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HECKE THEORY FOR $GL(3)$

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In view of the recent results of Gelfand, Kajdan, and one of the authors ([1], [6]), it appears likely that the results of [2] – the Hecke theory – will extend to all groups $GL(p)$. In this note, we present nearly complete results for the case $p = 3$. To avoid technical difficulties we restrict ourselves to the case of a function field.

1. Global computations

Let F be a commutative field, G (resp. G') the group $GL(p)$ (resp. $GL(p-1)$) regarded as an algebraic group defined over F . We regard G' as imbedded into G by the map

$$g \rightarrow \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote by Z (resp. Z') the center of G (resp. G') and by ${}^t g$ the transpose of a matrix g . The entries of a matrix g in G are written as g_{ij} , the first index being the column index.

In this section we take F to be a function field whose field of constants has cardinality Q . We let \mathbb{A} be the ring of adèles of F and \mathbb{I} the group of ideles. Then $G_{\mathbb{A}} = GL(p, \mathbb{A})$ is a locally compact group of which $G_F = GL(p, F)$ is a discrete subgroup. A cusp form on $G_{\mathbb{A}}$ is a complex-valued function ϕ , which satisfies the following three conditions:

(1.1) for all $g \in G_{\mathbb{A}}$, $a \in Z_{\mathbb{A}} = \mathbb{I}$, and $\gamma \in G_F$, $\phi(a\gamma g) = \omega(a)\phi(g)$,

where ω is a quasi-character of \mathbb{I}/F^{\times} ,

(1.2) the function ϕ is invariant on the right by a compact open subgroup of $G_{\mathbb{A}}$,

(1.3) if P is a proper F -parabolic subgroup of G and U its unipotent radical, then

$$\int_{U_F \backslash U_{\mathbb{A}}} \phi(ug) du = 0, \quad \text{for all } g \in G_{\mathbb{A}}.$$

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Let N be the group of p by p upper triangular matrices whose diagonal entries are one. Let $N' = N \cap G'$ be the corresponding subgroup of G' . Choose a non-trivial character ψ of \mathbb{A}/F and define a character θ of $N_{\mathbb{A}}$ by the formula

$$(1.4) \quad \theta(n) = \prod_{1 \leq i \leq p-1} \psi(n_{i+1, i}).$$

Let ϕ be a cusp form on $G_{\mathbb{A}}$. The function

$$(1.5) \quad W(g) = \int_{N_F \backslash N_{\mathbb{A}}} \phi(n g) \bar{\theta}(n) dn$$

is called the ‘Whittaker function’ attached to ϕ . It transforms on the left according to the formula

$$W(n g) = \theta(n) W(g), \quad \text{for } n \in N_{\mathbb{A}},$$

and the form ϕ has the following Fourier expansion:

$$(1.6) \quad \phi(g) = \sum_{\gamma \in N'_F \backslash G'_F} W(\gamma g).$$

Now let ϕ' be a cusp form on $G'_{\mathbb{A}}$ and let us compute the integral

$$(1.7) \quad \int_{G'_F \backslash G'_{\mathbb{A}}} \phi(g) \phi'(g) |\det g|^s dg,$$

where $s \in \mathbb{C}$ and dg is an invariant measure on $G'_F \backslash G'_{\mathbb{A}}$. Replacing ϕ by (1.6), we obtain

$$\begin{aligned} \int_{G'_F \backslash G'_{\mathbb{A}}} \sum_{N'_F \backslash G'_F} W(\gamma g) \phi'(g) |\det g|^s dg &= \int_{G'_F \backslash G'_{\mathbb{A}}} \sum_{N'_F \backslash G'_F} W(\gamma g) \phi'(\gamma g) |\det(\gamma g)|^s dg \\ &= \int_{N'_F \backslash G'_{\mathbb{A}}} W(g) \phi'(g) |\det g|^s dg \\ &= \int_{N'_{\mathbb{A}} \backslash G'_{\mathbb{A}}} |\det g|^s dg \int_{N'_F \backslash N'_{\mathbb{A}}} W(n g) \phi'(n g) dn \\ &\quad \int_{N'_{\mathbb{A}} \backslash G'_{\mathbb{A}}} |\det g|^s W(g) dg \int_{N'_F \backslash N'_{\mathbb{A}}} \theta(n) \phi'(n g) dn. \end{aligned}$$

Let W' be the Whittaker function attached to ϕ' and ε the $p-1$ by $p-1$ matrix defined by

$$\varepsilon_{ij} = \delta_{i, j} (-1)^i.$$

Since $\theta(\varepsilon n \varepsilon^{-1}) = \theta(n^{-1})$, in the last line the inner integral is actually $W'(\varepsilon g)$. Thus (1.7) is equal to

$$(1.8) \quad \int_{N_{\mathbb{A}} \backslash G_{\mathbb{A}}} W(g)W'(\varepsilon g)|\det g|^s dg.$$

More precisely, for Res sufficiently large, all the above integrals converge and are equal.

