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ON ADDITIVE FUNCTIONS

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1. Introduction

A function F defined on the set of positive integers is said to be additive if

$$F(nm) = F(n)+F(m) \tag{1}$$

whenever $(n, m) = 1$; F is called completely additive if (1) holds for every pair of positive integers n and m. Obviously a completely additive function is determined by its values on the set of primes. The function $F(n) = c \log n$ (c constant) is an example of a completely additive function. On the other hand Erdös [1] proved that if an additive function F satisfies one of the following two conditions

$$F(n+1) \geqq F(n) \qquad \text{for all } n \tag{2}$$

or

$$F(n+1)-F(n) = o(1) \qquad \text{as } n \to \infty \tag{3}$$

then F is necessarily of the form $F(n) = c \log n$ for some constant c. The same results have been found by several other authors.

Pisot and Schoenberg [2] considered the following situation. Let $p_1, \cdots, p_r$ denote r different primes, let A be the multiplicative semigroup generated by $p_1, \cdots, p_r$ and let F be an additive function defined on A. Pisot and Schoenberg proved that if F is non decreasing on A, then $F(n) = c \log n$ on A provided that $r \geqq 3$. (For $r = 2$ monotonic additive functions exist which are not of the form $c \log n$). Obviously this result is a generalisation of the first mentioned result of Erdös. Now one may ask if a condition similar to condition (3) would imply that $F(n) = c \log n$ on A. The present paper deals with this question.

We write $A = \{1 = n_1 < n_2 < n_3 < \cdots\}$ and restrict ourselves to functions F on A which are completely additive. In section 2 we give some results on the structure of A. For the proofs we refer to [3], [4] and [5]. In section 3 we prove that

$$F(n_{i+1})-F(n_i) = o(1) \qquad \text{as } i \to \infty$$

implies that $F(n) = c \log n$ on A. Subsequently we prove in section 4 that every completely additive function F on A satisfies

$$F(n_{i+1})-F(n_i) = O(\log (n_{i+1}-n_i)) \qquad \text{as } i \to \infty.$$

We conjecture that the result of section 3 can be improved considerably and that the condition

$$F(n_{i+1})-F(n_i) = o(\log (n_{i+1}-n_i)) \qquad \text{as } i \to \infty$$

is already sufficient to imply that $F(n) = c \log n$ on A. We are able to prove the conjecture for $r \leqq 5$; see section 5.

2. The structure of A

In the sequel we shall use the following notations. By $p_1, \cdots, p_r$ we denote r different primes ($r \geqq 2$). The multiplicative semigroup generated by them will be denoted by A; $1 = n_1 < n_2 < n_3 < \cdots$ are the elements of A in increasing order.

LEMMA 1: *There exist positive constants C_1, C_2 and N such that*

$$\frac{n_i}{(\log n_i)^{C_1}} < n_{i+1}-n_i < \frac{n_i}{(\log n_i)^{C_2}} \qquad \text{for } n_i \geqq N.$$

PROOF: The first inequality is the corollary of [3] Theorem 1. The second can be found in [4].

We shall use the following easy consequences of lemma 1:

(4) $$n_{i+1}-n_i \geqq \sqrt{n_i} \qquad \text{for } n_i \geqq N_1,$$

(5) $$n_{i+1} \leqq 2n_i \qquad \text{for } n_i \geqq N_2,$$

(6) $$\frac{\log n_{i+1}}{\log n_i} \to 1 \qquad \text{for } i \to \infty,$$

$$\frac{n_{i+1}}{n_i} \to 1 \qquad \text{for } i \to \infty. \tag{7}$$

LEMMA 2: *Let p be one of the primes* $\{p_1, \cdots, p_r\}$. *Then there exists an infinite number of pairs* n_i, n_{i+1} *such that* n_i *is a pure power of p and* n_{i+1} *is not divisible by p.*

PROOF: See [5] Theorem 2.

