} \alpha_n x_n(s) r_n(t)|^p dt \, ds$$

$$\geqq \lambda^{-p} \sum_{j=1}^{\infty} \int_{B_j} \left\{ \int_0^1 [\sum_{n=1}^{\infty} \alpha_n x_n(s) r_n(t)] r_j(t) \text{ sign } [\alpha_j x_j(s)] \, dt \right\}^p ds$$

$$= \lambda^{-p} \sum \int_{B_j} |\alpha_j x_j(s)|^p \, ds \geqq \lambda^{-p} \delta^p \sum |\alpha_j|^p.$$

Thus $K = \lambda^2\delta^{-1}$ is a permissible choice for K.

REMARK 1: Our original proof of Lemma 2 used Khinchine's inequality and produced a worse value for the constant K. T. Figiel pointed out to us that Khinchine's inequality is not needed.

REMARK 2: J. Lindenstrauss pointed out to us that the inequality

$$\left(\sum \int_{B_j} |\alpha_j x_j(s)|^p\right)^{1/p} \leqq \lambda ||\sum \alpha_j x_j||$$

in the proof of Lemma 2 is an immediate consequence of the following special case of a known (cf., e.g. [8, p. 22]) result concerning diagonals of linear operators:

Suppose $T : [x_n] \to (\sum X_n)_{l_p}$ is an operator, (x_n) is an unconditional basis with unconditional constant λ, and D is the diagonal of T; i.e. $D : [x_n] \to (\sum X_n)_{l_p}$ is defined by $D(\sum \alpha_n x_n) = \sum \alpha_n P_n T x_n$ where $P_n : (\sum X_n)_{l_p} \to X_n$ is the natural projection. Then $||D|| \leqq \lambda ||T||$.

(Of course, to apply this result to derive the mentioned inequality, one sets $X_n = L_p(B_n)$ and $T : [x_n] \to L_p(\cup B_n) = (\sum L_p(B_n))_{l_p}$ is the natural norm one projection.)

In the proof of the next theorem we use the truncation lemma of Enflo and Rosenthal [3]. Following their notation, we define for $x \in L_p$ and $0 < k < \infty$

$$^k x(t) = x(t), \text{ if } |x(t)| \leqq k$$
$$0, \text{ otherwise.}$$

Before proving Theorem 2, we mention some terminology. Given a f.d.d. (E_n) for X and integers $1 = p_1 < p_2 < \cdots$, let $X_n = [E_i]_{i=p_n}^{p_{n+1}-1}$. (X_n) is clearly a finite dimensional decomposition (f.d.d., in short) for X and is called a block f.d.d. of (E_n).

For $1 \leqq p < \infty$, let us recall (cf. [2]) that a f.d.d. (E_n) is *block p-Hilbertian* (respectively, *block p-Besselian*) provided there is a constant $0 < K < \infty$ so that

$$||\sum x_n|| \leqq K (\sum ||x_n||^p)^{1/p}$$
$$(\text{respectively, } ||\sum x_n|| \geqq K(\sum ||x_n||^p)^{1/p})$$

for any block f.d.d. (X_n) of (E_n) and any $x_n \in X_n$.

THEOREM 2: *Suppose X is a subspace of L_p $(1 < p < 2)$ and there exists $\beta < \infty$ so that every normalized basic sequence in X contains a subsequence which is β-equivalent to the unit vector basis of l_p. If X admits an unconditional f.d.d. (E_n), then X is isomorphic to a subspace of l_p.*

PROOF: By lemma 1 in section 2, we have that (E_n) is block p-Hilbertian. Suppose $0 < \delta < \beta^{-1}$. From [3] it follows that

(*) for each $k < \infty$, there is an integer m so that if $x \in [E_i]_{i=m}^{\infty}$ with $||x|| = 1$, then $||^k x - x|| > \delta$.

Indeed, if (*) were false, then there would exist $k < \infty$, integers $p_1 < p_2 < \cdots$ and $x_n \in [E_i]_{i=p_n}^{p_{n+1}-1}$ with $||x_n|| = 1$ and $||^k x_n - x_n|| \leqq \delta$. But by the hypothesis on X, there is a subsequence (x_{n_j}) of (x_n) so that (x_{n_j}) is β-equivalent to the unit vector basis of l_p. Thus for any sequence (α_j) of scalars, $|| \sum \alpha_j x_{n_j}|| \geqq \beta^{-1}(\sum |\alpha_j|^p)^{1/p}$. But then by Lemma 2.1 of [3], $||^k x_{n_j} - x_{n_j}|| > \delta$ for all but finitely many j. This establishes (*).

