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SQUARE-INTEGRABLE REPRESENTATIONS MOD Z OF UNIPOTENT GROUPS

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1. Introduction

Let G be a locally compact unimodular group, Z the center of G and $\mathrm{d}\dot{x}$ a Haar measure on G/Z. An irreducible unitary representation π of G on the Hilbert space $\mathscr{H}$, with scalar product $\langle , \rangle$, is called square-integrable mod Z if there exist $\xi, \eta \in \mathscr{H}$, both non-zero, such that

$$\int_{G/Z} |\langle \pi(x)\xi, \eta\rangle|^2 \mathrm{d}\dot{x} < \infty.$$

Let k be a locally compact field of characteristic zero. From now on, we denote by G the group of k-points of a unipotent algebraic group, defined over k. G is locally compact and unimodular. Let $\mathfrak{g}$ be the Lie algebra of G (G being viewed as a Lie group over k) and $\mathfrak{g}^*$ its dual as a k-vector space. G acts on $\mathfrak{g}$ by the adjoint representation and hence on $\mathfrak{g}^*$ by duality. According to Kirillov there is a one-to-one correspondence between the set of equivalence-classes of irreducible unitary representations of G and the set of G-orbits in $\mathfrak{g}^*$. Given $f \in \mathfrak{g}^*$, let π_f stand for a representative of the corresponding class of irreducible unitary representations. In a recent paper [9], C. C. Moore and J. Wolf characterize the representations π_f, which are square-integrable mod Z, in terms of f, at least for $k = \mathbb{R}, \mathbb{C}$, but is not too difficult to prove the same results for non-archimedean fields k of characteristic zero: one has Kirillov's theory and the Plancherel formula can be proved, together with the right normalizations of the measures on the orbits. In this paper we prove an additional result in case k is non-archimedean. Let π be an irreducible unitary representation of G on $\mathscr{H}$. A vector $\xi \in \mathscr{H}$ is called locally constant if the map

$$x \mapsto \pi(x)\xi \qquad (x \in G)$$

is locally constant. We have the following theorem: π is square-integrable mod Z if and only if for each pair ξ, η of locally constant vectors in $\mathscr{H}$ the coefficient $x \mapsto \langle \pi(x)\xi, \eta\rangle$ $(x \in G)$ is a locally constant complex-valued function with compact support mod Z.

2. Preliminaries

We recall here the definition of induced (unitary) representations, some analysis on locally compact fields k and the definition and some properties of locally constant vectors.

Induced representations

Let G be a locally compact group and H a closed subgroup of G. It is enough for our purposes to assume both G and H unimodular. It is well-known that the quotient space $H\backslash G$ carries then a G-invariant positive measure, which we denote by $d\dot{x}$ ($\dot{x}$ being the notation for elements of $H\backslash G$). Now let ρ be any (continuous) unitary representation of H on a Hilbert space V, with scalar product denoted by $\langle , \rangle$. Consider the vectorspace $\mathscr{H}_0$ of all continuous functions $f: G \to V$ satisfying

(i) $f(hx) = \rho(h)f(x)$ $(h \in H, x \in G)$

(ii) f has compact support mod H, i.e. there is a compact set $K \subset G$ such that Supp $f \subset HK$, K depending on f.

G acts on $\mathscr{H}_0$ by

$$\pi_0(x)f(y) = f(yx) \qquad (x, y \in G;\ f \in \mathscr{H}_0).$$

We define a scalar product on $\mathscr{H}_0$ as follows:

$$(f, g) = \int_{H\backslash G} \langle f(x), g(x)\rangle \mathrm{d}\dot{x}.$$

Since $\mathrm{d}\dot{x}$ is G-invariant, π_0 becomes a unitary representation of G on $\mathscr{H}_0$. Denote by $\mathscr{H}$ the completion of $\mathscr{H}_0$ and by π the extension of π_0 to $\mathscr{H}$. Then π is a (continuous) unitary representation of G on the Hilbert-space $\mathscr{H}$. We call π the representation of G induced by ρ and denote it also by

$$\pi = \operatorname*{ind}_{H\uparrow G} \rho.$$

Some analysis on locally compact fields

Let k be a locally compact, non-discrete, field. So $k = \mathbb{R}$, $\mathbb{C}$ or a $\mathfrak{p}$-adic field, with a discrete valuation. There is an absolute value on k, denoted by $|.|$, which we assume to be normalized in the following way. Let $\mathrm{d}x$ be an additive Haar measure on k. Then $\mathrm{d}(ax) = |a|\mathrm{d}x$ $(a \in k^*)$.

