COMPOSITIO MATHEMATICA

DAVID K. HSIEH

A characterization of the *B*-realcompact space

Compositio Mathematica, tome 29, nº 1 (1974), p. 63-66

<http://www.numdam.org/item?id=CM_1974__29_1_63_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 1, 1974, pag. 63–66
Noordhoff International Publishing
Printed in the Netherlands

A CHARACTERIZATION OF THE B-REALCOMPACT SPACE

David K. Hsieh

1. Introduction

Realcompact spaces are most often studied and characterized in the framework of the ring $C(X)$ of all continuous functions, both bounded and unbounded, defined on the space X. We abandon this traditional approach to realcompact spaces in favor of a more general yet simpler setting made available by the λ-compactification theory. Thus instead of the ring of all continuous functions, we consider a Banach algebra of functions; instead of both bounded and unbounded functions, we need only to work with the bounded functions. Frolik introduced in [1] the concept of a complete family for unbounded continuous functions and properties of Q-spaces (i.e. realcompact spaces) were derived in terms of complete families. Since we deal with only the bounded functions in the λ-compactification theory, Frolik's definition of complete family cannot be adopted here. In this article, we introduce without using unbounded functions the definition of B-complete family of bounded functions in terms of positive singular functions in a given Banach algebra B, and then characterize the B-realcompact space, which is a generalization of the realcompact space, in terms of the B-complete families.

2. Preliminaries

In general we adopt the notation and definitions in [3] and [4]. Thus we study an arbitrary set E and a Banach algebra B of bounded real-valued functions defined on E. The norm in B is defined by $\|f\| = \sup_{x\in E}|f(x)|$ for each f in B. B is called an admissible Banach algebra if B contains constants and B separates points in E. A positive cone in B is a collection of positive singular functions in B which is closed under addition, and multiplication by positive scalars. A maximal positive cone (m.p.c.) is a positive cone not properly contained in any other positive cone. A m.p.c. M is said to be strong if there exists a function f in M such that $f(x) \neq 0$ for each $x \in E$, otherwise M is said to be weak. A m.p.c. M is said to be free if there exists no point x in E such that $f(x) = 0$ for every f in M.

3. B-realcompact spaces and B-complete family of functions

In the following discussions, E will always denote a set and B an admissible Banach algebra on E.

3.1 DEFINITION: Let A be a subset of E and f be a non-negative function in B. f is said to be strongly bounded on A provided that either there is a point $\bar{x}$ in A such that $f(\bar{x}) = 0$ or there exists an $\varepsilon > 0$ such that $f(x) \geqq \varepsilon$ for each x in A.

NOTATION: When f is strongly bounded on A, we write $f \not\rightarrow 0$ on A.

3.2 DEFINITION: Let $\mathscr{F}$ be a filter of zero sets of functions in B. $\mathscr{F}$ is said to be a maximal ε-filter in (E, B) provided that the set $\mathscr{F}^- = \{f \in B: f \geqq 0, [f, \varepsilon] \in \mathscr{F}$ for each $\varepsilon > 0\}$ is a m.p.c. (maximal positive cone) in B where $[f, \varepsilon]$ denotes the set $\{x \in E: |f(x)| \leqq \varepsilon\}$.

For latter discussion, we shall need the following lemma which is proved in [4].

3.3 LEMMA: *$\mathscr{F}$ is a maximal ε-filter if and only if given a non-negative function f in B if for every $\varepsilon > 0$ the set $[f, \varepsilon] = \{x \in E: |f(x)| \leqq \varepsilon\}$ meets every member of $\mathscr{F}$ then $[f, \varepsilon] \in \mathscr{F}$ for every $\varepsilon > 0$.*

3.4 LEMMA: *Let M be a m.p.c. in B. Let $M^* = \{[f, \varepsilon]: \varepsilon > 0, f \in M\}$ where $[f, \varepsilon]$ denotes the set $\{x \in E: |f(x)| \leqq \varepsilon\}$. Then M^* is a maximal ε-filter.*

PROOF: Since M is a positive cone, by definition $[f, \varepsilon] \neq \emptyset$ for each $f \in M$. Now let $[f, \varepsilon]$ and $[g, \delta]$ be in M^*. Clearly $[f, \varepsilon] \cap [g, \delta] \supset [f+g, \varepsilon \wedge \delta]$.