Since ϕ and ϕ' are compactly supported modulo $G_F Z_{\mathbb{A}}$ and $G'_F Z'_{\mathbb{A}}$ respectively, the integral (1.7) is always convergent. Hence, the integral (1.8), which converges only in a half-space, is, as a function of s , a polynomial in Q^{-s} , Q^s .

Now, let w be the element of G_F defined by

$$w_{ij} = 0 \quad \text{if } i+j \neq p+1, \quad w_{p+1-i,i} = (-1)^{i-1} \quad (1 \leq i, j \leq p)$$

and let w' be the corresponding element of G'_F . Clearly

$$wNw^{-1} = {}^tN, \quad \theta(w^t n^{-1} w^{-1}) = \theta(n).$$

In particular, the functions $\tilde{\phi}$ and $\tilde{\phi}'$ defined by

$$\tilde{\phi}(g) = \phi(w^t g^{-1}) = \phi({}^t g^{-1}), \quad \tilde{\phi}'(g) = \phi'(w'^t g^{-1}) = \phi'({}^t g^{-1})$$

are automorphic forms on $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$ respectively, whose Whittaker functions are the functions \tilde{W} and \tilde{W}' given by

$$\tilde{W}(g) = W(w^t g^{-1}), \quad \tilde{W}'(g) = W'(w'^t g^{-1}).$$

Changing g into ${}^t g^{-1}$ in the integral (1.7), we easily obtain that (1.7) is also equal to the following integral:

$$(1.9) \quad \int_{G_F \backslash G_{\mathbb{A}}} \tilde{\phi}(g)\tilde{\phi}'(g)|\det g|^{-s} dg,$$

and conclude that (1.8) and the integral

$$(1.10) \quad \int_{N'_{\mathbb{A}} \backslash G'_{\mathbb{A}}} \tilde{W}(g)\tilde{W}'(g)|\det g|^{-s} dg$$

are equal in the sense of analytic continuation – as polynomials in Q^{-s} , Q^s .

2. Local conjectures

From now on and until Section 5, the field F will be a non-archimedean local field whose residual field has q elements. We denote by $|x|$ or $\alpha_F(x)$ or simply $\alpha(x)$ the module of an x in F . We denote by $M(p \times r, F)$ the space of matrices with p columns and r rows whose entries belong to F , and by $\mathcal{S}(p \times r, F)$ the space of locally constant compactly supported complex-valued functions on $M(p \times r, F)$.

Fix a non-trivial character ψ of F and let θ be the character of N_F defined by (1.4). Let π be an irreducible admissible representation of G_F . We say that π is non-degenerate if it can be realized by right translations in a space \mathcal{W} of functions W on G_F satisfying

$$W/ng) = \theta(n)W(g), \quad \text{for all } n \text{ in } N_F, g \text{ in } G_F.$$

If π is non-degenerate the space \mathcal{W} is unique and denoted by $\mathcal{W}(\pi, \psi)$. It is called the Whittaker model of π ([1], [6]).

Since $'g$ and g are always conjugate in G_F , the representation $\tilde{\pi}$ contra-redient to π is equivalent to the representation $g \rightarrow \pi('g^{-1})$ ([1]). In particular, if π is non-degenerate so is $\tilde{\pi}$. More precisely, if W belongs to $\mathcal{W}(\pi, \psi)$ then the function \tilde{W} defined by

$$\tilde{W}(g) = W(w'g^{-1})$$

belongs to $\mathcal{W}(\tilde{\pi}, \psi)$.

Let π and π' be non-degenerate representations of G_F and G'_F respectively. For $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi)$ and $s \in \mathbb{C}$, we set

$$\Psi(s, W, W') = \int_{N_F \backslash G'_F} W(g)W'(\varepsilon g)|\det g|^{s-\frac{1}{2}} dg.$$

The global computations of Section 1 lead to the formulation of the following conjectures.