Note that in case $k = \mathbb{R}$ we have the usual absolute value, in case $k = \mathbb{C}$ we have just the square of the usual one.

For $\mathfrak{p}$-adic fields k, we introduce some more notation. Let $\mathcal{O}$ be the ring of integers: $\mathcal{O} = \{x \in k : |x| \leqq 1\}$; $\mathcal{O}$ is a local ring with unique maximal ideal P, given by $P = \{x \in k : |x| < 1\}$. The residue-class field $\mathcal{O}/P$ has finitely many, say q, elements. P is a principal ideal with generator ϖ. So $P = \varpi\mathcal{O}$, $|\varpi| = q^{-1}$. Put $P^n = \varpi^n\mathcal{O} (n \in \mathbb{Z})$. Usually the Haar measure on k is normalized such that $\int_{\mathcal{O}} dx = 1$. Let us fix a non-trivial (continuous) character χ_0 of the additive group of k in the following way. If $k = \mathbb{R}$, choose $\chi_0(x) = e^{2\pi i x}$; if $k = \mathbb{C}$, choose $\chi_0(z) = e^{2\pi i \,\mathrm{Re}(z)}$; if k is a $\mathfrak{p}$-adic field, fix a character χ_0 such that $\chi_0(x) = 1$ for all $x \in \mathcal{O}$, $\chi_0(x) \neq 1$ for some $x \in P^{-1}$. It is well-known that each continuous additive character of k can be written in the form $x \mapsto \chi_0(xy)$ for some $y \in k$. Actually, $\hat{k}$ is naturally isomorphic to k.

For $f \in L^1(k)$ one defines its Fourier transform by

$$\hat{f}(y) = \int_k f(x)\bar{\chi}_0(xy)dx \qquad (y \in k).$$

Due to our normalizations of χ_0 and dx (on $\mathbb{R}$ and $\mathbb{C}$ we choose the usual Lebesgue measures), the Plancherel formula reads

$$\int_k |f(x)|^2 dx = \int_k |\hat{f}(y)|^2 dy \qquad (f \in L^1 \cap L^2).$$

For archimedean k ($k = \mathbb{R}, \mathbb{C}$) it is well-known that the Fourier transform is an isomorphism of the space $\mathcal{S}(k)$, the so-called Schwartz-space of rapidly decreasing C^∞-functions on k, endowed with its usual locally convex Hausdorff space structure. For non-archimedean k (k $\mathfrak{p}$-adic), we denote by $C_c^\infty(k)$ the space of locally constant complex-valued functions on k with compact support. Given $f \in C_c^\infty(k)$, there are $m, n \in \mathbb{Z}$ such that (i) Supp $f \subset P^n$ (ii) $f(x+y) = f(x)$ $(x \in k, y \in P^m)$. The following result is easy to prove:

$f \in C_c^\infty(k)$ is supported on P^n and constant on cosets of P^m
iff $\hat{f} \in C_c^\infty(k)$ is supported on P^{-m} and constant on cosets of P^{-n}.

Locally constant vectors

Let G be a locally compact, totally disconnected, group. It is known that G has arbitrarily small open compact subgroups. Let us denote by $C_c^\infty(G)$ the space of locally constant complex-valued functions on G with compact support.

Let V be any complex vector space and π a representation of G on V. π is called *admissible* (following Jacquet, Langlands) if

(i) for each $v \in V$ the map $x \mapsto \pi(x)v$ is locally constant,
(ii) given any *open* subgroup K of G, the space of vectors $v \in V$ which are left fixed by $\pi(K)$, is finite-dimensional.