LEMMA 3: *Let p and q be two different primes from* $\{p_1, \cdots, p_r\}$. *There exist infinitely many pairs* n_i, n_{i+1} *such that one of the numbers* n_i, n_{i+1} *is composed of p and q and the other is neither divisible by p nor by q.*

PROOF: See [5] Theorem 3.

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In this section we prove the following theorem.

THEOREM 1: *Let F be a completely additive function defined on A satisfying*

$$F(n_{i+1}) - F(n_i) = o(1) \quad \text{as} \quad i \to \infty, \tag{8}$$

then $F(n) = c \log n$ *on A for some constant c.*

PROOF: Put $F(p_\rho) = c_\rho \log p_\rho$ $(\rho = 1, \cdots, r)$. Without loss of generality we may assume that the $p_1, \cdots, p_r$ are arranged in such a way that $c_1 \geqq c_2 \geqq \cdots \geqq c_r$. We shall prove $c_1 = \cdots = c_r$, which obviously implies the assertion of the theorem.

Suppose $c_1 = c_2 = \cdots = c_s > c_{s+1} \geqq \cdots \geqq c_r$ for some $s \in \{1, \cdots, r-1\}$. Let n_i, n_{i+1} be a pair of consecutive elements of A such that n_i is composed of primes $p_1, \cdots, p_s$ only and n_{i+1} contains at least one prime from $\{p_{s+1}, \cdots, p_r\}$. It is evident that an infinite sequence of such pairs exists.

Put $n_{i+1} = p_1^{k_1} \cdots p_s^{k_s} p_{s+1}^{k_{s+1}} \cdots p_r^{k_r}$, then

$$\begin{aligned} F(n_{i+1}) &= c_1 k_1 \log p_1 + \cdots + c_s k_s \log p_s + c_{s+1} k_{s+1} \log p_{s+1} + \cdots \\ &\quad + c_r k_r \log p_r = c_1(k_1 \log p_1 + \cdots + k_r \log p_r) - R = c_1 \log n_{i+1} - R, \end{aligned}$$

where

$$R \geqq (c_s - c_{s+1}) \log 2. \tag{9}$$

Since $F(n_i) = c_1 \log n_i$ we obtain

$$F(n_i) - F(n_{i+1}) = c_1 \log \frac{n_i}{n_{i+1}} + R.$$

In view of (7) and (9) this is a contradiction with (8).

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Using the estimates of $n_{i+1} - n_i$ we can easily deduce the following theorem.

THEOREM 2: *Let F be a completely additive function on A. Then*

$$F(n_{i+1}) - F(n_i) = O(\log (n_{i+1} - n_i)) \quad \textit{as } i \to \infty.$$

PROOF: Put $F(p_\rho) = c_\rho \log p_\rho$ $(\rho = 1, \cdots, r)$ and $M = \max_\rho |c_\rho|$. Then we obtain for $n_i = p_1^{t_1} \cdots p_r^{t_r}$ that

$$|F(n_i)| = |t_1 c_1 \log p_1 + \cdots + t_r c_r \log p_r| \leqq M \log n_i .$$

Hence by (5),

$$|F(n_{i+1}) - F(n_i)| \leqq 2M \log n_{i+1} \leqq 2M \log 2n_i \qquad \text{for } n_i \geqq N_2 .$$

Therefore using (4) we obtain

$$\left| \frac{F(n_{i+1}) - F(n_i)}{\log (n_{i+1} - n_i)} \right| \leqq \frac{4M \log 2n_i}{\log\ n_i} = O(1) \qquad \text{for } n_i \geqq \max (N_1, N_2)$$

from which the theorem follows.

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For $r \leqq 5$ we can improve theorem 1. To that purpose we prove the following theorem.

THEOREM 3: *Let F be a completely additive function on A. Put $F(p_\rho) = c_\rho \log p_\rho$ $(\rho = 1, \cdots, r)$. We may suppose that $p_1, \cdots, p_r$ are arranged in such a way that $c_1 \geqq c_2 \geqq \cdots \geqq c_r$. If*

(10) $$F(n_{i+1})-F(n_i) = o(\log (n_{i+1}-n_i)) \qquad \text{as } i \to \infty,$$

then it follows that $c_1 = c_2$ *if* $r = 2$ *and* $c_1 = c_2 = c_3$ *and* $c_{r-2} = c_{r-1} = c_r$ *if* $r \geqq 3$.