Use the uniform absolute continuity of the unit balls of finite dimensional subspaces of L_p and (*) to choose $0 < k_1 < k_2 < \cdots$, integers $1 = m_1 < m_2 < \cdots$, and $\delta/2 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$ so that

(1) if $|A| < \varepsilon_n$ and $x \in [E_i]_{i=1}^{m_n-1}$, then

$$\left(\int_A |x|^p\right)^{1/p} < \varepsilon_{n-1}||x||$$

(2) $||^{k_n} x - x|| > \delta$ for $x \in [E_i]_{i=m_{n+1}}^{\infty}$ with $||x|| = 1$

(3) $k_n^{-p} < \varepsilon_n^{2^{-n}}$.

Set $X_n = [E_i]_{i=m_n}^{m_{n+1}-1}$. We will show that (X_{2n}) and (X_{2n-1}) are p-Besselian and hence l_p decompositions (since (E_n) is block p-Hilbertian). But (X_n) is an unconditional f.d.d., so it will also be an l_p decomposition. This, of course, implies the conclusion of the theorem.

So suppose $x_n \in X_{2n}$. Let $A_n = \{t : k_{2n-1}||x_n|| < |x_n(t)|\}$ and set

$$B_n = A_n \sim \bigcup_{j=n+1}^{\infty} A_j.$$

From (2) we have that

$$\left(\int_{A_n} |x_n|^p\right)^{1/p} > \delta||x_n||. \tag{4}$$

Now it is clear that $|A_n| \leqq k_{2n-1}^{-p}$, hence

$$|\bigcup_{j=n+1}^{\infty} A_j| < \varepsilon_{2n+1} \text{ by (3),}$$

whence from (1) and (4) it follows that

$$\left(\int_{B_n} |x_n|^p\right)^{1/p} > (\delta - \varepsilon_{2n})||x_n|| > (\delta/2)||x_n||. \tag{5}$$

But (B_n) is pairwise disjoint, so if λ is the unconditional constant of (E_n) and we set $K = K(\lambda, \delta/2)$ from Lemma 2, then we have from Lemma 2 that $||\sum x_n|| \geqq K^{-1}(\sum ||x_n||^p)^{1/p}$.

Thus (X_{2n}) is p-Besselian. Similarly, (X_{2n-1}) is p-Besselian, so the proof is complete.

4. The non-separable case

In this section we show that Theorem 1 has a non-separable analogue. (The possibility of such a theorem was suggested to us by J. Lindenstrauss.)

THEOREM 3: *Suppose X is a subspace of $L_P(\mu)(2 < p < \infty)$ for some measure μ. If no subspace of X is isomorphic to l_2, then X is isomorphic to a subspace of $l_P(\Gamma)$ for some set Γ.*

PROOF: It is known that X has a normalized Markuschevich basis $(x_\alpha, x_\alpha^*)_{\alpha\in A}$; i.e., $\|x_\alpha\| = 1$, $[x_\alpha] = X$, and (x_α^*) separates the points of X (so that $[x_\alpha^*] = X^*$ by reflexivity of X). Indeed, Markuschevich proved that separable spaces have Markuschevich bases so the general case follows from this result and transfinite induction by using Lindenstrauss' decomposition of general reflexive spaces via 'long sequences' of projections (cf. [6]).

As is well known, $L_P(\mu)$ is isometric to

$$(\sum_{\beta\in\Lambda} L_p(\mu_\beta))_{l_p(\Lambda)}$$

with each μ_β a finite measure (or, possibly, dim $L_p(\mu_\beta) = 1$). For $\beta\in\Lambda$ denote by P_β the natural projection of $L_p(\mu)$ onto $L_p(\mu_\beta)$.

Given $\beta\in\Lambda$, let $N_\beta = \{x_\alpha : P_\beta x_\alpha \neq 0\}$. Observe that N_β is countable. Indeed, otherwise there is $\varepsilon > 0$ and a sequence (x_{α_n}) in N_β with $P_\beta x_{\alpha_n} \in M_p(\varepsilon)$ and $\inf_n\|P_\beta x_{\alpha_n}\| > 0$. Now $x_{\alpha_n} \xrightarrow{w} 0$, hence $P_\beta x_{\alpha_n} \xrightarrow{w} 0$, whence by Corollary 5 of [5], we may assume by passing to a subsequence of $(P_\beta x_{\alpha_n})$ that $(P_\beta x_{\alpha_n})$ is equivalent to the unit vector basis of l_2. However, since X contains no subspace isomorphic to l_2 and $[x_{\alpha_n}]$ isometrically embeds into L_p, it also follows from [5] that some subsequence of (x_{α_n}) is equivalent to the unit vector basis of l_p. This, of course, is impossible for $p > 2$.