We do not go into detail about this kind of representation. We just need one result.

Let π be a (continuous) unitary representation of G on the Hilbert space $\mathscr{H}$. A vector $\xi \in \mathscr{H}$ is called *locally constant* if the map $x \mapsto \pi(x)\xi$ is locally constant. $\pi(C_c^\infty(G))\mathscr{H}$ is dense in $\mathscr{H}$ and consists entirely of locally constant vectors: the locally constant vectors are dense in $\mathscr{H}$. They span a G-invariant subspace V of $\mathscr{H}$. Let us denote by π_0 the restriction of π to V. Then we have the following result. Assume π_0 to be admissible. Then π is (topologically) irreducible iff π_0 is (algebraically) irreducible. In particular, each $v \in V$ can be written as

$$v = \sum_{i=1}^{n} \lambda_i \pi(g_i) v_0 \qquad (v_0 \in V \text{ fixed, } v_0 \neq 0)$$

for some $g_1, \cdots, g_n \in G$; $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$.

3. Square-integrable representations mod Z

Let G be any locally compact unimodular group with center Z. By Z_0 we denote a closed subgroup of Z. G/Z_0 carries a Haar measure, which is denoted by $d\dot{x}$.

Let π be an irreducible unitary representation of G on a Hilbert space, called $\mathscr{H}(\pi)$. By Schur's Lemma, we have $\pi(z) = \lambda_\pi(z)1 (z \in Z)$ where λ_π is a continuous homomorphism $Z \to \mathbb{C}^*$.

π is called *square-integrable mod* Z_0 if there exist $\xi, \eta \in \mathscr{H}(\pi) - \{0\}$, such that

$$\int_{G/Z_0} |\langle \pi(\dot{x})\xi, \eta \rangle|^2 d\dot{x} < \infty.$$

We note that a necessary condition for existence of such π is given by: Z/Z_0 is compact. But, for the time being, we do not make this assumption. We have, similar to [1], Théorème 5.15(a) and [9], Theorem A the following slightly modified theorem:

THEOREM: *Let π be an irreducible unitary representation of G on $\mathscr{H}(\pi)$. The following three conditions are equivalent*:

(i) *There exist $\xi, \eta \in \mathscr{H}(\pi) - \{0\}$ such that* $\int_{G/Z_0} |\langle \pi(x)\xi, \eta \rangle|^2 d\dot{x} < \infty$.
(ii) *For all $\xi, \eta \in \mathscr{H}(\pi)$ one has* $\int_{G/Z_0} |\langle \pi(x)\xi, \eta \rangle|^2 d\dot{x} < \infty$.
(iii) *π is a direct summand of* $\mathrm{ind}_{Z_0 \uparrow G} \lambda_\pi$.

The proof is similar to [1], Démonstration of Théorème 5.15(a).

4. Kirillov's theorem

Let k be a locally compact field of characteristic zero. By $\mathbf{G}$ we denote a unipotent algebraic group, defined over k, with Lie algebra $\mathfrak{g}$.

Let G, $\mathfrak{g}$ be the sets of k-points of $\mathbf{G}$, $\mathfrak{g}$ respectively. We have the k-isomorphism of algebraic varieties $\exp: \mathfrak{g} \to \mathbf{G}$, which maps $\mathfrak{g}$ onto G. Let 'log' denote its inverse.

We shall simply call $\mathfrak{g}$ the Lie algebra of G, which is even correct if G is viewed as a Lie group.

Let Z be the center of G, its Lie algebra $\mathfrak{z}$. One has $\exp \mathfrak{z} = Z$. More generally: the exponential of a subalgebra of $\mathfrak{g}$ is a closed subgroup of G (in the ordinary topology), the exponential of an ideal in $\mathfrak{g}$ is a normal subgroup of G. This are easy consequences of the Campbell-Hausdorff formula. (cf. [5], ch V, 5).