- (2.1) For Res sufficiently large the integrals $\Psi(s, W, W')$ and $\Psi(s, \tilde{W}, \tilde{W}')$ are absolutely convergent.
- (2.2) They are rational functions of q^{-s} . More precisely, for $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi)$ the integrals $\Psi(s, W, W')$ span a fractional ideal $\mathbb{C}[q^{-s}, q^s]L(s, \pi \times \pi')$ of the ring $\mathbb{C}[q^{-s}, q^s]$; the factor $L(s, \pi \times \pi')$ has the form $1/P(q^{-s})$ where $P \in \mathbb{C}[X]$ and $P(0) = 1$.

There is a similar factor $L(s, \tilde{\pi} \times \tilde{\pi}')$.

- (2.3) There is a factor $\varepsilon(s, \pi \times \pi', \psi)$ of the form cq^{-ms} such that
- $$\begin{aligned} \Psi(1-s, \tilde{W}, \tilde{W}')/L(1-s, \tilde{\pi} \times \tilde{\pi}') \\ = \varepsilon(s, \pi \times \pi', \psi)\omega'(-1)^p \Psi(s, W, W')/L(s, \pi \times \pi') \end{aligned}$$
- for $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi)$, where ω' is the quasi-character of F^\times such that $\pi'(a1_{p-1}) = \omega'(a) \cdot 1$.

Let us emphasize that in the above statements the pairs (π, π') and $(\tilde{\pi}, \tilde{\pi}')$ play symmetrical roles, since, in fact, \tilde{W} (resp. \tilde{W}') belongs to $\mathcal{W}(\tilde{\pi}, \psi)$ (resp. $\mathcal{W}(\tilde{\pi}', \psi)$) if W (resp. W') belongs to $\mathcal{W}(\pi, \psi)$ (resp. $\mathcal{W}(\pi', \psi)$).

These conjectures have been proved for $p = 2$ (see below), and, in [1], for all p under the additional assumption that π is supercuspidal. Indeed,

under this assumption, the restriction of a W in $\mathcal{W}(\pi, \psi)$ to G'_F belongs to the space $C_c^\infty(G', \theta)$ of all maps f from G'_F to \mathbb{C} which, on the left, transform according to

$$f(ng) = \theta(n)f(g), \quad n \in N'_F,$$

and are locally constant, and compactly supported modulo N'_F . In fact, the map $W \rightarrow W|_{G'_F}$ is a bijection of $\mathcal{W}(\pi, \psi)$ onto $C_c^\infty(G', \theta)$. It follows that the integrals are convergent for all s and satisfy (2.2) with $L(s, \pi \times \pi') = L(s, \tilde{\pi} \times \tilde{\pi}') = 1$. In order to prove (2.3) one need only show that there is a constant c such that, for all W in $\mathcal{W}(\pi, \psi)$ and W' in $\mathcal{W}(\pi', \psi)$,

$$\int_{N'_F \backslash G'_F} \tilde{W}(g)\tilde{W}'(\varepsilon g)dg = c \int_{N'_F \backslash G'_F} W(g)W'(\varepsilon g)dg.$$

Indeed, the left hand side defines a bilinear form B on the product $C_c^\infty(G', \theta) \times \mathcal{W}(\pi', \psi)$. It satisfies the following invariance condition:

$$B(\rho(g)f, \pi'(g)W') = B(f, W'), \quad \text{where } (\rho(g)f)(h) = f(hg).$$

Therefore, it must of the form

$$B(f, W') = \int_{N'_F \backslash G'_F} \lambda(\pi'(g)W')f(g)dg,$$

where λ is a linear form on $\mathcal{W}(\pi', \psi)$ such that

$$\lambda(\pi'(n)W') = \bar{\theta}(n)\lambda(W'), \quad \text{for } n \in N'_F.$$

The uniqueness of the Whittaker model shows that λ must be proportional to the linear form $W' \rightarrow W'(\varepsilon)$ and our assertion follows.

In the next sections we call the factor $L(s, \pi \times \pi')$ the 'g.c.d.' of the integrals $\Psi(s, W, W')$ and set

$$\varepsilon'(s, \pi \times \pi', \psi) = \varepsilon(s, \pi \times \pi', \psi)L(1-s, \tilde{\pi} \times \tilde{\pi}')/L(s, \pi \times \pi').$$

We shall use also the same notation and terminology for other integrals which have similar properties.