COROLLARY: *If* $r = 2, 3, 4$ *or* 5 *then condition* (10) *is a sufficient one for a completely additive function on* A *to imply* $F(n) = c \log n$ *on* A. *Obviously this is an improvement of theorem* 1.

PROOF OF THEOREM 3: Suppose $c_1 > c_2$. Choose $\varepsilon > 0$ such that

(11) $$c_2+\varepsilon < c_1.$$

We obtain from (10) that $F(n_i) \leqq F(n_{i+1})+\varepsilon \log n_{i+1}$, if i is sufficiently large.

By lemma 2 there exists an infinite number of pairs n_i, n_{i+1} such that n_i is a pure power of p_1 and n_{i+1} is not divisible by p_1. Then $F(n_i) = c_1 \log n_i$ and $c_2 \geqq \cdots \geqq c_r$ implies $F(n_{i+1}) \leqq c_2 \log n_{i+1}$. Hence

$$c_1 \log n_i \leqq (c_2+\varepsilon) \log n_{i+1}$$

from which it follows by (6),

$$c_1 \leqq c_2+\varepsilon.$$

This is a contradiction with (11). Thus $c_1 = c_2$. Applying this result to the function $-F$ we obtain $c_r = c_{r-1}$.

Suppose $r \geqq 3$ and $c_1 = c_2 > c_3$. Choose $\varepsilon > 0$ such that

(12) $$c_3+\varepsilon < c_1.$$

We obtain from (10) that

(13) $$F(n_i) \leqq F(n_{i+1})+\varepsilon \log n_{i+1}$$

and

(14) $$F(n_{i+1}) \leqq F(n_i)+\varepsilon \log n_{i+1}$$

if i is sufficiently large. By lemma 3 there exist infinitely many pairs n_i, n_{i+1} such that one of them is composed of p_1 and p_2 and the other is composed of $p_3, \cdots, p_r$.

Suppose first that n_i is composed of p_1, p_2. Then $F(n_i) = c_1 \log n_i$ and since $c_3 \geqq \cdots \geqq c_r$, we have $F(n_{i+1}) \leqq c_3 \log n_{i+1}$. Hence, by (13),

$$c_1 \log n_i \leqq c_3 \log n_{i+1} + \varepsilon \log n_{i+1} \qquad (i \geqq i_0). \tag{15}$$

Suppose on the other hand that n_{i+1} is composed of p_1, p_2. Then

$$F(n_{i+1}) = c_1 \log n_{i+1} \quad \text{and} \quad F(n_i) \leqq c_3 \log n_i.$$

Therefore by (14)

$$c_1 \log n_{i+1} \leqq c_3 \log n_i + \varepsilon \log n_{i+1} \qquad (i \geqq i_0). \tag{16}$$

Since there is an infinite number of integers i for which (15) or (16) holds we obtain by (6)

$$c_1 \leqq c_3 + \varepsilon.$$

This is a contradiction with (12). Therefore $c_1 = c_2 = c_3$. Applying this result to the function $-F$ we obtain $c_r = c_{r-1} = c_{r-2}$.

REMARK: In [5] the following conjecture is made:

Let $p_1, \cdots, p_r$ be different primes and $\{n_i\}$ the increasing sequence of integers composed of these primes. Let t be fixed, $1 \leqq t \leqq r-1$. Then there exist infinitely many pairs n_i, n_{i+1} such that one of the numbers n_i, n_{i+1} is composed of $p_1, \cdots, p_t$ and the other is composed of $p_{t+1}, \cdots, p_r$.

Under the assumption that this conjecture is true, we can prove, for all r, that the condition

$$F(n_{i+1}) - F(n_i) = o(\log (n_{i+1} - n_i)) \qquad \text{as } i \to \infty$$

implies that $F(n) = c \log n$ on A for some constant c.

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