Finally we show the following: if $Z(h) \supset [f, \varepsilon]$ for some $[f, \varepsilon]$ in M^* then $Z(h) \in M^*$. First we define a sequence g_n in B as below; for each positive integer n, let

$$g_n = \left[|h(x)| + \frac{\varepsilon}{f(x) \vee 1/n} \right] \wedge [|h(x)| + 1].$$

It can be readily verified that g_n converges uiformly to a function $g(x)$ where $g(x) = 1$ if $x \in [f, \varepsilon]$ and $g(x) = |h(x)| + \varepsilon / f(x)$ if $x \notin [f, \varepsilon]$.Since B is a Banach algebra with the norm defined by $\|f\| = \sup \{|f(x)|: x \in E\}$, g is in B. In fact g is a non-negative function in B. We now claim gf is in M. This claim follows from the observation: (i) $gf \leqq \|g\| f$; (ii) $\|g\| f \in M$; and (iii) M is a m.p.c. However $Z(h)$ is precisely the set $[gf, \varepsilon]$. Hence $Z(h) \in M$. This completes the proof that M^* is a filter of zero sets. Since M is a m.p.c., M^* is a maximal ε-filter by definition.

3.5 Definition: A subfamily of non-negative functions $H \subset B$ is said to be complete provided that given a maximal ε-filter $\mathscr{F}$, if for every $f \in H$ there exists a $Z_f \in \mathscr{F}$ such that $f \not\rightarrow 0$ on Z_f then $\bigcap \mathscr{F} \neq \emptyset$.

3.6 Theorem: *Suppose $\mathscr{F}$ is a maximal ε-filter, and suppose that for each sequence $\{F_n\}$ in $\mathscr{F}$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Then for each non-negative function f in B there is a Z_f in $\mathscr{F}$ such that $f \not\rightarrow 0$ on Z_f.*

Proof: For each positive integer n, let $[f, 1/n]$ denote the set $\{x \in E: |f(x)| \leqq 1/n\}$. We consider two cases.

Case I: Suppose there is an integer k such that $[f, 1/k] \notin \mathscr{F}$. By Lemma 3.3, there is some integer t and there is a $Z_f \in \mathscr{F}$ such that $[f, 1/t] \cap Z_f = \emptyset$. Hence $f > 1/t$ on Z_f.

Case II: On the other hand, suppose $[f, 1/n] \in \mathscr{F}$ for each n. Since each sequence in $\mathscr{F}$ has non-empty intersection, there is a point $\bar{x}$ in $\bigcap_{n=1}^{\infty} [f, 1/n]$. Clearly $f(\bar{x}) = 0$. Therefore in either case $f \not\rightarrow 0$ on some member of $\mathscr{F}$.

3.7 Definition: E is said to be B-realcompact provided that every free m.p.s. in B is strong.

Remark: It can be readily verified that a topological space X is realcompact if and only if X is $C^*(X)$-realcompact where $C^*(X)$ is the Banach algebra of all bounded real-valued continuous functions.

3.8 Theorem: *The following statements are equivalent*:

(i) *E is B-realcompact.*

(ii) *If $\mathscr{F}$ is a maximal ε-filter in (E, B) such that each sequence in $\mathscr{F}$ has non-empty intersection, then $\bigcap \mathscr{F} \neq \emptyset$.*

(iii) *The set $B^+ = \{f \in B: f \geqq 0\}$ is a B-complete family.*

(iv) *$G = \{f \in B: f \rhd 0$ on $E\}$ is B-complete, where $f \rhd 0$ means that $f > 0$ and for each $\varepsilon > 0$ there exists $x \in E$ such that $f(x) < \varepsilon$.*

(v) *There exists a B-complete family H in B.*

Proof:

(i) $\Rightarrow$ (ii). Suppose there exists a maximal ε-filter $\mathscr{F}$ such that each sequence in $\mathscr{F}$ has non-empty intersection, but $\bigcap \mathscr{F} = \emptyset$. By definition 3.2, the set $\mathscr{F}^- = \{ f \in B: f \geqq 0, [f, \varepsilon] \in \mathscr{F}$ for each $\varepsilon > 0\}$ is a m.p.c. in B. Since $\bigcap \mathscr{F} = \emptyset$, $\mathscr{F}^-$ is a free m.p.c. For each $f \in \mathscr{F}^-$, $[f, 1/n] \in \mathscr{F}$ for every integer n. Thus $\bigcap_{n=1}^{\infty} [f, 1/n] \neq \emptyset$. Suppose that y is in $\bigcap_{n=1}^{\infty} [f, 1/n]$. Clearly $f(y) = 0$. That is $\mathscr{F}^-$ is a weak free m.p.c. in B. Hence E is not B-realcompact.