3. The case $p = 2$

In this section we review the case $p = 2$. Accordingly, $G = GL(2)$, $G' = GL(1)$, $G'_F = F^\times$,

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$w' = 1$. Moreover, $w^t g^{-1} w^{-1} = \det g^{-1} \cdot g$ for g in G_F .

Let π and π' be as above. Let ω be the quasi-character of F^\times such that

$$\pi \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega(a) \cdot 1.$$

Then $\tilde{\pi}$ is equivalent to the representation $\pi \otimes \omega^{-1}$ (i.e. the representation $g \rightarrow \pi(g)\omega^{-1}(\det g)$). Moreover, for W in $\mathcal{W}(\pi, \psi)$ the function \tilde{W} is given by

$$\tilde{W}(g) = W(gw)\omega^{-1}(\det g).$$

On the other hand, the representation π' is just a quasi-character of F^\times and the function W' coincides with the function π' . Therefore the integral $\Psi(s, W, W')$ reduces to the integral

$$\int_{F^\times} W \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} \pi'(a) d^\times a$$

and the integral $\Psi(s, \tilde{W}, \tilde{W}')$ to

$$\int_{F^\times} W \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w \right] |a|^{s-\frac{1}{2}} \omega^{-1} \pi'^{-1}(a) d^\times a.$$

Hence, the results of [2] show that (2.1) to (2.3) are true with

$$L(s, \pi \times \pi') = L(s, \pi \otimes \pi'), \quad \varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \pi \otimes \pi', \psi).$$

We shall need the fact that these factors are related to other integrals as well. Indeed, let V be the space of π , \tilde{V} the space of $\tilde{\pi}$, and $\langle \cdot, \cdot \rangle$ the invariant bilinear form on $V \times \tilde{V}$. A coefficient of π is a function f of the form

$$f(g) = \langle \pi(g)v, \tilde{v} \rangle$$

where v is in V and \tilde{v} in \tilde{V} . The function \check{f} defined by $\check{f}(g) = f(g^{-1})$ is a coefficient of $\tilde{\pi}$.

Let f be a coefficient of π , Φ a function in $\mathcal{S}(2 \times 2, F)$ and s a complex number; we set

$$Z(\Phi, s, f) = \int_{G_F} f(x) |\det x|^s \Phi(x) d^\times x,$$

where $d^\times x$ is a multiplicative Haar measure on G_F . These integrals converge for Res large, the 'g.c.d.' of the integrals $Z(\Phi, s + \frac{1}{2}, f)$ is the factor $L(s, \pi) = L(s, \pi \times 1)$, and the integrals satisfy the functional equation

$$(3.1) \quad Z(\hat{\Phi}, \frac{3}{2} - s, \check{f}) = \varepsilon'(s, \pi, \psi) Z(\Phi, s + \frac{1}{2}, f)$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ :

$$\hat{\Phi}(x) = \int_{M(2, F)} \Phi(y) \psi(\text{Tr}(yx)) dy,$$

Tr denoting the trace of an element of $M(2 \times 2, F)$ and dy being the self-dual Haar measure on $M(2 \times 2, F)$.

At least when π is supercuspidal, this can be derived from the following lemma combined with the fact that on $V \times \tilde{V} = \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tilde{\pi}, \psi)$ the invariant bilinear form is given by

$$\langle W_1, W_2 \rangle = \int_{F^\times} W_1 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} W_2 \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} d^\times a.$$

LEMMA (3.2): *Suppose Φ is in $\mathcal{S}(2 \times 2, F)$. Then*

$$\begin{aligned} \int_{F^3} \Phi \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix} \right] \psi(-b)\psi(-x) da db dx \\ = \int_F \hat{\Phi} \left[w^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right] \psi(x) dx. \end{aligned}$$

The lemma itself is a simple consequence of the Fourier theorem in one variable.