For $\alpha\in A$, let $\Lambda_\alpha = \{\beta\in\Lambda : P_\beta x_\alpha \neq 0\}$. Since

$$x_\alpha = \sum_{\beta\in\Lambda_\alpha} P_\beta x_\alpha$$

it is c lear that Λ_α is countable. Now for $\beta\in\Lambda$, define N_β^n and L_β^n inductively by

$$N_\beta^1 = N_\beta;\ L_\beta^1 = \bigcup_{x_\alpha\in N_\beta} \Lambda_\alpha;$$

$$N_\beta^{n+1} = \bigcup\{N_\gamma : \gamma\in L_\beta^n\}$$

$$L_\beta^{n+1} = \bigcup\{\Lambda_\alpha : x_\alpha\in N_\beta^{n+1}\}.$$

Let

$$A_\beta = \bigcup_{n=1}^{\infty} N_\beta^n;\ \Gamma_\beta = \bigcup_{n=1}^{\infty} L_\beta^n.$$

Certainly A_β and Γ_β are countable, while for $\beta_1 \neq \beta_2$, either $A_{\beta_1} = A_{\beta_2}$ and $\Gamma_{\beta_1} = \Gamma_{\beta_2}$ or $A_{\beta_1} \cap A_{\beta_2} = \phi = \Gamma_{\beta_1} \cap \Gamma_{\beta_2}$. Thus A can be written as a disjoint union $\cup_{\gamma\in\Gamma} B_\gamma$ and there are disjoint sets $(C_\gamma)_{\gamma\in\Gamma} \subseteq A$ so that

(i) B_γ and C_γ are countable

(ii) $X_\gamma \equiv [x_\alpha]_{\alpha\in B_\gamma} \subseteq (\sum_{\beta\in C_\gamma} L_p(\mu_\beta))_{l_p(C_\gamma)}$

By (i) X_γ is separable and thus is λ-isomorphic to a subspace of l_p (where λ depends only on p) by Theorem 1. But $X = (\sum X_\gamma)_{l_p(\Gamma)}$ by (ii), so X is λ-isomorphic to a subspace of $(\sum l_p)_{l_p(\Gamma)} = l_p(\Gamma)$.

REMARK 1: Suppose X is a Banach space every separable subspace of which isomorphically embeds into l_p. Then X embeds into $L_p(\mu)$ for some measure μ by the results of Lindenstrauss and Pelczynski, [7]. If $2 < p < \infty$, then X embeds into $l_p(\Gamma)$ for some Γ by Theorem 3. We don't know if this result is valid also for $1 \leqq p < 2$. (It does follow easily from the truncation lemma of [3] that such an X is separable if $1 \leqq p < 2$ and μ is finite.)

Added in Proof

REMARK 2: Since the truncation lemma of Enflo and Rosenthal [3] is valid for L_1, the proof of Theorem 2 shows that if $X \subseteq L_1$ has an unconditional f.d.d. (E_n) and there is $\lambda > 0$ so that every normalized basic sequence in X has a subsequence λ-equivalent to the unit vector basis of l_1, then there is a blocking (X_n) of (E_n) with (X_n) and l_1 decomposition. In particular, an unconditional f.d.d. for a subspace of l_1 can be blocked to be an l_1 decomposition.

REMARK 3: THEOREM 4: *Let $2 < p < \infty$, let μ be a finite measure, and let X be a non-separable subspace of $L_p(\mu)$. Let $\alpha <$ dens X if the density character of X is the limit of an increasing sequence of smaller cardinals, and set $\alpha =$ dens X otherwise. Then $l_2(\alpha)$ is isomorphic to a subspace of X.*

PROOF: Assume w.l.o.g. that μ is product measure on $\{-1, 1\}^I$ with measure $\{-1\}$ = measure $\{1\} = \frac{1}{2}$ in each coordinate, and that $\{i \in I$: x depends on i for some $x \in X\} = I$ (so that dens $X = |I|$). It is sufficient to show that there is an unconditional basis set A in X with $|A| =$ dens X. Indeed, we then have that $B \equiv A \cap M_p(\varepsilon)$ has cardinality at least α for some $\varepsilon < 0$, hence by Corollary 4 of [5], B is equivalent to the unit vector basis for l_2 $(|B|)$.

By Zorn's lemma there is a subset A of X maximal with respect to

(*) every finite subset of A can be ordered to form a martingale difference sequence.

A is an unconditional basis set by the Burkholder-Gundy Theorem (cf. D. L. Burkholder: Distribution function inequalities for martingales. *Ann. of Prob.* 1 (1973) 19–42, so we complete the proof by showing $|A| = \text{dens } X$.

Let $J = \{i \in I : x \text{ depends on } i \text{ for some } x \in A\}$. Since each x depends on only countably many i, $|J| = |A|$ (the argument below shows that A is infinite so this follows). Letting P be the conditional expectation projection on $L_p(\mu)$ determined by the sub-sigma algebra generated by J, we have that $Px = x$ for each $x \in A$ and dens range $P = |J|$. Now if dens $X > |J|$, then by the reflexivity of X the restriction of P to Xcannot be one to one. Choosing $x \in X$ with $x \neq 0$ and $Px = 0$, we have that $A \cup \{x\}$ satisfies (*), which contradicts the maximality of A. Thus $|J| = \text{dens } X = |I|$, and the proof is complete.

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