Let us fix a non-trivial character χ_0 of k as in Section 2. G acts on $\mathfrak{g}$ by means of the adjoint representation Ad and hence, by duality, on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$ as a k-vector space. Let $f \in \mathfrak{g}^*$ and denote by $\mathfrak{h}_f$ any subalgebra of $\mathfrak{g}$ such that $f(X) = 0$ for all $X \in [\mathfrak{h}_f, \mathfrak{h}_f]$. Let $H_f = \exp \mathfrak{h}_f$, the subgroup of G corresponding to $\mathfrak{h}_f$. Then $\tau : x \mapsto \chi_0(f(\log x))$ is a 1-dimensional unitary representation of H_f.

Denote by $\pi_{f, \mathfrak{h}_f}$ the unitary representation induced by τ. We have the following theorem:

THEOREM:

(i) *$\pi_{f, \mathfrak{h}_f}$ is irreducible iff $\mathfrak{h}_f$ is maximal with respect to the property that f vanishes on $[\mathfrak{h}_f, \mathfrak{h}_f]$.*

(ii) *Let $f, g \in \mathfrak{g}^*$ and suppose $\pi_{f, \mathfrak{h}_f}$ and $\pi_{g, \mathfrak{h}_g}$ irreducible. $\pi_{f, \mathfrak{h}_f}$ is equivalent to $\pi_{g, \mathfrak{h}_g}$ iff f and g are in the same G-orbit in $\mathfrak{g}^*$.*

(iii) *Each irreducible unitary representation of G is of the form $\pi_{f, \mathfrak{h}_f}$ for some $f \in \mathfrak{g}^*$.*

This theorem was first proved by Kirillov for real groups ([6]). C. C. Moore observed that the same proof went through for all locally compact fields of characteristic zero ([8], Theorem 3). Actually we shall apply not only the above theorem, but also make use of a few lemma's, which play a role in Kirillov's proof. By the above theorem, there is a one-to-one correspondence between the set of equivalence-classes of irreducible unitary representations of G and the set of G-orbits in $\mathfrak{g}^*$.

5. The main theorem

In case of reductive $\mathfrak{p}$-adic groups G, Harish-Chandra introduced the notion of supercuspidal unitary representation ([3], Part I, § 3).

Later on he proved that supercuspidal representations are completely characterized in the set of irreducible unitary representations of G by the following property: for each pair of locally constant vectors ξ, η the coefficient $x \mapsto \langle \pi(x)\xi, \eta \rangle$ has compact support mod Z, where Z is the center of G ([4], Theorem 6). Keeping this in mind, we can easily define the notion of supercuspidal representation for each locally compact, totally disconnected group, G. We shall not do it.

Note that supercuspidal representations are square-integrable mod Z. The converse is not true in general, due to the existence of so-called special representations (cf. [7]).

Now let k be a $\mathfrak{p}$-*adic* field of characteristic zero and let G be as in Section 4. Our purpose is to prove the following theorem:

THEOREM: *Let π be an irreducible unitary representation of G on $\mathscr{H}(\pi)$. Then π is square-integrable mod Z iff for each pair of locally constant vectors $\xi, \eta \in \mathscr{H}(\pi)$ the coefficient $x \mapsto \langle \pi(x)\xi, \eta \rangle$ has compact support mod Z.*

PROOF: Let π be an irreducible unitary representation of G on $\mathscr{H}(\pi)$. Since the locally constant vectors are dense in $\mathscr{H}(\pi)$, one implication is obvious.

Let us now assume that π is square-integrable mod Z and prove the other implication. As observed by Moore, it follows from a result of Fell ([2]), that the restriction of π to the space of locally constant vectors is admissible (Section 2)[1]. Therefore, it is enough to prove the theorem for a fixed pair of non-zero locally constant vectors ξ and η in $\mathscr{H}(\pi)$ (cf. Section 2). We use induction on $\dim G$.

If $\dim G = 1$, G is abelian and the theorem is evident.

Suppose $\dim G > 1$. We have two cases: $\dim \mathfrak{z} = 1$ and $\dim \mathfrak{z} > 1$.