Finally, we shall need the results of [3] that we now state in a form appropriate to our purposes. Let π_1 and π_2 be non-degenerate representations of $G_F = GL(2, F)$. For $\Phi \in \mathcal{S}(1 \times 2, F)$, $W_1 \in \mathcal{W}(\pi_1, \psi)$, and $W_2 \in \mathcal{W}(\pi_2, \psi)$, we set

$$\Psi(s, W_1, W_2, \phi) = \int_{N_F \backslash G_F} W_1(g) W_2(\varepsilon g) \phi[(0, 1)g] |\det g|^s dg$$

where

$$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then again, the integrals are convergent for Res large, have a g.c.d. $L(s, \pi \times \pi')$, and satisfy a functional equation

$$(3.3) \quad \Psi(1-s, \tilde{W}_1, \tilde{W}_2, \hat{\phi}) = \omega_2(-1) \varepsilon'(s, \pi_1 \times \pi_2, \psi) \Psi(s, W_1, W_2, \phi)$$

where $\hat{\phi}$ is the Fourier transform of ϕ :

$$\hat{\phi}(x, y) = \int_{F^2} \phi(u, v) \psi(ux + vy) du dv,$$

and ω_i , for $i = 1, 2$, is the quasi-character of F^\times defined by

$$\pi_i \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega_i(a) \cdot 1.$$

4. The case $p = 3$

In this section $p = 3$; accordingly, $G = GL(3)$, $G' = GL(2)$,

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad w' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In addition, we let U be the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Fix an admissible irreducible representation σ of G'_F and a quasi-character μ of F^\times . Let \mathcal{H} be the space of σ and denote by \mathcal{V} the space of all locally constant mappings f from G_F to \mathcal{H} , such that

$$f \left[\begin{pmatrix} m & 0 \\ u & a \end{pmatrix} g \right] = \sigma(m) |\det m|^{-\frac{1}{2}} \mu \alpha(a) f(g)$$

for all m in $G'_F = GL(2, F)$, a in F^\times , and g in G_F . We call π_0 the representation of G_F in \mathcal{V} defined by right translations. From the results of [4], one can prove that any irreducible admissible representation of G_F which is not supercuspidal is a component of π_0 , for a suitable choice of σ and μ . Furthermore, according to a result of Rodier, if π_0 has a non-degenerate component, then σ itself is non-degenerate. So, from now on, we assume σ to be so. We also let \mathcal{H} be the space $\mathcal{W}(\sigma, \psi)$. Then, one can show that π_0 has exactly one non-degenerate component, π say.

In order to show that (2.1) to (2.3) are true for any non-degenerate representation of G_F , it suffices to show that they are true for such a π . In these notes, we assume in addition that π_0 is irreducible and sketch a proof of the fact that $\pi = \pi_0$ is then non-degenerate and that the conjectures are true for π and $\tilde{\pi}$.

Let $\mathcal{S} = \mathcal{S}(3 \times 2, F) \otimes \mathcal{H}$ be the space of all locally constant compactly supported maps from $M(3 \times 2, F)$ to \mathcal{H} . If Φ is in \mathcal{S} , then the function f defined by

$$(4.1) \quad f(g) = \mu \alpha(\det g) \int_{G'_F} \mu \alpha^{\frac{3}{2}}(\det m) \sigma(m^{-1}) \Phi(vmg) d^\times m,$$

where

$$v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is an element of \mathcal{V} – provided the integral converges. For $g \in G_F$, $f(g)$ is

a function on G'_F whose value at $h \in G'_F$ is denoted by $f(g; h)$. A similar notation is used for Φ . Accordingly, $f(g; e)$ is simply

$$f(g; e) = \mu\alpha(\det g) \int_{G'_F} \mu\alpha^{\frac{3}{2}}(\det m)\Phi(vmg; m^{-1})d^\times m.$$

Set now

$$(4.2) \quad W(g) = \int_{U_F} f(ug; e)\bar{\theta}(u)du.$$

Then, when W is defined, it satisfies

$$W(ng) = \theta(n)W(g), \quad \text{for } n \in N_F.$$

Suppose now that Φ has the form

$$(4.3) \quad \Phi \left[\begin{pmatrix} a & b & x \\ c & d & y \end{pmatrix}; g \right] = \phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi_2(x, y)W_1(g),$$

where ϕ_1 is in $\mathcal{S}(2 \times 2, F)$, ϕ_2 in $\mathcal{S}(2 \times 1, F)$, and W_1 in $\mathcal{W}(\sigma, \psi)$. Then, for $g \in G'_F$,

$$\begin{aligned} W(g) &= \int_{U_F} f(gg^{-1}ug; e)\bar{\theta}(u)du \\ &= \alpha^{\frac{1}{2}}(\det g) \int_{U_F} f(u; g)\bar{\theta}(gug^{-1})du \\ &= \alpha^{\frac{1}{2}}(\det g) \int_{G'_F \cdot F^2} \phi_1(m)\phi_2[(x, y)^t m]W_1(gm^{-1})\psi[(0, -1)g^t(x, y)] \\ &\quad \times u\alpha^{\frac{3}{2}}(\det m)d^\times m dx dy. \end{aligned}$$