(ii) $\Rightarrow$ (i). Suppose E is not B-realcompact. There exists a weak free m.p.c. M in B. Let $M^* = \{[f, \varepsilon]: \varepsilon > 0, f \in M\}$ where $[f, \varepsilon]$ denotes the set $\{x \in E: |f(x)| \leqq \varepsilon\}$. It follows from Lemma 3.4 that M^* is a maximal ε-filter. Let $\{[f_n, \varepsilon_n]\}$ be a sequence in M^*. We may assume $\|f_n\| = 1$ for each n. Hence $f = \sum_{n=1}^{\infty} f_n/2^n$ is in M. Since M is a weak m.p.c., there exists $\bar{x}$ in E such that $f(\bar{x}) = 0$. Hence $f_n(\bar{x}) = 0$ for each n. It follows that $\bar{x} \in \bigcap_{n=1}^{\infty} [f_n, \varepsilon_n] \neq \emptyset$. But $\bigcap M^* = \emptyset$ *as* M is a free m.p.c.

(iii) $\Rightarrow$ (ii). It follows directly from Theorem 3.6 and Definition 3.5.

(ii) $\Rightarrow$ (iii). Let $\mathscr{F}$ be a maximal ε-filter. Suppose for each $f \in B^+$, there exists Z_f in $\mathscr{F}$ such that $f \not\Rightarrow 0$ on Z_f. To show that B^+ is B-complete, it suffices, in view of (ii), to show each sequence in $\mathscr{F}$ has non-empty intersection. Assume the contrary: suppose that there exists a sequence $\{Z_n\}$ in $\mathscr{F}$ such that $\bigcap_{n=1}^{\infty} Z_n = \emptyset$. Since $\mathscr{F}$ is a maximal ε-filter, $\mathscr{F}^- = \{f \in B: f \geqq 0, [f, \varepsilon] \in \mathscr{F}$ for every $\varepsilon > 0\}$ is a m.p.c. in B. Write $Z_n = Z(f_n)$, the zero set of f_n, with $f_n \in B^+$ and $\|f_n\| \leqq 1$. Obviously $[f_n, \varepsilon] \in \mathscr{F}$ for each $\varepsilon > 0$ and each n. Thus $f_n \in \mathscr{F}^-$ for each n. Define $f = \sum_{n=1}^{\infty} f_n/2^n$. Clearly f is in $\mathscr{F}^-$ as $\mathscr{F}^-$ is a m.p.c. But $\bigcap_{n=1}^{\infty} Z(f_n) = \emptyset$, it follows that $f > 0$. Now recall that $Z_f \in \mathscr{F}$, and let $\varepsilon > 0$ be given. Choose n so large that $1/2^n < \varepsilon$. Since $\mathscr{F}$ is a filter, there exists $\bar{x}$ in $Z_1 \cap \cdots \cap Z_n \cap Z_f$. Clearly $f(\bar{x}) \leqq 1/2^n < \varepsilon$. This contradicts that $f \not\Rightarrow 0$ on Z_f.

(iii) $\Rightarrow$ (iv). Suppose $\mathscr{F}$ is a maximal ε-filter and that for each $f \in G$ there exists a $Z_f \in \mathscr{F}$ with $f \not\Rightarrow 0$ on Z_f. To show $\bigcap \mathscr{F} \neq \emptyset$, let $g \in B^+ \sim G$. Being the zero set of the zero function, E is in $\mathscr{F}$. Clearly $g \not\Rightarrow 0$ on E which is in $\mathscr{F}$. Since B^+ is B-complete $\bigcap \mathscr{F} \neq \emptyset$.

(iv) $\Rightarrow$ (v). Trivial.

(v) $\Rightarrow$ (ii). It follows immediately from Theorem 3.6 and Definition 3.5.

REFERENCES

[1] Z. Frolik: Applications of complete families of continuous functions to the theory of Q-spaces. *Czech. Math. J. 11* (1961) 115–132.

[2] L. Gillman and M. Jerison: *Rinfs of Continuous Functions*. Van Nostrand, Princeton, 1960.

[3] D. Hsieh: Topological properties associated with the λ-map. *Bull. de l'Acad. Pol. des Sc. XX* (1972) 1–6.

[4] D. Hsieh: The ε-filters and maximal positive cones (to appear).

[5] E. Lorch: Compactification, Baire functions, and Daniell integration. *Acta Sc. Math. (Szeged) XXIV* (1963) 204–218.

[6] J. Mack: Countable paracompactness and weak normality properties. *Trans. Amer. Math. Soc. 148* (1970) 365–272.

(Oblatum 8–XI–1973 & 13–V–1974)

Mathematics Department
City University of New York
The City College
New York, N.Y. 10031 U.S.A.