(i) $\dim \mathfrak{z} > 1$. According to Section 4, let $\pi = \pi_{f, \mathfrak{h}_f}$ for some $f \in \mathfrak{g}^*$. There is a subalgebra $\mathfrak{z}^0 \subset \mathfrak{z}$, $\mathfrak{z}^0 \neq 0$ such that $\mathfrak{z}^0 \subset \operatorname{Ker}(f)$. Put $Z^0 = \exp \mathfrak{z}^0$. Then $Z^0 \subset \operatorname{Ker}(\pi)$ since $\pi(z) = \chi_0(f(\log z))(z \in Z)$, according to the theorem of Section 4. Denote by π^0 the representation π of G, pulled down to G/Z^0. Of course, the locally constant vectors with respect to π are the same as those with respect to π^0. Since π has a non-zero matrix coefficient which is square-integrable mod Z, π^0 has one which is square-integrable mod Z/Z_0. Hence center (G/Z^0) mod Z/Z^0 is compact, hence equals $\{1\}$, hence center $(G/Z^0) = Z/Z^0$. By induction hypothesis

$$x \mapsto \langle \pi^0(x)\xi, \eta \rangle \qquad (x \in G/Z^0)$$

[1] See also [3], Part I, § 1. Fell's result holds for any irreducible unitary representation π of G.

has compact support mod Z/Z^0 for all locally constant vectors $\xi, \eta \in \mathscr{H}(\pi)$. Therefore $x \mapsto \langle \pi(x)\xi, \eta\rangle$ $(x \in G)$ has compact support mod Z.

(ii) $\dim \mathfrak{z} = 1$.

We may assume that $f \neq 0$ on $\mathfrak{z}(\pi = \pi_{f, \mathfrak{h}_f})$, otherwise the method of (i) applies again. Let X be an element of $\mathfrak{g}$ such that $[X, \mathfrak{g}] \subset \mathfrak{z}$, $X \notin \mathfrak{z}$. Put $\mathfrak{g}_0 = \{U : [U, X] = 0\}$. Then $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$ of codimension one. We may arrange that $f(X) = 0$ and we assume this done. Put $G_0 = \exp \mathfrak{g}_0$. G_0 is a closed normal subgroup of G. Let f_0 be the restriction of f to $\mathfrak{g}_0$ and denote by π_0 an irreducible unitary representation of G_0 corresponding to f_0 by Kirillov's theorem (Section 4). It is part of the Kirillov theory that π is equivalent to $\mathrm{ind}_{G_0 \uparrow G}\pi_0$ ([10]).

Choose $Y \notin \mathfrak{g}_0$ such that $f([X, Y]) = 1$. Put $G_1 = \exp (sY)_{s \in k}$. G_1 is a closed 1-parameter subgroup of G such that $G = G_0 \cdot G_1$ and $G_0 \cap G_1 = \{e\}$. Put $\mathfrak{z}_0 = \mathfrak{z} + (X)$ and $Z_0 = \exp \mathfrak{z}_0$.

So $Z_0 = Z \cdot (\exp tX)_{t \in k} \cdot Z_0$ is contained in the center of $\mathfrak{g}_0$. Let us consider the space $\mathscr{H}(\pi)$ of π. It can be identified with the space $L^2(G_1, \mathscr{H}(\pi_0))$. The group G acts on $\mathscr{H}(\pi)$ as follows:

$$\begin{aligned}\pi(x)\varphi(\xi) &= \varphi(\xi x) = \varphi(\xi x_0 x_1) = \varphi(\xi x_0 \xi^{-1}\xi x_1)\\ &= \pi_0(\xi x_0 \xi^{-1})\varphi(\xi x_1)\end{aligned}$$

$$(x \in G;\ x = x_0 x_1 (x_0 \in G_0,\ x_1 \in G_1),\ \varphi \in L^2(G_1, \mathscr{H}(\pi_0)),\ \xi \in G_1)$$