Let ϕ be the co-Fourier-transform of ϕ_2 :

$$\phi(u, v) = \int_{F^2} \phi_2(x, y)\psi[-(u, v)^t(x, y)]dx dy.$$

Then, we may write the above integral as

$$(4.4) \quad W(g) = \alpha^{\frac{1}{2}}(\det g) \int_{G'_F} \phi_1(m)\phi[(0, 1)gm^{-1}]W_1(gm^{-1})\mu\alpha^{\frac{1}{2}}(\det m)d^\times m,$$

an integral which converges 'better' than (4.2). It can be used to define W even though (4.2) might not converge. Then, it can be shown that π is non-degenerate and W belongs to $\mathcal{W}(\pi, \psi)$. Actually, since the space \mathcal{S} is spanned by functions of the form (4.3), the space $\mathcal{W}(\pi, \psi)$ is spanned by functions W for which $W(g)$, $g \in G'_F$, is given by (4.4).

We also need an integral representation for $\bar{W}(g)$, $g \in G'_F$. Now, if g is in G'_F ,

$$\begin{aligned}\tilde{W}(g) &= W(w'g^{-1}) = \int_{U_F} f(uw'g^{-1}; e)\bar{\theta}(u)du \\ &= \mu\alpha(\det g^{-1}) \int_{G'_F \cdot U_F} \mu\alpha^{\frac{3}{2}}(\det m)\Phi[vmuw'g^{-1}; m^{-1}]\bar{\theta}(u)d^\times m du.\end{aligned}$$

More explicitly,

$$u = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta(u) = \psi(b), \quad du = dadb,$$

and $vmuw'g^{-1}$ is the 3 by 2 matrix obtained by the juxtaposition of the 2 by 2 matrix

$$m \begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix} {}^t g^{-1}$$

and the 1 by 2 matrix $m \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence, the above integral may be written in the form

$$\begin{aligned}\mu\alpha(\det g^{-1}) \int_{G'_F \cdot F^2} \mu\alpha^{\frac{3}{2}}(\det m)\phi_1 \left[m \begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix} {}^t g^{-1} \right] \phi_2[(1, 0){}^t m] \\ \times W_1(m^{-1})\psi(-b)dadb d^\times m.\end{aligned}$$

The integral on G'_F can be realized as an integral on N'_F and then on G'_F/N'_F . In this way, one obtains

$$\begin{aligned}\mu\alpha(\det g^{-1}) \int_{G'_F/N'_F} \mu\alpha^{\frac{3}{2}}(\det m)\phi_2[(1, 0){}^t m]W_1(m^{-1})d^\times m \\ \times \int \phi_1 \left[mn \begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix} {}^t g^{-1} \right] \psi(-b)dadb\bar{\theta}(n)dn.\end{aligned}$$

Using Lemma (3.2) to transform the inner integral, one arrives at

$$\begin{aligned}\alpha\mu^{-1}(\det g) \int_{G'_F/N'_F} \mu\alpha^{-\frac{3}{2}}(\det m)\phi_2[(1, 0){}^t m]W_1(m^{-1})d^\times m \\ \times \int_{N'_F} \hat{\phi}_1[{}^t gw'^{-1}nm^{-1}]\theta(n)dn.\end{aligned}$$

At this point, one puts back together the integrations on G'_F/N'_F and N'_F and changes m into ${}^t m^{-1}{}^t gw'^{-1}$ to arrive at the following formula:

$$(4.5) \quad \tilde{W}(g) = \alpha^{\frac{1}{2}}(\det g) \int_{G'_F} \mu^{-1}\alpha^{\frac{3}{2}}(\det m)\hat{\phi}_1({}^t m)\hat{\phi}[(0, 1)gm^{-1}] \\ \times \tilde{W}_1(gm^{-1})d^\times m.$$

Here $\hat{\phi}_1'(m) = \hat{\phi}_1({}^t m)$ and $\hat{\phi} = \phi_2$ is the Fourier transform of ϕ .