Fix a locally constant vector $v \in \mathscr{H}(\pi_0)$ such that $x_0 \mapsto \langle \pi_0(x_0)v, v\rangle$ does not vanish on G_0. Choose $\psi \in C_c^\infty(G_1)$, $\psi \neq 0$ and put

$$\psi_v(x_1) = \psi(x_1)v \qquad (x_1 \in G_1).$$

Then $\psi_v \in \mathscr{H}(\pi)$. We have

$$\langle \pi(x)\psi_v, \psi_v\rangle = \int_{G_1} \psi(\xi x_1)\bar{\psi}(\xi)\langle \pi_0(\xi x_0 \xi^{-1})v, v\rangle \mathrm{d}\xi$$

$$(x = x_0 x_1,\ x_0 \in G_0,\ x_1 \in G_1).$$

Let us choose a complementary subspace $\mathfrak{a}_0$ of $\mathfrak{z}_0$ in $\mathfrak{g}_0$ and put $A_0 = \exp \mathfrak{a}_0$. Then $G_0 = A_0 \cdot Z_0 = A_0 .\ (\exp tX)_{t \in k} \cdot Z$. Let us write $x_0 = a_0 \cdot \exp tX \cdot z$ $(x_0 \in G_0,\ a_0 \in A_0,\ t \in k,\ z \in Z)$. Then we have for $\xi \in G_1$:

$$\langle \pi_0(\xi x_0 \xi^{-1})v, v\rangle = \lambda_{\pi_0}(\xi \exp tX \cdot \xi^{-1})\lambda_{\pi_0}(z)\langle \pi_0(\xi a_0 \xi^{-1})v, v\rangle.$$

Hence, for $x = x_0 x_1 = a_0 \cdot \exp tX \cdot z \cdot x_1$,

$$\begin{aligned}&\langle \pi(x)\psi_v, \psi_v\rangle\\ &\quad = \lambda_{\pi_0}(z) \int_{G_1} \psi(\xi x_1)\bar{\psi}(\xi)\langle \pi_0(\xi a_0 \xi^{-1})v, v\rangle \lambda_{\pi_0}(\xi \cdot \exp tX \cdot \xi^{-1})\mathrm{d}\xi.\end{aligned}$$

Now $\xi = \exp(sY)$ for some $s \in k$. Therefore:

$$\lambda_{\pi 0}(\xi \cdot \exp tX \cdot \xi^{-1}) = \lambda_{\pi 0}(\exp e^{s \operatorname{ad} Y}(tX)) = \chi_0(f_0(e^{s \operatorname{ad} Y}(tX)))$$
$$= \chi_0(f_0\{tX + st[Y, X]\}) = \bar{\chi}_0(st).$$

So we get:

$$(1) \qquad \langle \pi(x)\psi_v, \psi_v \rangle = \lambda_{\pi_0}(z) \int_k \psi(\exp(s+s_1)Y)\bar{\psi}(\exp sY) \times \langle \pi_0(\exp sY \cdot a_0 \cdot \exp(-sY))v, v \rangle \bar{\chi}_0(st) \mathrm{d}s$$
$$(x = a_0 \cdot \exp tX \cdot \exp s_1 Y \cdot z)$$

The Haar measure on G is given by $\mathrm{d}a_0 \mathrm{d}t \mathrm{d}s_1 \mathrm{d}z$, corresponding to the decomposition $x = a_0 \cdot \exp tX \cdot \exp s_1 Y \cdot z$, as above; $\mathrm{d}a_0$ stands for the exponential of a canonical additive Haar measure on $\mathfrak{a}_0$. By assumption $\int_{G/Z} |\langle \pi(x)\psi_v, \psi_v \rangle|^2 \mathrm{d}\dot{x} < \infty$ (cf. Section 3, Theorem, (ii)). Hence the following integral converges:

$$\int_k \int_{A_0} \int_k \left| \int_k \psi(\exp(s+s_1)Y)\bar{\psi}(\exp sY) \times \langle \pi_0(\exp sY \cdot a_0 \cdot \exp(-sY))v, v \rangle \bar{\chi}_0(st) \mathrm{d}s \right|^2 \mathrm{d}t \, \mathrm{d}a_0 \, \mathrm{d}s_1$$

But this equals, applying Plancherel's formula on k (cf. Section 2):

$$\int_{A_0} \int_k \int_k |\psi(\exp(s+s_1)Y)\bar{\psi}(\exp sY) \times \langle \pi_0(\exp sY \cdot a_0 \cdot \exp(-sY))v, v \rangle|^2 \mathrm{d}s \, \mathrm{d}s_1 \, \mathrm{d}a_0.$$

This implies for almost all $s \in k$:

$$\int_{A_0} |\langle \pi_0(\exp sY \cdot a_0 \cdot \exp(-sY))v, v \rangle|^2 \mathrm{d}a_0 < \infty,$$

or

$$\int_{G_0/Z_0} |\langle \pi_0(\exp sY \cdot x_0 \cdot \exp(-sY))v, v \rangle|^2 \mathrm{d}\dot{x}_0 < \infty,$$

since $\exp sY \cdot Z_0 \cdot \exp(-sY) = Z_0$. But there exists a positive constant $c(s)$ such that

$$\int_{G_0/Z_0} g(\exp sY \cdot x_0 \exp(-sY)) \mathrm{d}\dot{x}_0 = c(s) \int_{G_0/Z_0} g(x_0) \mathrm{d}\dot{x}_0$$

for all continuous complex-valued functions g on G_0/Z_0 with compact support, and hence, by extension, for $g \in L^1(G_0/Z_0)$. This implies easily:

$$\int_{G_0/Z_0} |\langle \pi_0(x_0)v, v\rangle|^2 \mathrm{d}\dot{x}_0 < \infty.$$

Hence π_0 is square-integrable mod Z_0. Therefore center $(G_0)/Z_0$ is compact, hence center $(G_0) = Z_0$. By induction hypothesis

$$x_0 \mapsto \langle \pi_0(x_0)v, v\rangle \qquad (x_0 \in G_0)$$

has compact support mod Z_0.

Now consider the above expression (1) for $\langle \pi(x)\psi_v, \psi_v\rangle$. Notice that ψ_v is a locally constant vector in $\mathscr{H}(\pi)$, different from 0. We shall show that

$$x \mapsto \langle \pi(x)\psi_v, \psi_v\rangle$$

vanishes outside a compact set mod Z.

The function

$$(s, s_1, a_0) \mapsto \psi(\exp{(s+s_1)Y})\bar{\psi}(\exp sY)\langle \pi_0(\exp sY \cdot a_0 \cdot \exp{(-sY)})v, v\rangle$$

vanishes for s, s_1, a_0 outside an open compact set of $k \times k \times A_0$. Moreover, there is $n \in \mathbb{Z}$, independent of the choice of a_0 and s_1, such that

$$s \mapsto \psi(\exp{(s+s_1)Y})\bar{\psi}(\exp sY)\langle \pi_0(\exp sY \cdot a_0 \cdot \exp{(-sY)})v, v\rangle$$

is locally constant on cosets of P^n. Applying the Fourier transform yields that

$$x \mapsto \langle \pi(x)\psi_v, \psi_v\rangle$$

vanishes for t outside a compact set ($x = a_0 \cdot \exp tX \cdot \exp s_1 Y \cdot z$) (cf. Section 2). Hence

$$x \mapsto \langle \pi(x)\psi_v, \psi_v\rangle$$

vanishes outside a compact set mod Z.

This completes the proof of the theorem.

REMARK: Let $k = \mathbb{R}$ and let $\mathfrak{g}_0$ be any subspace complementary to $\mathfrak{z}$. With similar methods one proves the following: π is square-integrable mod Z iff there exists $\xi \neq 0$, $\xi \in \mathscr{H}(\pi)$ such that $X \mapsto \langle \pi(\exp X)\xi, \xi\rangle$ ($X \in \mathfrak{g}_0$) belongs to $\mathscr{S}(\mathfrak{g}_0)$, the space of Schwartz-functions on $\mathfrak{g}_0$. This more careful formulation is due to the fact that we do not have a clear idea about the notion of admissible representation in the real case. In particular we do not see a good candidate for the space of locally constant vectors with respect to an (irreducible) unitary representation of G.

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