Now let π' be a non-degenerate representation of G'_F . For W' in $\mathcal{W}(\pi', \psi)$, the integral $\Psi(s, W, W')$ may be written as a double integral with respect to $g \in N'_F \backslash G'_F$ and $m \in G'_F$. After changing g into gm we find that

$$(4.6) \quad \Psi(s, W, W') = \int_{G'_F} \phi_1(m) \mu \alpha^{s+\frac{1}{2}}(\det m) f(s, m) d^\times m$$

where

$$f(s, m) = \Psi(s, W_1, \pi'(m)W', \phi).$$

Since, for a fixed s , the function $m \rightarrow f(s, m)$ is a coefficient of π' , it is not too hard to see that (2.1) and (2.2) are true with

$$L(s, \pi \times \pi') = L(s, \pi' \times \sigma) L(s, \pi' \otimes \mu).$$

Similarly, one finds that

$$(4.7) \quad \Psi(s, \tilde{W}, \tilde{W}') = \int_{G'_F} \hat{\phi}_1(m) \mu^{-1} \alpha^{s+\frac{1}{2}}(m) f'(s, {}^t m) d^\times m$$

where

$$f'(s, m) = \Psi(s, \tilde{W}_1, \tilde{\pi}'(m)\tilde{W}', \hat{\phi}).$$

Now one observes that

$$\tilde{\pi}'(m)\tilde{W}' = (\pi'({}^t m^{-1})W')^\sim$$

and therefore that, by (3.3),

$$f'(1-s, {}^t m) = \omega'(-1) \varepsilon'(s, \pi \times \pi', \psi) f(s, m^{-1}).$$

Combining this with (3.1), one arrives at the functional equation (2.3) with

$$\varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \pi' \times \sigma, \psi) \varepsilon(s, \pi' \otimes \mu, \psi).$$

Hence (2.1) to (2.3) are now completely proved for $p = 3$. For instance, if μ , π , and π' are 'unramified' – contain the trivial representation of a maximal compact subgroup, the various factors have the following form:

$$(4.5) \quad \begin{aligned} L(s, \mu) &= (1 - cq^{-s})^{-1}, & L(s, \sigma) &= (1 - aq^{-s})^{-1}(1 - bq^{-s})^{-1}, \\ L(s, \pi) &= (1 - a'q^{-s})^{-1}(1 - b'q^{-s})^{-1}, \\ L(s, \pi \times \pi')^{-1} &= (1 - a'cq^{-s})(1 - b'cq^{-s})(1 - aa'q^{-s})(1 - ba'q^{-s})(1 - ab'q^{-s}) \\ &\quad \times (1 - bb'q^{-s}). \end{aligned}$$

If, in addition, the character ψ is of order zero, then the ε factor is one.

5. Conclusion

Let us go back to the situation and the notations of Section 1. Let π be an admissible irreducible representation of $G_{\mathbb{A}}$. Then, π is, in a certain sense, an infinite tensor product $\otimes \pi_v$, where, for each place v , π_v is an admissible irreducible representation of $G_v = GL(3, F_v)$; almost all the representations π_v are unramified. We shall assume that π is non-degenerate (i.e. that each π_v is non-degenerate) and also that there is a character ω of \mathbb{A}/F^\times such that

$$\pi \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \omega(a) \cdot 1 \quad \text{for } a \text{ in } \mathbb{A}.$$

Let also π' be a representation of $G'_{\mathbb{A}}$ satisfying the same assumptions. Then, we define

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v), \quad \varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

The first product is an infinite Euler product whose factors are almost all of the form (4.5); if, for instance, the representations are pre-unitary, then it converges for Res sufficiently large. In the second factor, we have set $\psi(x) = \prod \psi_v(x_v)$; the product has only a finite number of terms $\neq 1$ and is actually independent of the choice of ψ .

Suppose now that π and π' are contained in the corresponding space of cusp-forms. They are then both pre-unitary and non-degenerate. The factor $L(s, \pi \times \pi')$ is a polynomial in Q^{-s} , Q^s , and satisfies the functional equation

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1-s, \tilde{\pi} \times \tilde{\pi}').$$

The proof follows step by step the proof of (11.1) in [3].

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